Anisotropy analysis of the system of air pores in polar ice

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Workshop
Spatial and spatio-temporal modelling in biology, ecology and geosciences
Smögen, August 21-25, 2006

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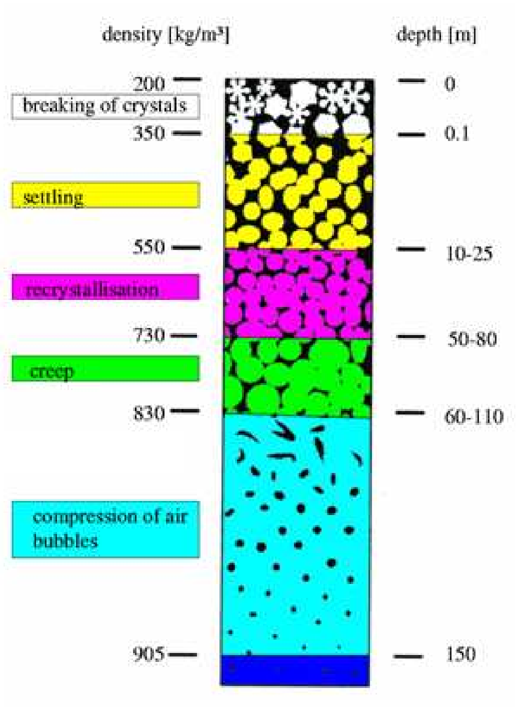
Motivation



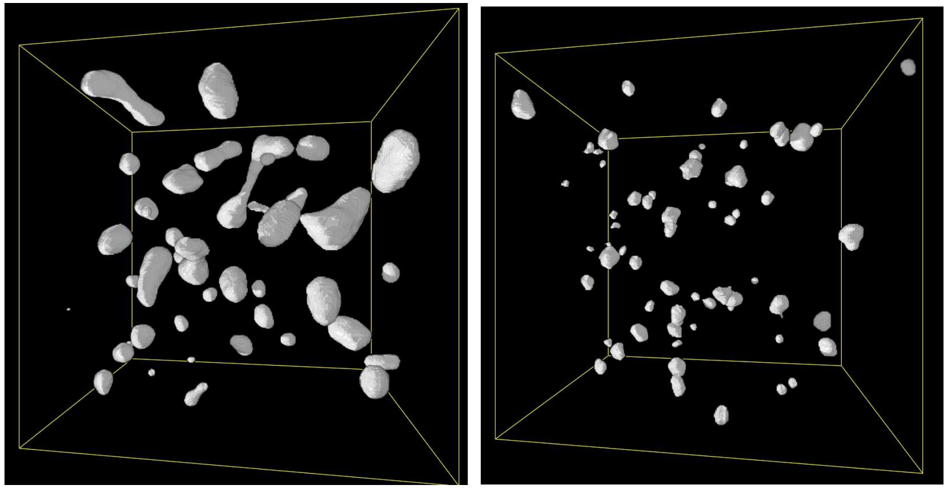
Polar ice is compacted (sintered) snow. During the compression air pores are isolated in the ice.

Image: Freitag 1), Kipfstuhl 1), Stauffer 2)
1) Alfred-Wegener-Institute for polar and marine research, Bremerhaven
2) University Bern

Motivation

Aim

investigation of the spatial arrangement of air pores in the ice



Question

Does the location of the pores tell anything about the movements of the ice?

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Data

Original data

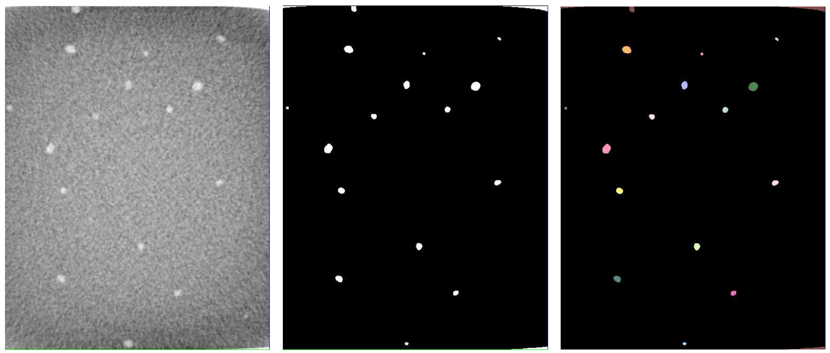
tomographic images of ice cores

4 images per depth

number	250	610	910	1209
depth	137.5	335.5	500.5	664.95

Image processing

smoothing -> binarization -> labelling -> postprocessing



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Methods in 2d

- point-pair rose density (Stoyan, Benes, 1991) and directional K-function (Stoyan, Kendall, Mecke, 1995)
- directional pair correlation function (Stoyan, Stoyan, 1992)
- Fry-method (Fry, 1979)
- 0-contour of the density $\frac{d}{d\varphi}K(r,\varphi)$ (SKM, 1995)

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Ideas for 3d

Requirements

- measurement of characteristics easy and robust
- methods for visualisation of results

Consider

- nearest neighbour directions (Fisher et. al., 1987)
  -> nearest neighbours and directions only
- directional G-function
  -> only nearest neighbours, directions and distances
- directional K-function / Fry-method
  -> all points, directions and distances

Nearest neighbour directions

Investigate set of directions from a point to its nearest neighbour.

