Slepian models for moving averages driven by a non-Gaussian noise

the Slepian noise approach

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SYNOPSIS

- **Slepian models** are derived describing a stochastic process observed at level crossings of a moving average driven by a **Laplace noise**.
- The approach is through a Gibbs sampler of a **Slepian model for the Laplace noise** and it allows for simultaneously studying a number of stochastic characteristics observed at the level crossing instants.
- It is observed that the behavior of the process at **high level crossings** is fundamentally different from that in the Gaussian case.
- The shape of **extreme episodes** resembles the (asymmetric) kernel while for Gaussian model the shape is given by the correlation function of which is symmetric in time.

Biased sampling distribution

- $N(T, A)$ – "number" of times the field X takes value u in $[0, T]$ and at the same time a possibly another process Y has a property A
- For ergodic stationary processes

$$\lim_{T\to\infty} \frac{N(T, A)}{N(T)} = \frac{\mathbb{E}\left[\{Y \in A\}|\dot{X}(0)|\middle|X(0) = u\right]}{\mathbb{E}\left[|\dot{X}(0)|\middle|X(0) = u\right]},$$

- RHS represents the biased sampling distribution when sampling is made over the u-level contour $\mathcal{C}_u = \{\boldsymbol{\tau} : X(\boldsymbol{\tau}) = 0\}$ (argument of the process X can be multivariate)

Rice formula for the crossing intensity

Rice formula – general case

$$\mu^+(u) = \mathbb{E}\left(\dot{X}^+(0) | X(0) = u\right) f_{X(0)}(u)$$

Gaussian case

$$\mu^+(u) = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{u^2}{2\lambda_0}},$$

λ_0, λ_2 – spectral moments

Application – upper bound for maximum in the Gaussian case

$$\mathbb{P}(M_T > u) \le \Phi(u/\sqrt{\lambda_0}) + T \cdot \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{u^2}{2\lambda_0}}$$

Non-Gaussian case – Laplace moving average

- By means of stochastic integrals we define

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{s}) d\Lambda(\mathbf{s}).$$

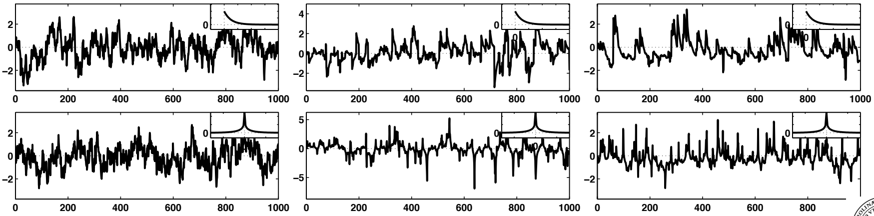
- $\Lambda(A)$ has the generalized asymmetric Laplace distribution

$$\phi(t) = \frac{1}{\left(1 - i\mu t + \frac{\sigma^2}{2} t^2\right)^{\lambda(A)}},$$

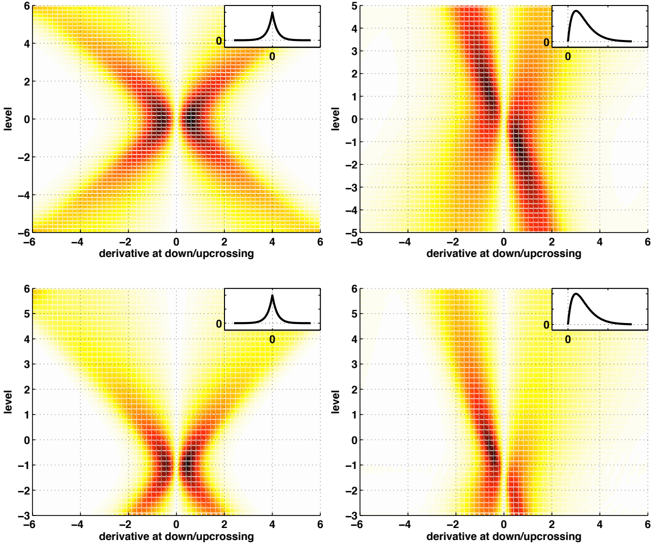
  where λ is the Lebesgue measure in $\mathbb{R}^d$.
- If $d = 1$, then $\Lambda(-\infty, x] = B(\Gamma(x))$, where B is a Brownian motion with drift and Γ is a gamma process.
- Conditionally on Γ the process X_t can be viewed as a non-stationary Gaussian process.

