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FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

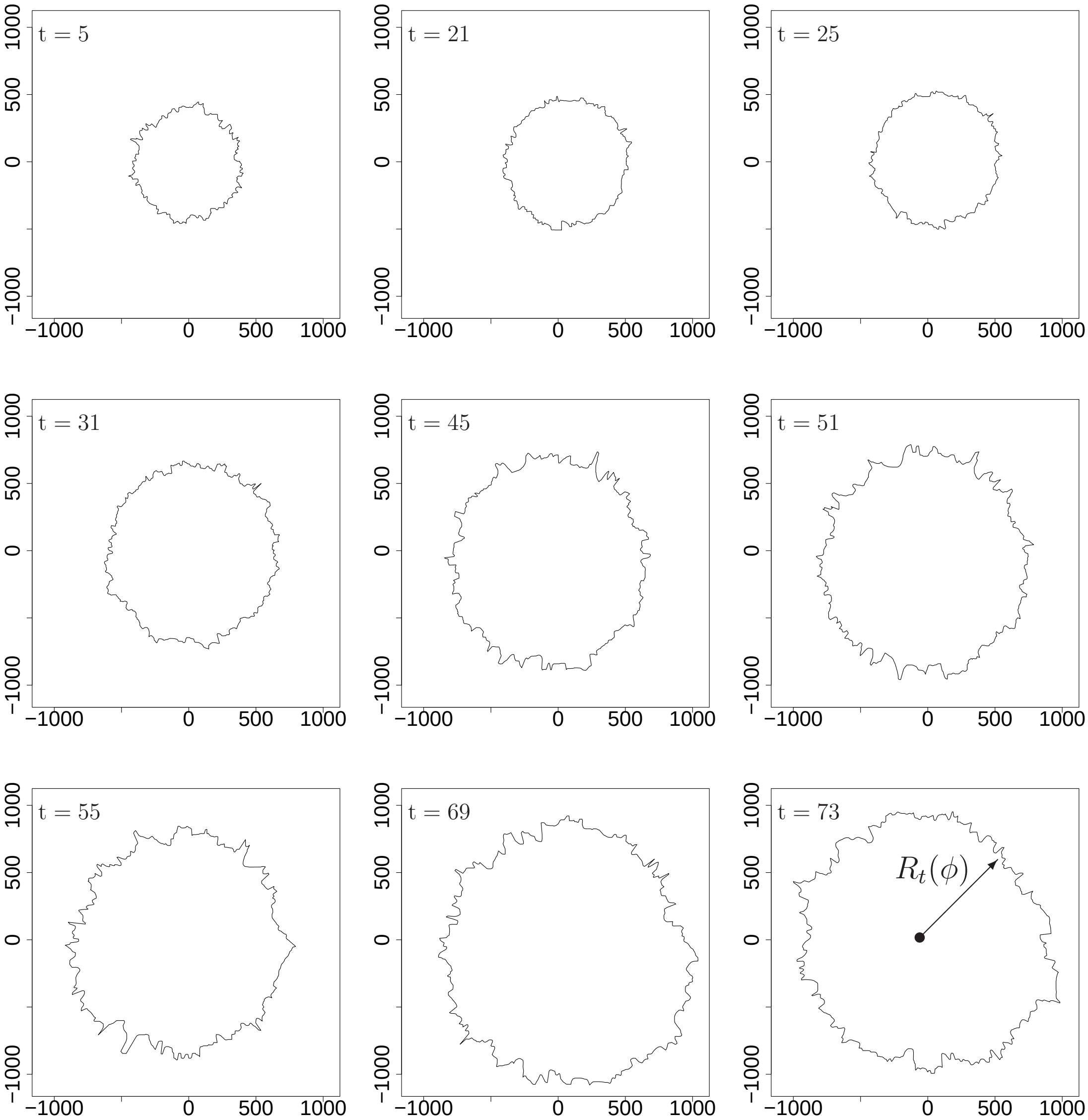
Self-scaling tumor growth

Jürgen Schmiegel

Thiele Centre for Applied Mathematics
in Natural Sciences
Department of Mathematical Sciences
University of Aarhus

star-shaped tumor profiles:

$$R_t(\phi) = \max\{R : c_0 + Re_\phi \in Y_t\}$$



spatial correlators:

$$c_{n_1,n_2}(t,\Delta\phi) \equiv \frac{\mathrm{E}\{r_t(\phi)^{n_1} r_t(\phi+\Delta\phi)^{n_2}\}}{\mathrm{E}\{r_t(\phi)^{n_1}\}\mathrm{E}\{r_t(\phi+\Delta\phi)^{n_2}\}}$$

where E{ } denotes the expectation and

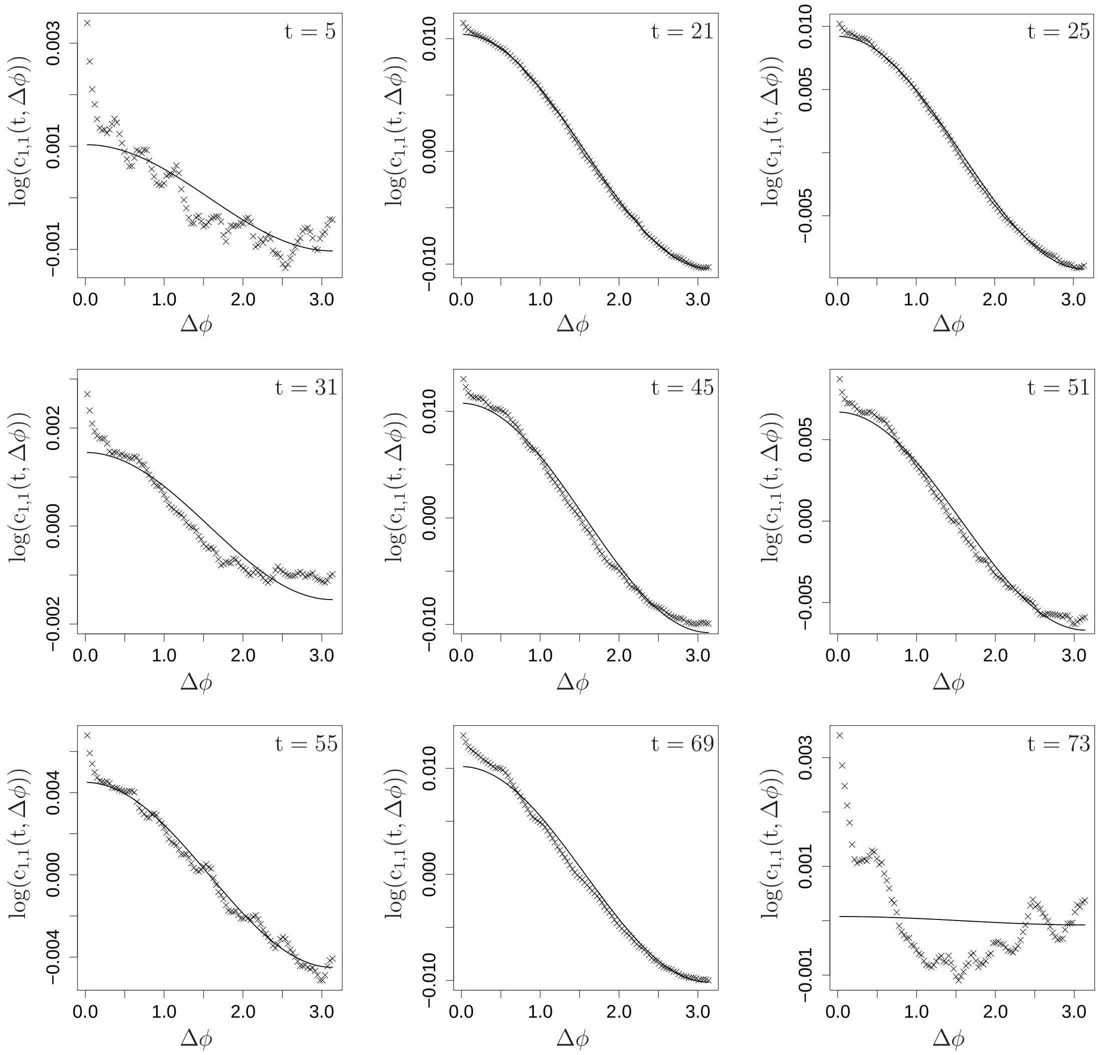
$$r_t(\phi) \equiv \frac{R_t(\phi)}{\mathrm{E}\{R_t(\phi)\}}$$

is the normalized radius function.