Cartesian coordinates $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$

Cylindrical coordinates $(\varphi_1, z_1), \ldots, (\varphi_n, z_n)$

Uniform distribution on unit sphere

$$U = \begin{pmatrix} \cos(\Phi)\sqrt{1-Z^2} \\ \sin(\Phi)\sqrt{1-Z^2} \\ Z \end{pmatrix},$$

with $Z \sim U(0,1)$ and $\Phi \sim U(0, 2\pi)$.

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p-values of a χ^2-test (10 groups)

	250_2	250_3	250_4	250_5
Φ	0.8075	0.0243	0.5812	0.6446
Z	0.3224	0.5547	0.2757	0.1079
	610_1	610_2	610_3	610_4
Φ	0.4830	0.5712	0.4388	0.4894
Z	$1.6e-5$	0.0001	0.0365	0.0038
	910_1	910_2	910_3	910_4
Φ	0.8847	0.5924	0.5341	0.22835
Z	0.0052	0.1160	0.6090	0.0005
	1209_1	1209_3	1209_4	
Φ	0.7359	0.6934	0.0005	
Z	$1.9e-5$	0.1794	0.0215	

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Gine's statistic

Gine's statistic

Tests uniform distribution against models which are symmetric to center of the sphere.
ψ_{ij} angle between u_i and u_j.

$$G_n = \frac{n}{2} - \frac{4}{n\pi} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sin(\psi_{ij})$$

1.413(20%), 1.816(10%), 2.207(5%), 3.090(1%)

250_2	0.41	610_1	1.45	910_1	1.84	1209_1	1.68
250_3	0.88	610_2	2.16	910_2	0.97	1209_3	1.27
250_4	1.01	610_3	0.89	910_3	0.74	1209_4	2.29
250_5	0.60	610_4	1.83	910_4	2.11		

Orientation matrix

Given n unit vectors $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$.

Orientation matrix

$$T = \begin{pmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i z_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i z_i \\ \sum x_i z_i & \sum y_i z_i & \sum z_i^2 \end{pmatrix}.$$

Normalized eigenvalues $\tau_1 \leq \tau_2 \leq \tau_3$

Shape parameter $\gamma = \frac{\log\left(\frac{\tau_3}{\tau_2}\right)}{\log\left(\frac{\tau_2}{\tau_1}\right)}$

Strength parameter $\zeta = \log\left(\frac{\tau_3}{\tau_1}\right)$

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Results

		τ_1	τ_2	τ_3	per	γ	ζ
250_2	87	0.2312	0.3673	0.4016	0.4178 (10%)	0.19	0.55
250_3	140	0.2707	0.3327	0.3965	0.4000 (10%)	0.85	0.38
250_4	104	0.2421	0.3365	0.4215	0.4106 (10%)	0.68	0.55
250_5	112	0.2900	0.3348	0.3752	0.4078 (10%)	0.79	0.26
610_1	179	0.2635	0.2873	0.4492	0.4109 (1%)	5.16	0.53
610_2	173	0.2379	0.2970	0.4652	0.4123 (1%)	2.02	0.67
610_3	119	0.2471	0.3154	0.4375	0.4285 (1%)	1.34	0.57
610_4	250	0.2484	0.3437	0.4079	0.3990 (1%)	0.53	0.50
910_1	284	0.2717	0.3095	0.4188	0.3949 (1%)	2.32	0.43
910_2	237	0.2765	0.3104	0.4132	0.4008 (1%)	2.47	0.40
910_3	193	0.2811	0.3077	0.4112	0.4081 (1%)	3.21	0.38
910_4	218	0.2779	0.2889	0.4332	0.4036 (1%)	10.42	0.44
1209_1	205	0.2617	0.2888	0.4495	0.4058 (1%)	4.48	0.54
1209_3	236	0.2847	0.3037	0.4117	0.4009 (1%)	4.71	0.37
1209_4	211	0.2524	0.2840	0.4636	0.4048 (1%)	4.16	0.61

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Principal axis

Axis about which the moment of inertia is least

Eigenvector with respect to τ_3

	610_1	610_2	610_3	610_4	910_1	910_2	910_3	910_4	1209_1	1209_3	1209_4
x	0.03	0.08	0.16	-0.08	0.09	-0.12	0.51	0.01	-0.01	-0.25	0.99
y	-0.25	0.06	-0.08	-0.02	-0.12	-0.24	0.05	-0.03	-0.20	-0.47	0.1
z	0.97	1.00	-0.98	1.00	0.99	0.96	-0.86	1.00	0.98	0.84	0.12

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Pressed point processes

Transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ cz \end{pmatrix}, 0 < c < 1$$

Aim

- detection of the distortion
- estimation of the parameter c

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Simulations

Test of the methods using simulated data

Matern hard-core process, R=0.05

Dataset I $\lambda_I = 1000$, $W_I = [0,1]^3$, $c = 1$

Dataset II $\lambda_{II} = 500$, $W_{II} = [0,1]^2 \times [0,2]$, $c = 1/2$

Dataset III $\lambda_{III} = 333$, $W_{III} = [0,1]^2 \times [0,3]$, $c = 1/3$

-> three point processes of intensity 1000 within the unit cube

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Directional G-function

Distribution function of the distance to the nearest neighbour conditioned on directions

Let (R, Φ, Θ) be the spherical coordinates of the vector from the typical point to its nearest neighbour and $\vartheta_M \in [0, \pi]$.

$$G_g(l) := \mathbb{P}(R \leq l \,|\, \Theta \in [0, \vartheta_M] \cup [\pi - \vartheta_M, \pi]).$$

Might be inaccurate for low intensities.

