Slepian models for moving averages driven by a non-Gaussian noise

the Slepian noise approach

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Slepian models are derived describing a stochastic process observed at level crossings of a moving average driven by a Laplace noise.

The approach is through a Gibbs sampler of a Slepian model for the Laplace noise and it allows for simultaneously studying a number of stochastic characteristics observed at the level crossing instants.

It is observed that the behavior of the process at high level crossings is fundamentally different from that in the Gaussian case.

The shape of extreme episodes resembles the (asymmetric) kernel while for Gaussian model the shape is given by the correlation function of which is symmetric in time.
Biased sampling distribution

- \( N(T, A) \) — “number” of times the field \( X \) takes value \( u \) in \([0, T]\) and at the same time a possibly another process \( Y \) has a property \( A \)
- For ergodic stationary processes

\[
\lim_{T \to \infty} \frac{N(T, A)}{N(T)} = \frac{\mathbb{E} \left[ \{ Y \in A \} | \dot{X}(0) \parallel X(0) = u \right]}{\mathbb{E} \left[ |\dot{X}(0)| | X(0) = u \right]},
\]

- RHS represents the biased sampling distribution when sampling is made over the \( u \)-level contour \( C_u = \{ \tau : X(\tau) = 0 \} \) (argument of the process \( X \) can be multivariate)
Rice's formula

Rice formula for the crossing intensity

Rice formula – general case

\[ \mu^+(u) = \mathbb{E} \left( \dot{X}^+(0) \mid X(0) = u \right) f_{X(0)}(u) \]

Gaussian case

\[ \mu^+(u) = \frac{1}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{u^2}{2\lambda_0}}, \]

\( \lambda_0, \lambda_2 \) – spectral moments

Application – upper bound for maximum in the Gaussian case

\[ \mathbb{P}(M_T > u) \leq \Phi(u/\sqrt{\lambda_0}) + T \cdot \frac{1}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{u^2}{2\lambda_0}} \]
Non-Gaussian case – Laplace moving average

- By means of stochastic integrals we define

\[ X(\tau) = \int_{\mathbb{R}^d} f(\tau - s)d\lambda(s). \]

- \( \lambda(A) \) has the generalized asymmetric Laplace distribution

\[ \phi(t) = \frac{1}{\left(1 - i\mu t + \frac{\sigma^2}{2} t^2\right)^\lambda(A)}, \]

where \( \lambda \) is the Lebesgue measure in \( \mathbb{R}^d \).

- If \( d = 1 \), then \( \lambda(-\infty, x] = B(\Gamma(x)) \), where \( B \) is a Brownian motion with drift and \( \Gamma \) is a gamma process.

- Conditionally on \( \Gamma \) the process \( X_t \) can be viewed as a non-stationary Gaussian process.
Non-Gaussian case – why it is interesting

- For a covariance $R$ a matching symmetric kernel is given by $f_{sym} = \mathcal{F}^{-1} \sqrt{\mathcal{F} R}$,
- Symmetric kernels cannot produce front-back asymmetries even if the moving average process is not Gaussian.
- Ornstein-Uhlenbeck autocorrelation $e^{-|x|}$ can be obtained both by symmetric and asymmetric kernels.

![Graphs showing different types of data distributions and their transformations.](image-url)
Rice’s formula

Slope distributions at level crossings
Biased sampling in non-Gaussian case – why it is difficult

- Biased sampling distribution

\[
P^u(A) = \frac{\mathbb{E}\left[ \{Y \in A \} \left| \dot{X}(0), X(0) = u \right. \right]}{\mathbb{E}\left[ \left| \dot{X}(0) \right| \left| X(0) = u \right. \right]},
\]

requires joint distribution of \( Y(\cdot) \) and \( (\dot{X}(0), X(0)) \).

- This can be difficult if \( (Y, X) \) are not jointly Gaussian

- When \( X \) is non-Gaussian even the denominator can be a problem
Illustration of difficulties – crossing intensity

- The joint distribution of $X(0)$ and $\dot{X}(0)$

$$
\phi_{X,\dot{X}}(\xi_1, \xi_2) = \exp \left( - \int_0^\infty \ln \left( 1 + \frac{1}{2} (\xi_1^2 + \xi_2^2 \lambda^2) \right) dF(\lambda) \right) .
$$

- The process and its derivative are uncorrelated but **not independent** as it is in a Gaussian case

- By inverse Fourier transform

$$
f_{X,\dot{X}}(u, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi_1 u + \xi_2 z)} \phi_{X,\dot{X}}(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2 .
$$

- The crossing intensity can be computed by evaluating the integral

$$
\mu^+(u) = \frac{1}{4\pi^2} \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z e^{-i(\xi_1 u + \xi_2 z)} \phi_{X,\dot{X}}(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2 \, dz .
$$
Representing biased sampling distributions