Non-Gaussian case – why it is interesting

- For a covariance R a matching symmetric kernel is given by $f_{sym} = \mathcal{F}^{-1}\sqrt{\mathcal{F}R}$,
- Symmetric kernels can not produce front-back asymmetries even if the moving average process is not Gaussian
- Ornstein-Uhlenbeck autocorrelation $e^{-|x|}$ can be obtained both by symmetric and asymmetric kernels



Slope distributions at level crossings



Biased sampling in non-Gaussian case – why it is difficult

- Biased sampling distribution

$$P^u(A) = \frac{\mathbb{E}\left[\{Y \in A\}|\dot{X}(0)|\middle|X(0) = u\right]}{\mathbb{E}\left[|\dot{X}(0)|\middle|X(0) = u\right]},$$

requires joint distribution of $Y(\cdot)$ and $(\dot{X}(0), X(0))$.

- This can be difficult if (Y, X) are not jointly Gaussian
- When X is non-Gaussian even the denominator can be a problem

Illustration of difficulties – crossing intensity

- The joint distribution of $X(0)$ and $\dot{X}(0)$

$$\phi_{X,\dot{X}}(\xi_1,\xi_2) = \exp\left(-\int_0^\infty \ln\left(1+\frac{1}{2}(\xi_1^2+\xi_2^2\lambda^2)\right)\, dF(\lambda)\right).$$

- the process and its derivative are uncorrelated but **not independent** as it is in a Gaussian case
- by inverse Fourier transform

$$f_{X,\dot{X}}(u,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 u+\xi_2 z)}\phi_{X,\dot{X}}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2.$$

- the crossing intensity can be computed by evaluating the integral

$$\mu^+(u) = \frac{1}{4\pi^2}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z e^{-i(\xi_1 u+\xi_2 z)}\phi_{X,\dot{X}}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2\, dz$$

Representing biased sampling distributions

- X – a stationary process having a.s. absolutely continuous sample and $f_{X,\dot{X}}$ of $X(0), \dot{X}(0)$ exists.
- upcrossing set within interval $[0,1]$ is defined as

$$\mathcal{C}(u) = \{s \in [0,1] : X(s) = u, \dot{X}(s) > u\}.$$

- $N(u)$ – the number of elements in $\mathcal{C}(u)$.
- A – a property of trajectories of another stationary stochastic process Y
- $N(A|u)$ – the number of $s \in \mathcal{C}(u)$ such that $Y(s + \cdot) \in A$.
- Crossing level distributions of Y:

$$P^u(A) = \frac{E[N(A|u)]}{E[N(u)]}.$$

- **Slepian model for P^u**– any stochastic process Y_u with distribution given by the upcrossing distribution

$$P(Y_u \in A) = P^u(A).$$

Slepian model in the Gaussian case

- Z – a stationary Gaussian process Z
- the Slepian model process Z_u around u-upcrossing of Z is given by

$$Z_u(t) = u\, r(t) - R\, \dot{r}(t) + \Delta(t),$$

  - r be the covariance function of Z,
  - R is a standard Rayleigh variable
  - a non-stationary Gaussian process Δ having covariance

$$r(t,s) = r(t-s) - r(t)r(s) - r'(t)r'(s).$$

  - R and Δ are independent

Random scaling – a simple non-Gaussian case

- for non-random scaling $X(t) = \sqrt{k}\, Z(t)$, $t \in \mathbb{R}$, the Slepian model is

$$X_u(t) = u\, r(t) - \sqrt{k} R r'(t) + \sqrt{k}\Delta(t)$$

- random scaling K, (assume a **gamma distribution**), then $X(t) = \sqrt{K}\, Z(t)$,
- Slepian model for X:

$$X_u(t) = u\, r(t) - \sqrt{K} R\, \dot{r}(t) + \sqrt{K}\Delta(t)$$

- **this is not the case!!!**
- $X(t)$ conditionally on $(K = k, \dot{X} = z, X = u)$ is represented by $u\, r(t) - z\, \dot{r}(t) + \sqrt{k}\Delta(t)$
- (k, z) has to be replaced by the Slepian model $(K_u, \dot{X}_u)$.
- K_u is 'biased' to account for the fact that the behavior observed at u up-crossings for specific u makes certain scalings more likely than other – **'sampling bias'**.
- a correct Slepian model for X_u is given by

$$X_u(t) = u\, r(t) - \sqrt{K_u} R\ \dot{r}(t) + \sqrt{K_u}\Delta(t),$$

$$f_{K_u}(k) = \frac{\beta^{p/2}}{2^{1-p/2} u^p K_p(\sqrt{2\beta} u)} \cdot k^{p-1} \exp\left(-\frac{2\beta k + u^2/k}{2}\right),$$

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Moving average process

- A moving average process is a convolution of a kernel function with a infinitesimal “white noise” process having variance equal to the discretization step.
- Gaussian moving average (GMA):

$$X(t) = \int_{-\infty}^{\infty} g(s-t)\, dB(s)$$

- Slepian model $dB_u(x)$ for the noise $dB(x)$ at the crossing levels u of X

$$B_u(t) = F_{u,g}(t) + G_g(t) + B(t),$$

  where the **non-random compenent** is

$$F_{u,g}(t) = u \int_0^t g,$$

  the **kernel only dependent random component**

$$G_g(t) = \left(R - \int \dot{g}\, dB\right) \cdot \int_0^t \dot{g} - \int g\, dB \cdot \int_0^t g,$$

  and purely **random noise** represented by Brownian motion $B(t)$.

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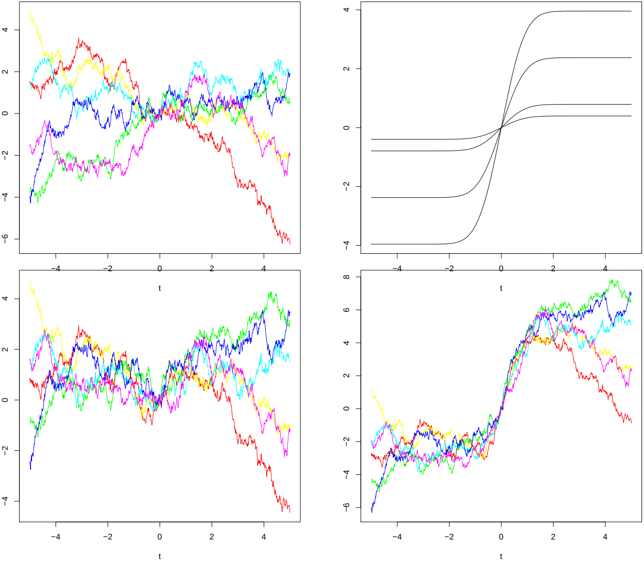
Simulated Slepian noise



Top: Brownian motion (left), Deterministic part F_u: $u = 0.5, 1, 3, 5$ (right),
Bottom: B_u for: $u = 0.5$ (left), $u = 5$ (right).

Non-Gaussian process at crossing of Gaussian one

- $Y_1(t)$, $t \in \mathbb{R}$– filtered original process $X(t)$:

$$Y_1(t) = \int h(s-t)\, dX(s) = \int h * g(s-t)\, dB(s).$$

- Laplace motion – subordination of the Brownian motion by the gamma motion K, ($K(1)$ has the gamma distribution with shape τ and scale $1/\tau$)
- Laplace moving average

$$Y_2(t) = \int f(s-t)\, dB \circ K(s).$$

- Jointly $Y(t) = (Y_1(t), Y_2(t))$ and $X(t)$ are not Gaussian
- Biased sampling distributions can be evaluated numerically, although it can be a computational chalenge