Let $(\tilde{R}, \tilde{\Phi}, \tilde{\Theta})$ be the spherical coordinates of the vector from the typical point to its nearest neighbour within a double cone defined by ϑ_M.

$$G_l(l) := \mathbb{P}(\tilde{R} \leq l)$$

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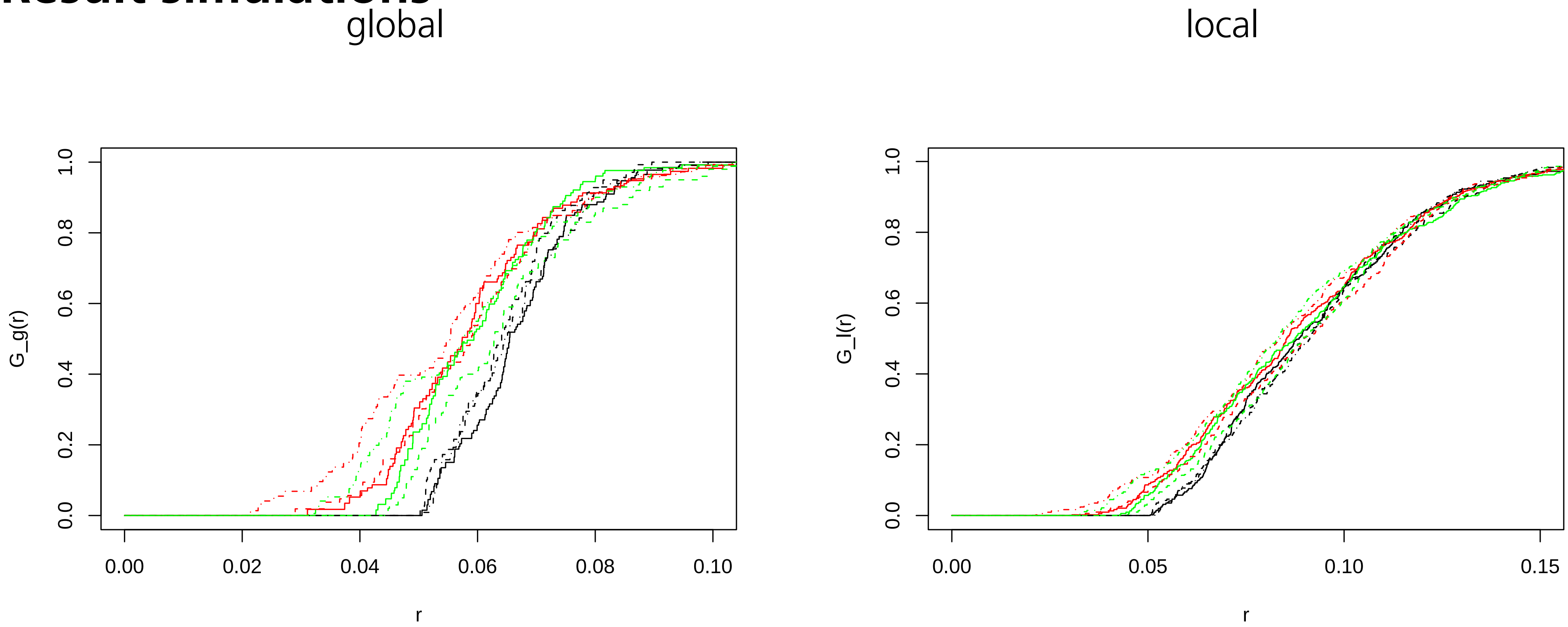
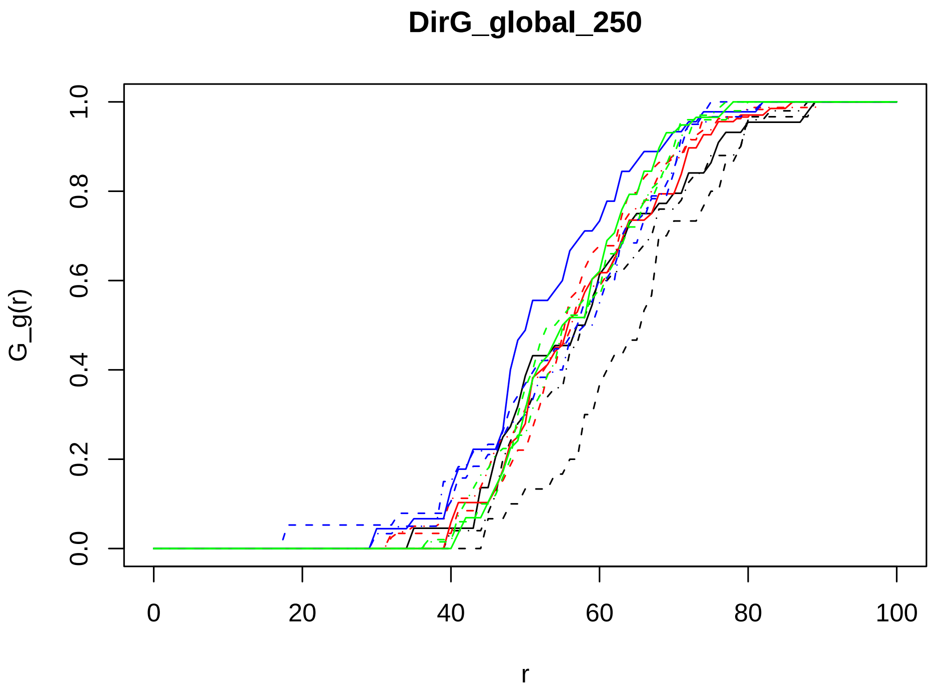
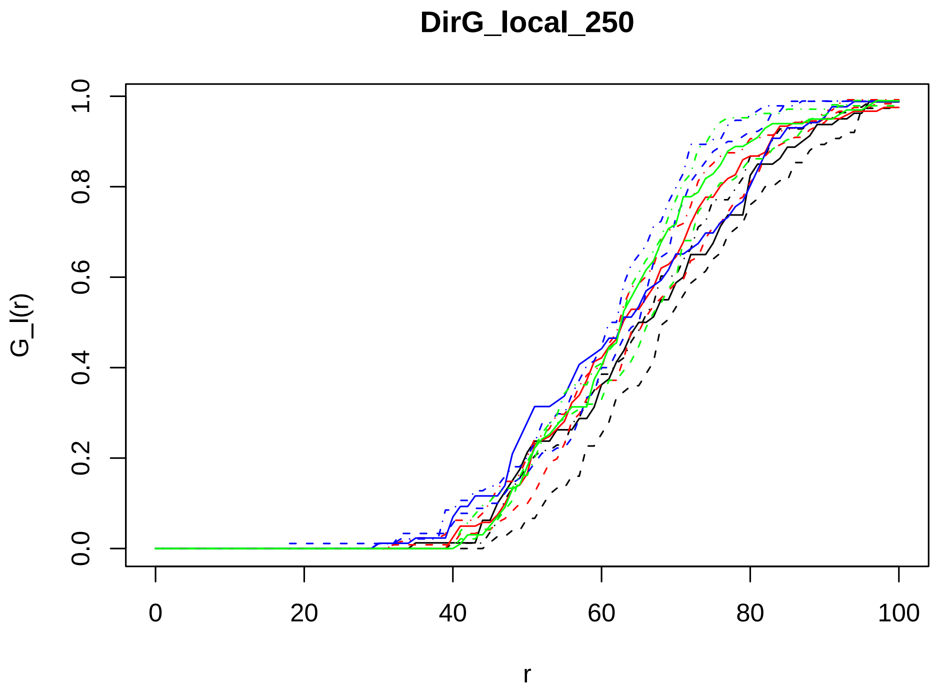
Result simulations