- $X$ – a stationary process having a.s. absolutely continuous sample and $f_{X, \dot{X}}$ of $X(0), \dot{X}(0)$ exists.
- Upcrossing set within interval $[0, 1]$ is defined as
  \[ C(u) = \{ s \in [0, 1] : X(s) = u, \dot{X}(s) > u \} . \]
- $N(u)$ – the number of elements in $C(u)$.
- $A$ – a property of trajectories of another stationary stochastic process $Y$
- $N(A|u)$ – the number of $s \in C(u)$ such that $Y(s + \cdot) \in A$.
- Crossing level distributions of $Y$:
  \[ P^u(A) = \frac{E[N(A|u)]}{E[N(u)]} . \]
- **Slepian model for $P^u$** – any stochastic process $Y_u$ with distribution given by the upcrossing distribution
  \[ P(Y_u \in A) = P^u(A) . \]
Slepian model in the Gaussian case

- $Z$ – a stationary Gaussian process $Z$
- the Slepian model process $Z_u$ around $u$-upcrossing of $Z$ is given by
  \[ Z_u(t) = u r(t) - R \dot{r}(t) + \Delta(t), \]

- $r$ be the covariance function of $Z$,
- $R$ is a standard Rayleigh variable
- a non-stationary Gaussian process $\Delta$ having covariance
  \[ r(t, s) = r(t - s) - r(t)r(s) - r'(t)r'(s). \]
- $R$ and $\Delta$ are independent
Random scaling – a simple non-Gaussian case

- for non-random scaling $X(t) = \sqrt{k} Z(t)$, $t \in \mathbb{R}$, the Slepian model is
  \[ X_u(t) = u r(t) - \sqrt{k} R r'(t) + \sqrt{k} \Delta(t) \]

- random scaling $K$, (assume a **gamma distribution**), then $X(t) = \sqrt{K} Z(t)$,

- Slepian model for $X$:
  \[ X_u(t) = u r(t) - \sqrt{K} R \dot{r}(t) + \sqrt{K} \Delta(t) \]

- this is not the case!!!
- $X(t)$ conditionally on $(K = k, \dot{X} = z, X = u)$ is represented by $u r(t) - z \dot{r}(t) + \sqrt{k} \Delta(t)$
- $(k, z)$ has to be replaced by the Slepian model $(K_u, \dot{X}_u)$.
- $K_u$ is ‘biased’ to account for the fact that the behavior observed at $u$ up-crossings for specific $u$ makes certain scalings more likely than other – ‘sampling bias’.
- a correct Slepian model for $X_u$ is given by
  \[ X_u(t) = u r(t) - \sqrt{K_u} R \dot{r}(t) + \sqrt{K_u} \Delta(t), \]
  \[ f_{K_u}(k) = \frac{\beta^{p/2}}{2^{1-p/2} u^p K_p(\sqrt{2\beta u})} \cdot k^{p-1} \exp\left(-\frac{2\beta k + u^2/k}{2}\right), \]
Moving average process

- A moving average process is a convolution of a kernel function with an infinitesimal “white noise” process having variance equal to the discretization step.
- Gaussian moving average (GMA):
  \[ X(t) = \int_{-\infty}^{\infty} g(s - t) \, dB(s) \]
- Slepian model \( dB_u(x) \) for the noise \( dB(x) \) at the crossing levels \( u \) of \( X \)
  \[ B_u(t) = F_{u,g}(t) + G_g(t) + B(t), \]
  where the non-random component is
  \[ F_{u,g}(t) = u \int_{0}^{t} g, \]
  the kernel only dependent random component
  \[ G_g(t) = \left( R - \int \dot{g} \, dB \right) \cdot \int_{0}^{t} \dot{g} - \int g \, dB \cdot \int_{0}^{t} g, \]
  and purely random noise represented by Brownian motion \( B(t) \).
Simulated Slepian noise

*Top:* Brownian motion (left), Deterministic part $F_u$: $u = 0.5, 1, 3, 5$ (right),
*Bottom:* $B_u$ for: $u = 0.5$ (left), $u = 5$ (right).
Non-Gaussian process at crossing of Gaussian one

- $Y_1(t)$, $t \in \mathbb{R}$ – filtered original process $X(t)$:
  \[ Y_1(t) = \int h(s - t) \, dX(s) = \int h \ast g(s - t) \, dB(s). \]

- Laplace motion – subordination of the Brownian motion by the gamma motion $K$, ($K(1)$ has the gamma distribution with shape $\tau$ and scale $1/\tau$)
- Laplace moving average
  \[ Y_2(t) = \int f(s - t) \, dB \circ K(s). \]

- Jointly $Y(t) = (Y_1(t), Y_2(t))$ and $X(t)$ are not Gaussian
- Biased sampling distributions can be evaluated numerically, although it can be a computational challenge

- How to get a Slepian model in this case with some manageable structure?
From Slepian noise to Slepian model