spatial correlators: cosine law

$$\log(c_{n_1,n_2}(t,\Delta\phi)) = b_{n_1,n_2}(t)\cos(\Delta\phi)$$

for $\Delta\phi > \phi_0(t)$ (critical angle)



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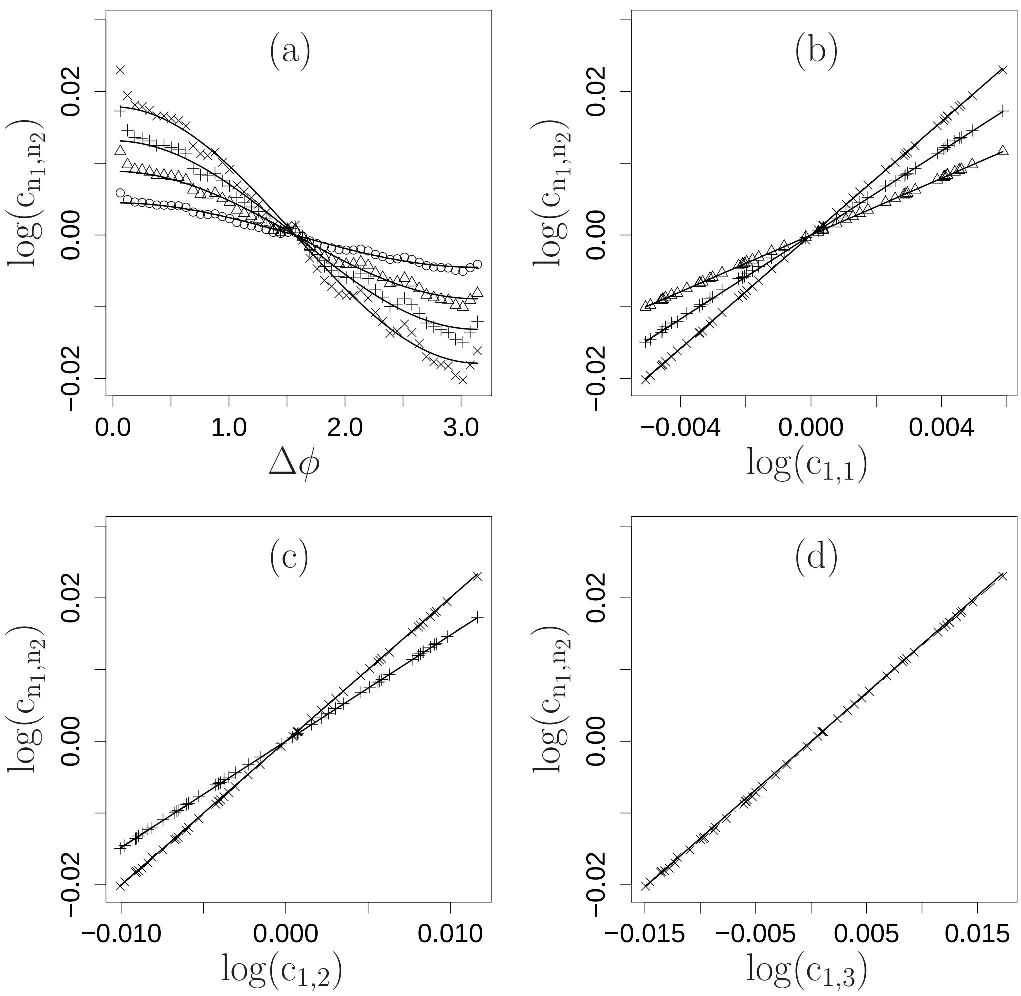
spatial correlators: self-scaling

$$c_{n_1,n_2}(t, \Delta\phi) = (c_{m_1,m_2}(t, \Delta\phi))^{k_t[m_1,m_2;n_1,n_2]}$$

The self-scaling exponents $k_t[m_1, m_2; n_1, n_2]$ are independent of $\Delta\phi$ and given by

$$k_t[m_1, m_2; n_1, n_2] = \frac{b_{n_1,n_2}(t)}{b_{m_1,m_2}(t)}$$



spatial correlators: self-scaling

Self-scaling holds for all angular distances $\Delta\phi$, including the deviations from the cosine law below the critical angle $\phi_0(t)$. Therefore, we may expect the deviations at small angular distances to be of the form