black: I, green: II, red: III, $\vartheta_M = \frac{\pi}{3}$

Result ice 250





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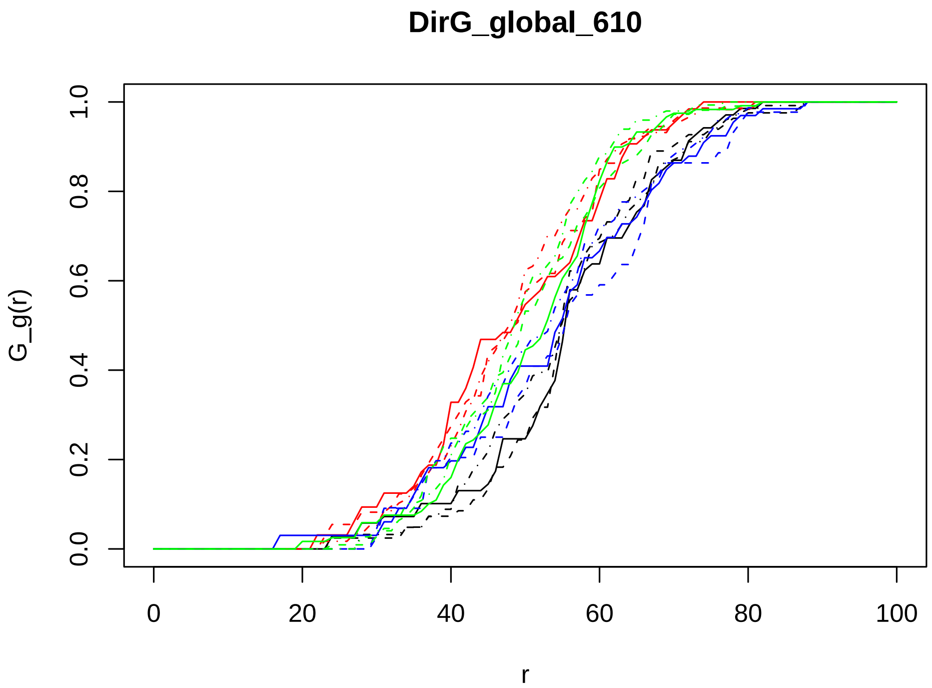
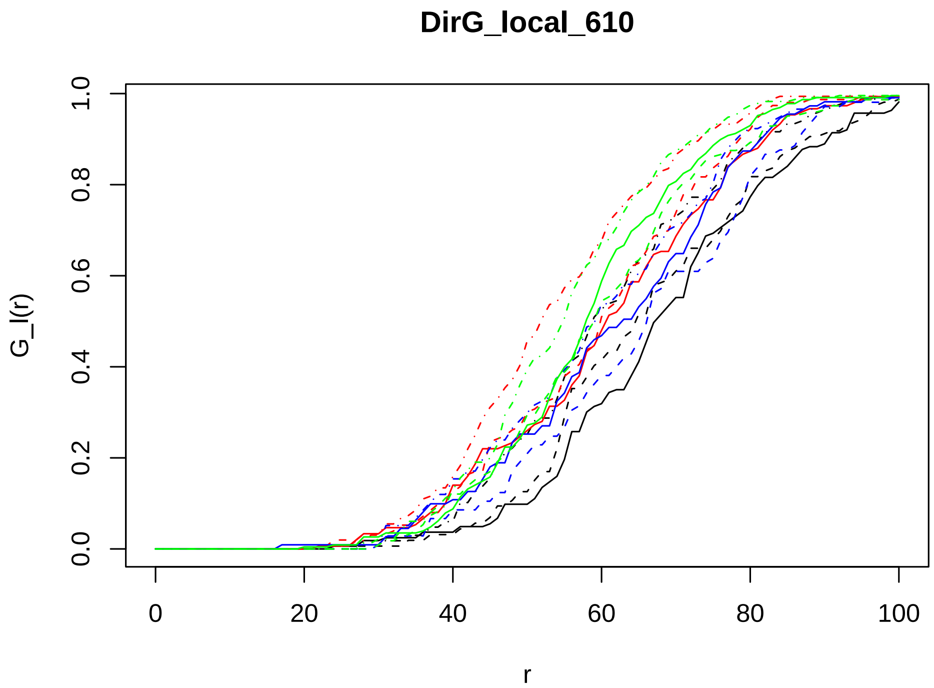
Result ice 610