- By using the Slepian noise $B_u(t)$ one can provide with any Slepian model for processes functionally dependent on this noise.
- **Bivariate Slepian model**

\[
Y_{1u}(t) = \int h(s - t) \, dX(s) = \int h \ast g(s - t) \, dB_u(s),
\]

\[
Y_{2u}(t) = \int f(s - t) \, dB_u \circ K(s).
\]

- The main benefit is that all computational difficulties are in evaluating $B_u$, and thus avoiding computation of the joint distribution of the processes with complex structure.
Samples from joint Slepian model

**Left:** Samples from GMA \( X = Y_1 \) (top) and LMA \( Y_2 \) (bottom). Six samples of BM and for \( Y_2 \) a single sample of gamma process

**Middle:** Joint Slepian model \((Y_{1u}, Y_{2u})\) at the crossings of \( X \) at \( u = 0.5 \).

**Right:** Analogous samples at \( u = 5 \).
More challenging problem of finding of a Slepian model at crossings of non-Gaussian process – a moving average driven by a non-Gaussian noise $dL(s)$ – **Laplace motion**, i.e.

$$X(t) = \int f(s - t) \, dL(s) = \int f(s - t) \, dB \circ K(s),$$

where $K(t)$ is a gamma process.

- Gamma process as a subordinator is by convenience – other classes of Lévy processes are possible.

- Laplace motion is a pure jump process and generally there is no explicit expression for the joint distribution of the process and its derivative.

**Strategy:**
- find the Slepian model for the Laplace noise $B \circ K$,
- obtaining the Slepian models by replacing the noise in the original moving averages by the Slepian noise.
Details of the approach — Slepian for \((Y_u, K_u, X_u)\)

- Given \((K_u, \dot{X}_u) = (k, \dot{x})\), the process \(Y_u(t)\) has the conditional distribution of \(Y(t)\) given \(K = k, \dot{X} = \dot{x}\), and \(X = u\), i.e. \(Y(t|k, \dot{x}, u)\). This has Gaussian distribution.

- Instead of \((K_u, \dot{X}_u)\) it happens easier to consider \((L_u, K_u, \dot{X}_u)\) by using the Gibbs sampler with the conditionals

\[
K_u|L_u, \dot{X}_u \quad \text{and} \quad L_u, \dot{X}_u|K_u
\]

- The first distribution is sampled from the generalized inverse Gaussian, the second is obtained by two conditionals: \(L_u|\dot{X}_u, K_u\) which is Gaussian, and \(\dot{X}_u|K_u\) which is can be expressed by the distribution which we called the tilted Rayleigh and for which an effective simulation algorithm has been developed

\[
x \cdot e^{(x-a)^2/b}
\]
Example

Left: Samples from BM (top) and LM (bottom)
Right: Slepian model $L_u$ at the crossings of $X$ at $u = 0.5$ (top) and $u = 5$ (bottom).
**Gaussian vs. Laplace Slepian model**

*Left:* Slepian noise BM (top) and LM (bottom), $u = 5$

*Right:* Slepian model $X_u$ at its own crossings at $u = 0.5$ (top) and $u = 5$ (bottom).
Quarter Vehicle and Road Variability

Modeling of true loads is difficult since tires filter nonlinearly the road profile, with very uncertain factors, e.g. tire’s pressure, wear etc.

Response $X(x)$ is difficult to model and one sometimes considers $X(x)$ as an external input.

It is of interest to study and model both the response and the road profile at locations when $X$ reaches some extreme level i.e. Slepian models for $X_u$ and $R_u$ and $Y_u$, when $X$ upcrosses $u$. 
Two moving averages – the Gaussian and Laplace ones.

The Laplace in the top three graphs of each column and the Gaussian case in the bottom three graphs of each column. Left: Road profile $R(x)$ (middle) and responses $X(x)$ (top), $Y(x)$ (bottom). Right: Slepian models $X_u(x)$, $R_u(x)$ and $Y_u(x)$ around the $u = 7$ upcrossing of $X(x)$ in the Laplace case (top three graphs) and around the $u = 4.5$ upcrossing of $X(x)$ for the Gaussian case (bottom three graphs).
Conclusions

- The level crossings distributions are important in studying extremal behavior of stochastic processes.
- Generalized Rice’s formula is utilized to obtain effectively level crossing distributions.
- They are also important for studying asymmetries in the records, which requires non-Gaussian models.
- Slepian model is a convenient way of representing level crossing distributions.
- Slepian model is quite straightforward in the Gaussian case.
- Non-Gaussian models require special care.
- An approach through Slepian model for noise process is investigated.
- It allows for simultaneous study different models arising from such a noise.
- A method of simulating from Slepian model in the case of Laplace noise is obtained exploring the dependence of the noise on the subordinating Gamma noise.
- Crossing level behavior for non-Gaussian models is fundamentally different from the Gaussian counterpart.
Thank you!