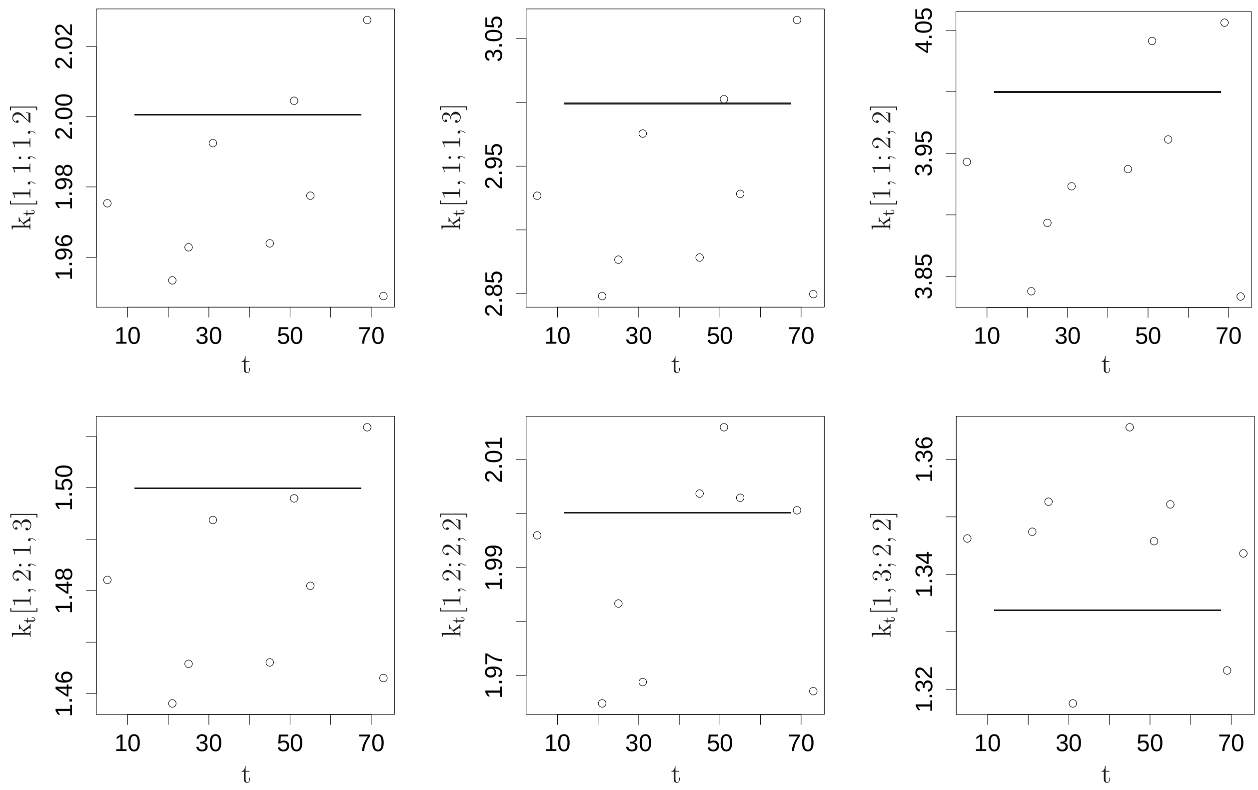
$$\log(c_{n_1,n_2}(t,\Delta\phi)) - b_{n_1,n_2}(t)\cos(\Delta\phi) = d_{n_1,n_2}(t) f_t(\Delta\phi)$$

for $\Delta\phi < \phi_0(t)$ and

$$\frac{d_{n_1,n_2}(t)}{d_{m_1,m_2}(t)} = \frac{b_{n_1,n_2}(t)}{b_{m_1,m_2}(t)} = k_t[m_1,m_2;n_1,n_2].$$

self-scaling exponents:

$$k_t[m_1, m_2; n_1, n_2] = \frac{n_1 n_2}{m_1 m_2}$$



modelling framework: Lévy bases

Let Z be a factorisable and homogeneous Lévy basis on $\mathbb{R} \times \mathbb{R}$ such that $Z(A)$ is infinitely divisible for any $A \subset \mathbb{R} \times \mathbb{R}$. Then we have the fundamental relation

$$\mathrm{E}\left\{\exp\left\{\int_A h(a) Z(\mathrm{d}a)\right\}\right\} = \exp\left\{\int_A \mathrm{K}[h(a)]\mathrm{d}a\right\}$$

where h is any integrable deterministic function, and K denotes the cumulant function of $Z(\mathrm{d}a)$, defined by