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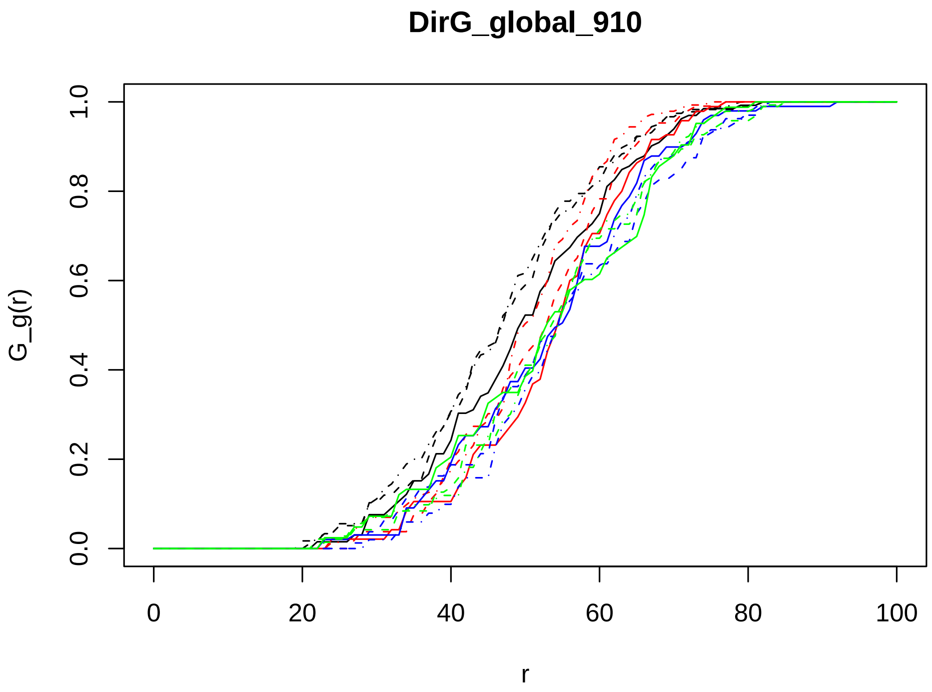
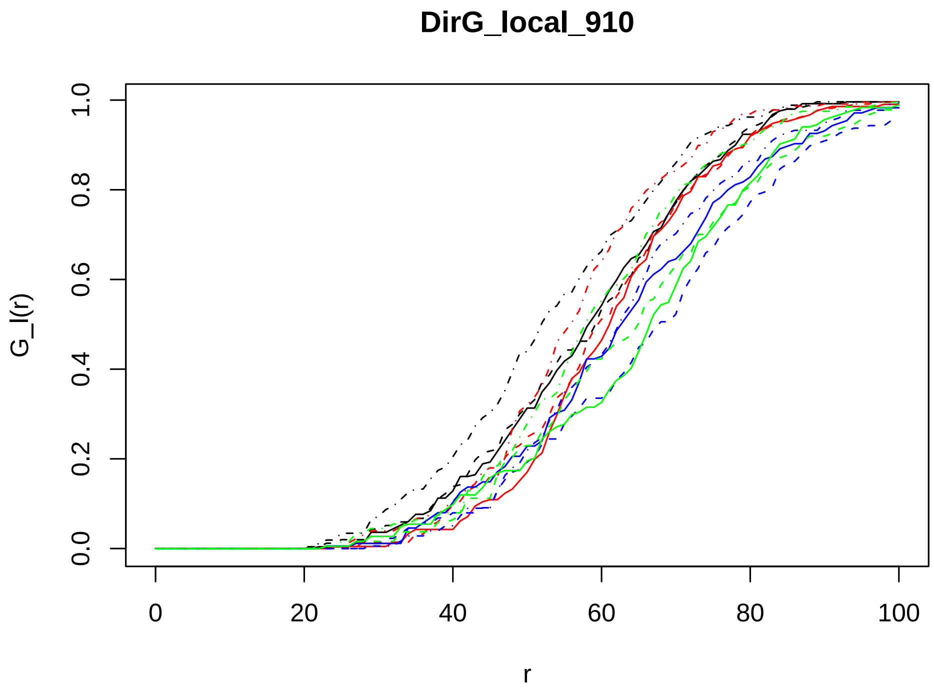
Result ice 910