- **How to get a Slepian model in this case with some manageable structiure?**

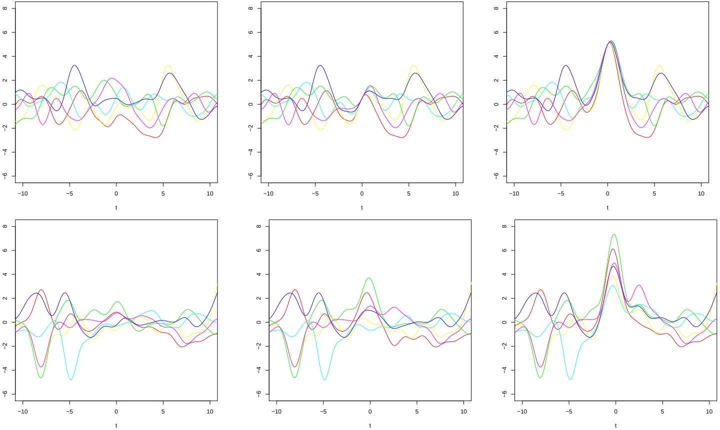
From Slepian noise to Slepian model

- By using the Slepian noise $B_u(t)$ one can provide with any Slepian model for processes functionally dependent on this noise
- **Bivariate Slepian model**

$$
\begin{aligned}
Y_{1u}(t) &= \int h(s-t)\, dX(s) = \int h * g(s-t)\, dB_u(s), \\
Y_{2u}(t) &= \int f(s-t)\, dB_u \circ K(s).
\end{aligned}
$$

- The main benefit is that all computational difficulties are in evaluating B_u, and thus avoiding computation of the joint distribution of the processes with complex structure.

Samples from joint Slepian model



Left: Samples from GMA $X = Y_1$ – *(top)* and LMA Y_2 – *(bottom)*.
Six samples of BM and for Y_2 a single sample of gamma process

Middle: Joint Slepian model (Y_{1u}, Y_{2u}) at the crossings of X at $u = 0.5$.

Right: Analogous samples at $u = 5$.

Moving averages driven by non-Gaussian noise

- More challenging problem of finding of a Slepian model at crossings of non-Gaussian process – a moving average driven by a non-Gaussian noise $dL(s)$ – **Laplace motion**, i.e.

$$X(t) = \int f(s-t)\, dL(s) = \int f(s-t)\, dB \circ K(s),$$

  where $K(t)$ is a gamma process.
- Gamma process as a subordinator is by convenience – other classes of Lévy processes are possible.
- Laplace motion is a pure jump process and generally there is no explicit expression for the joint distribution of the process and its derivative.
- **Strategy**:
  - find the Slepian model for the Laplace noise $B \circ K$,
  - obtaining the Slepian models by replacing the noise in the original moving averages by the Slepian noise.

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Details of the approach — Slepian for (Y_u, K_u, X_u)

- Given $(K_u, \dot{X}_u) = (k, \dot{x})$, the process $Y_u(t)$ has the conditional distribution of $Y(t)$ given $K = k$, $\dot{X} = \dot{x}$, and $X = u$, i.e. $Y(t|k, \dot{x}, u)$. This has gaussian distribution..
- Instead of $(K_u, \dot{X}_u)$ it happens easier to consider $(L_u, K_u, \dot{X}_u)$ by using the Gibbs sampler with the conditionals

$$K_u|L_u, \dot{X}_u \quad \text{and} \quad L_u, \dot{X}_u|K_u$$

- The first distribution is sampled from the generalized inverse Gaussian, the second is obtained by two conditionals: $L_u|\dot{X}_u, K_u$ which is Gaussian, and $\dot{X}_u|K_u$ which is can be expressed by the distribution which we called the **tilted Rayleigh** and for which an effective simulation algorithm has been developed

$$x \cdot e^{(x-a)^2/b}$$

Example

Left: Samples from BM (top) and LM (bottom)

Right: Slepian model L_u at the crossings of X at $u = 0.5$ (top) and $u = 5$ (bottom).