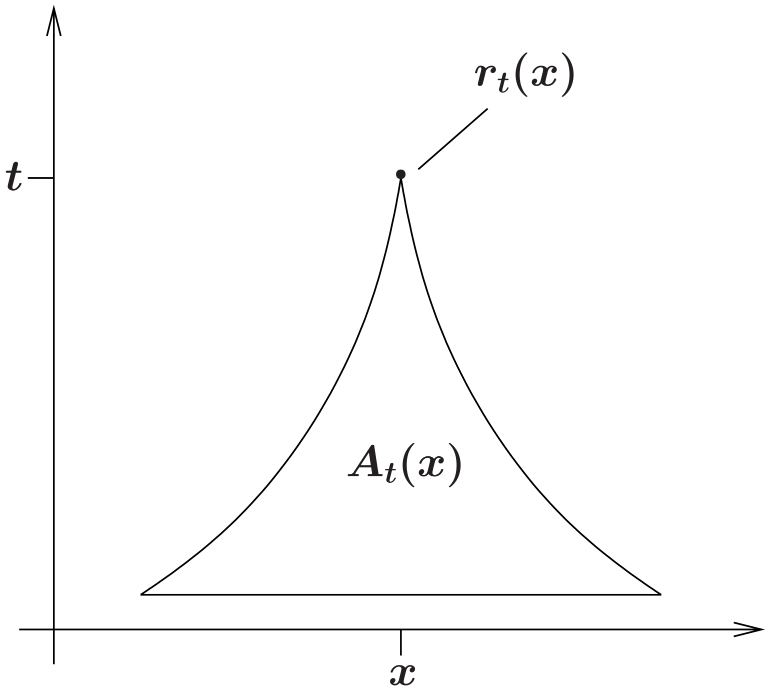
$$\mathrm{K}[\xi]\mathrm{d}a = \ln \mathrm{E}\left\{\exp\left\{\xi Z(\mathrm{d}a)\right\}\right\}.$$

exponential ambit process:

Let

$$r_t(x) = \exp\left\{\int_{A_t(x)} h(t-s, x-\rho) Z(\mathrm{d}s \times \mathrm{d}\rho)\right\}$$