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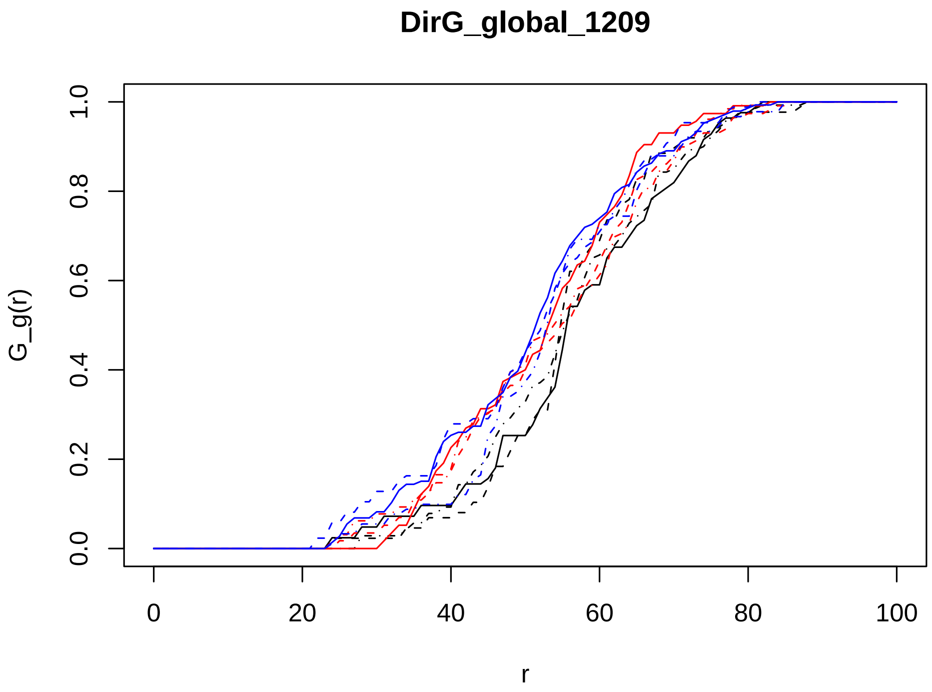
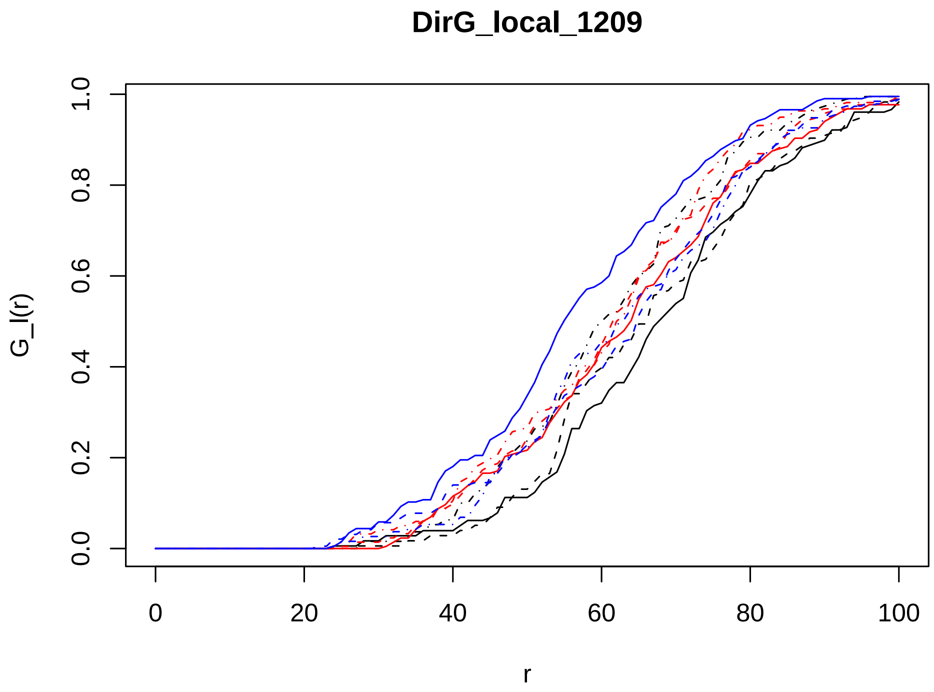
Result ice 1209





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Fry-method

method used in 2d by geologists (Fry,1979)

Assumption

Originally centers of objects are isotropic and obey some hard core condition.