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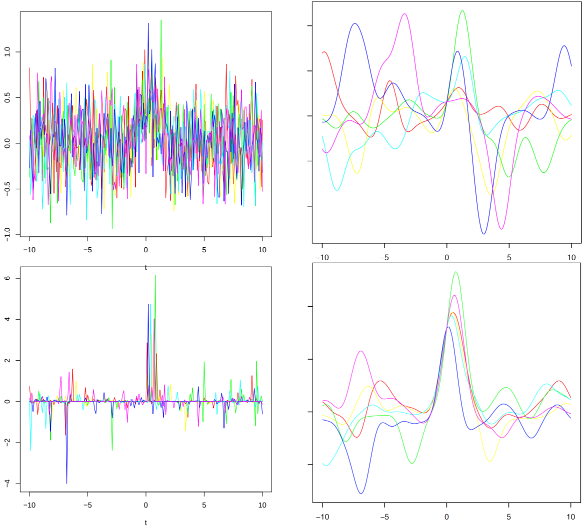
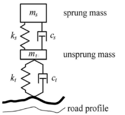
Gaussian vs. Laplace Slepian model



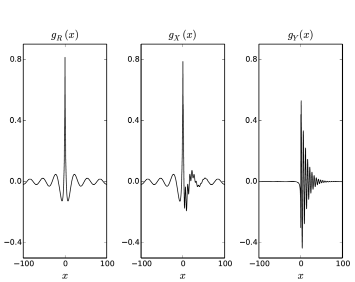
Left: Slepian noise BM (top) and LM (bottom), $u = 5$

Right: Slepian model X_u at its own crossings at $u = 0.5$ (top) and $u = 5$ (bottom).

Quarter Vehicle and Road Variability



Symbol	Value	Unit
m_s	3400	kg
k_s	270 000	N/m
c_s	6000	Ns/m
m_t	350	kg
k_t	950000	N/m
c_t	300	Ns/m



Quarter vehicle model: R road profile, X mass m_t response, Y mass m_s response

- Modeling of true loads is difficult since tires filter nonlinearly the road profile, with very uncertain factors, e.g. tire's pressure, wear etc.
- Response $X(x)$ is difficult to model and one sometimes considers $X(x)$ as an external input.
- It is of interest to study and model both the response and the road profile at locations when X reaches some extreme level i.e. Slepian models for X_u and R_u and Y_u, when X upcrosses u.

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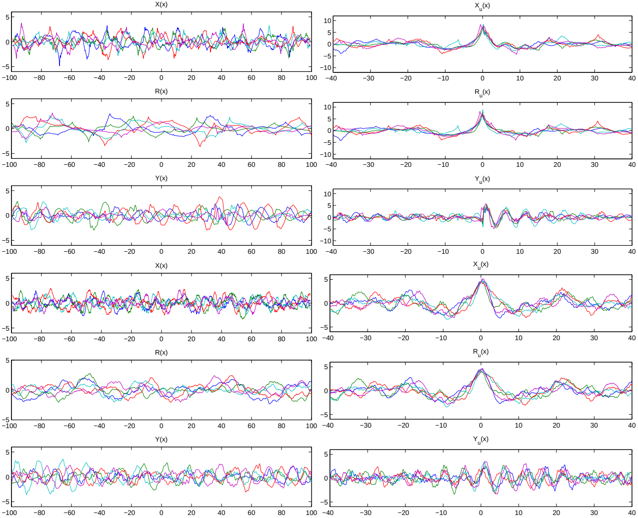
Two moving averages – the Gaussian and Laplace ones.



The Laplace in the top three graphs of each column and the Gaussian case in the bottom three graphs of each column. *Left:* Road profile $R(x)$ (middle) and responses $X(x)$ (top), $Y(x)$ (bottom). *Right:* Slepian models $X_u(x)$, $R_u(x)$ and $Y_u(x)$ around the $u = 7$ upcrossing of $X(x)$ in the Laplace case *(top three graphs)* and around the $u = 4.5$ upcrossing of $X(x)$ for the Gaussian case *(bottom three graphs)*.

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Conclusions

- The level crossings distributions are important in studying extremal behavior of stochastic processes
- Generalized Rice's formula is utilized to obtain effectively level crossing distributions
- They are also important for studying asymmetries in the records, which requires non-Gaussian models
- Slepian model is a convenient way of representing level crossing distributions
- Slepian model is quite straightforward in the Gaussian case
- Non-Gaussian models requires special care
- An approach through Slepian model for noise process is investigated
- It allows for simultaneous study different models arising from such a noise.
- A method of simulating from Slepian model in the case of Laplace noise is obtained exploring the dependence of the noise on the subordinating Gamma noise
- Crossing level behavior for non-Gaussian models is fundamentally different from the Gaussian counterpart

Thank you!