where $A_t(x) \in \mathbb{R}^2$ is called the associated *ambit set.*



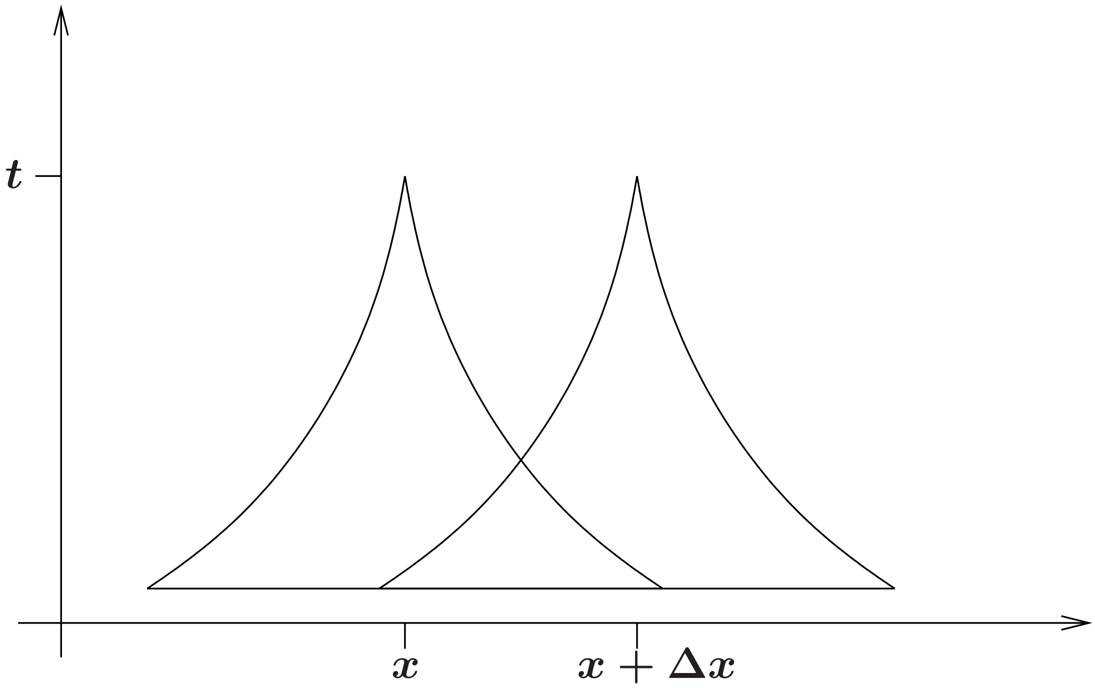
ambit process: correlator

$$c_{n_1,n_2}(t,\Delta x) = \exp\left\{\int_{A(\Delta x)} \overline{\mathrm{K}}_{n_1,n_2}[\Delta x,\rho,s]\mathrm{d}s\mathrm{d}\rho\right\}$$

where

$$A(\Delta x) = A_t(x) \cap A_t(x+\Delta x)$$

$$\overline{\mathrm{K}}_{n_1,n_2}[\Delta x,\rho,s] =$$
$$\mathrm{K}[n_1 h(t-s,x-\rho) + n_2 h(t-s,x+\Delta x-\rho)] -$$
$$\mathrm{K}[n_1 h(t-s,x-\rho)] - \mathrm{K}[n_2 h(t-s,x+\Delta x-\rho)]$$



simple example: $h \equiv 1$

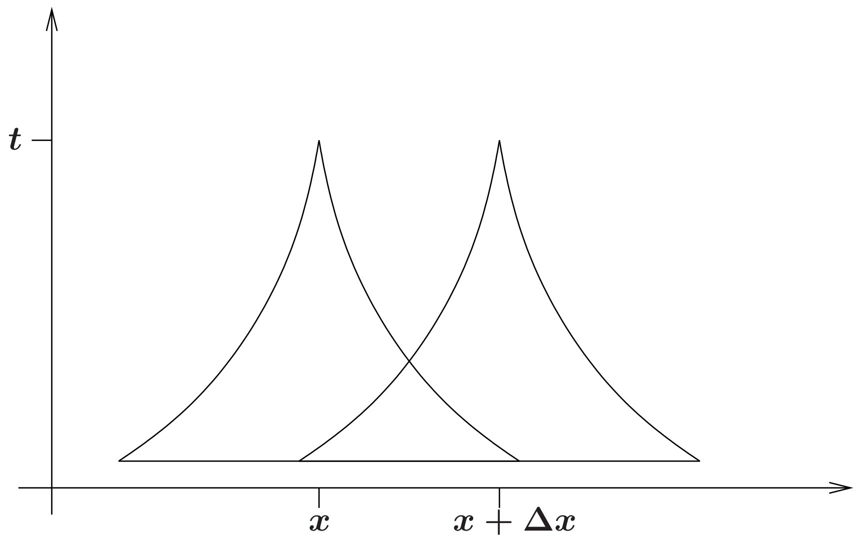
Let $A_t(x) = (x, t) + A_0(0)$ and

$$r_t(x) = \exp\left\{\int_{A_t(x)} Z(\mathrm{d}s \times \mathrm{d}\rho)\right\}.$$

Then

$$c_{n_1,n_2}(t, \Delta x) = \exp\left\{\overline{\mathrm{K}}_{n_1,n_2}\mathrm{V}(\Delta x)\right\}$$