Idea

plot all center-to-center-vectors

isotropic -> circular void in the center

strain -> elliptical void in the center

-> fit ellipse to central void

Fry vs. SKM

partition of the unit sphere, handling of outliers

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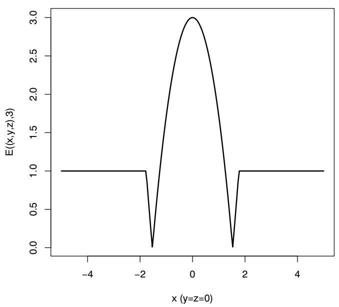
Fitting of the ellipsoid

Requirements

- ellipsoid should model void in the Fry plot
- ellipsoid should contain little or no points
- ellipsoid's shape should not be influenced by points far away

Cost function



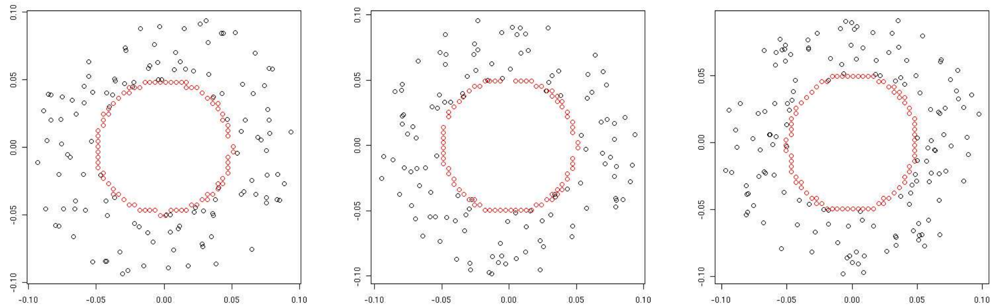
Let S be a 3×3 diagonal shape matrix and R a rotation of $\mathbb{R}^3$.

$$E(X, \lambda) = \sum_{x \in X} |\min(1, \lambda(||SRx||^2 - 1))|$$

-> Nelder-Mead nonlinear optimization algorithm

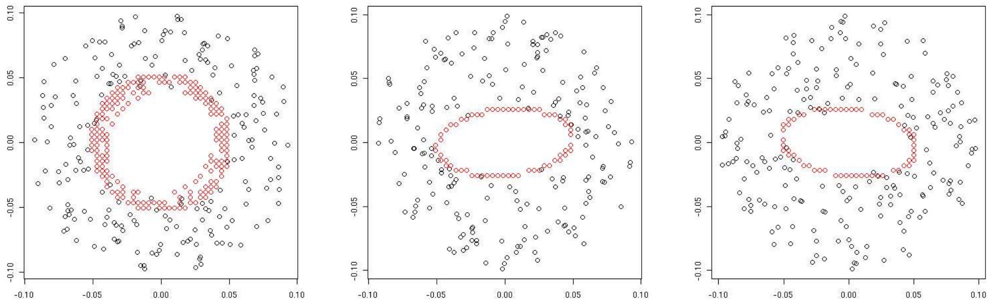
Results simulations

	r_x	φ_x	ϑ_x	r_y	φ_y	ϑ_y	r_z	φ_z	ϑ_z
I	**0.050**	179.568	-1.32	**0.050**	-90.85	18.67	**0.053**	-86.50	-71.26
II	**0.051**	13.01	4.49	**0.054**	102.35	-7.78	**0.027**	132.75	81.00
III	**0.052**	32.39	-7.09	**0.059**	-56.26	10.76	**0.022**	89.61	77.06

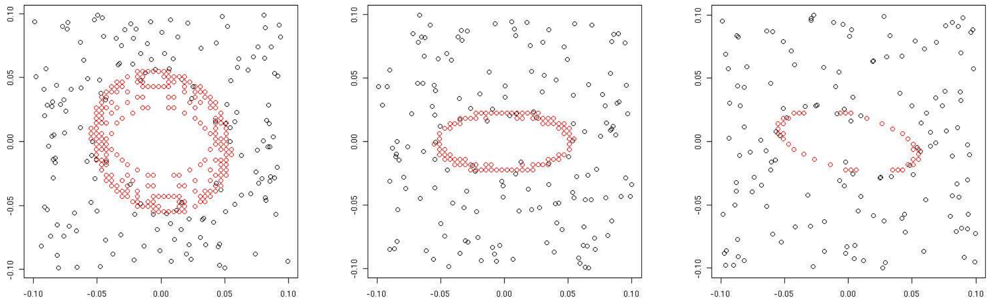


I, xy, xz, yz

Results simulations



II, xy, xz, yz



III, xy, xz, yz

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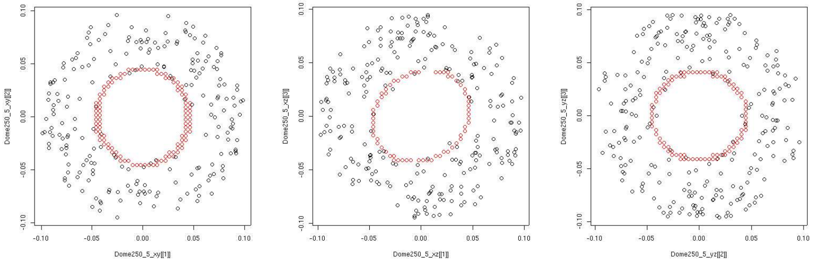
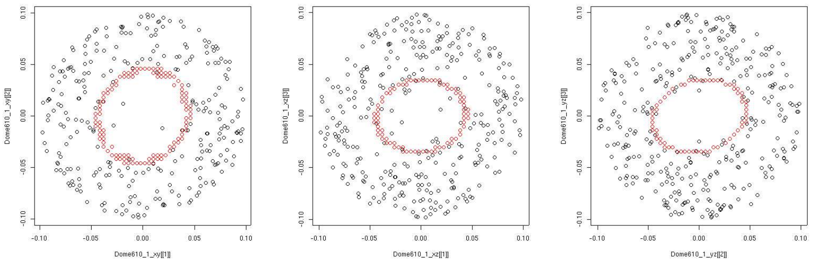
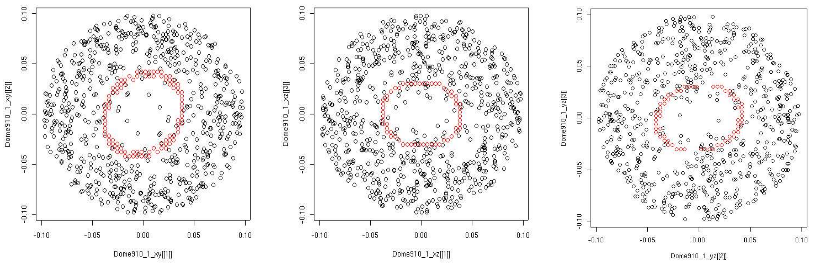
Results ice



250_5**, xy, xz, yz**



610_1**, xy, xz, yz**



910_1**, xy, xz, yz**

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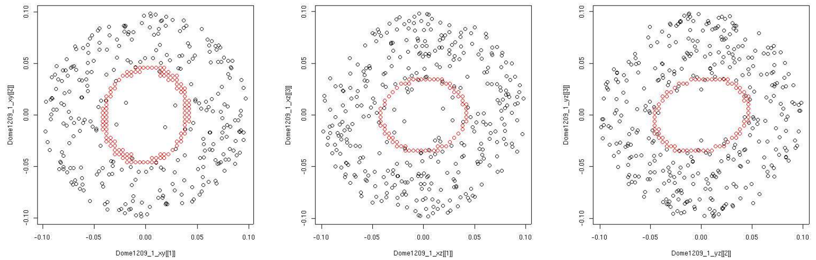
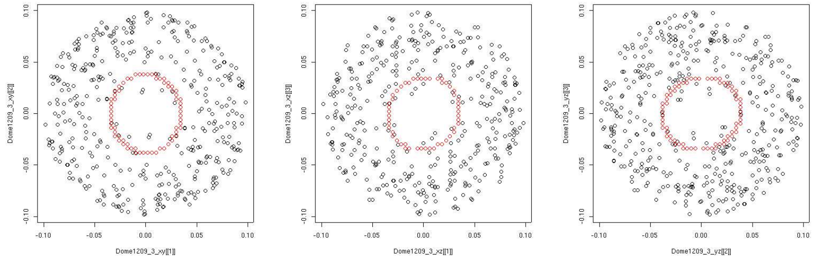
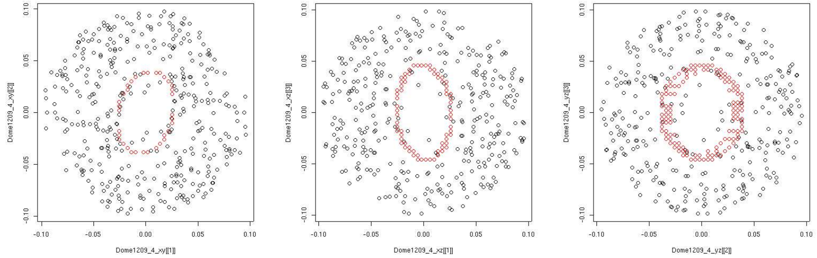
Results ice



1209_1, **xy, xz, yz**



1209_3, **xy, xz, yz**



1209_4, **xy, xz, yz**

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Conclusion

Result

- isotropy in upper layers
- anisotropy in z-direction in deeper layers
- have a closer look at behaviour far down

In general

further methods for directional analysis of 3d data should be developed

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References

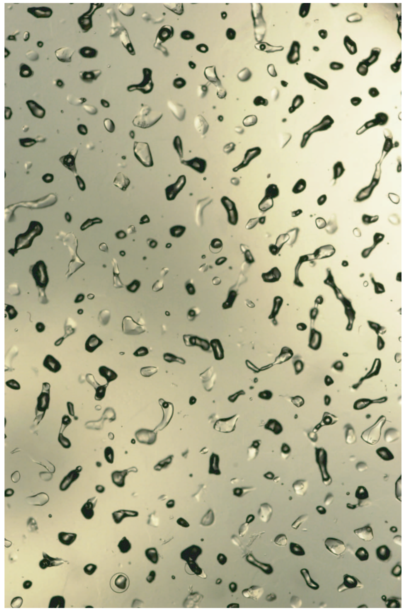

Image: Freitag 1), Kipfstuhl 1), Stauffer 2)
1) Alfred-Wegener-Institute for polar and marine research, Bremerhaven
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