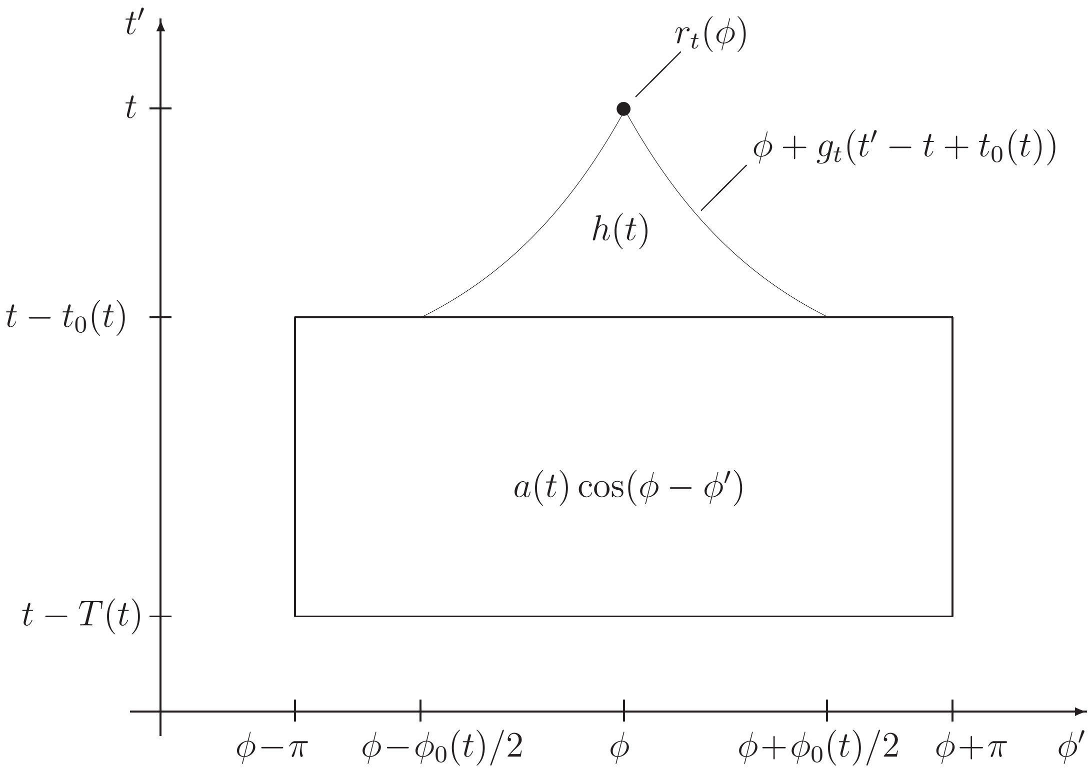
where $\mathrm{V}(\Delta x) = \mathrm{Vol}(A_t(x) \cap A_t(x + \Delta x))$
and $\overline{\mathrm{K}}_{n_1,n_2} = \mathrm{K}[n_1 + n_2] - \mathrm{K}[n_1] - \mathrm{K}[n_2]$.



tumor growth: log normal model

$$r_t(\phi) = \exp\left\{ a(t) \int_{t-T(t)}^{t-t_0(t)} \int_{\phi-\pi}^{\phi+\pi} \cos(\phi - \phi') Z(\mathrm{d}t' \times \mathrm{d}\phi') + h(t) \int_{t-t_0(t)}^{t} \int_{\phi - g_t(t'-t+t_0(t))}^{\phi + g_t(t'-t+t_0(t))} Z(\mathrm{d}t' \times \mathrm{d}\phi') \right\}$$

where Z is a normal Lévy basis with mean $\mu \mathrm{d}t\mathrm{d}\phi$ and variance $\sigma^2 \mathrm{d}t\mathrm{d}\phi$.



log normal model: correlators

Spatial correlators follow the observed cosine law with deviations at small scales

$$\log(c_{n_1,n_2}(t,\Delta\phi)) = d_{n_1,n_2}(t) f_t(\Delta\phi) 1_{[0,\phi_0(t)]}(\Delta\phi) + b_{n_1,n_2}(t)\cos(\Delta\phi),$$

where

$$b_{n_1,n_2}(t) = n_1 n_2 a(t)^2 \pi \left(T(t) - t_0(t)\right)$$

$$d_{n_1,n_2}(t) = n_1 n_2 h(t)^2$$

$$f_t(\Delta\phi) = \int_0^{g_t^{(-1)}(\Delta\phi/2)} (2g_t(s) - \Delta\phi) \mathrm{d}s.$$

The critical angle is given by

$$\phi_0(t) = 2g_t(0)$$

and

$$k_t[m_1, m_2; n_1, n_2] = \frac{n_1 n_2}{m_1 m_2}.$$

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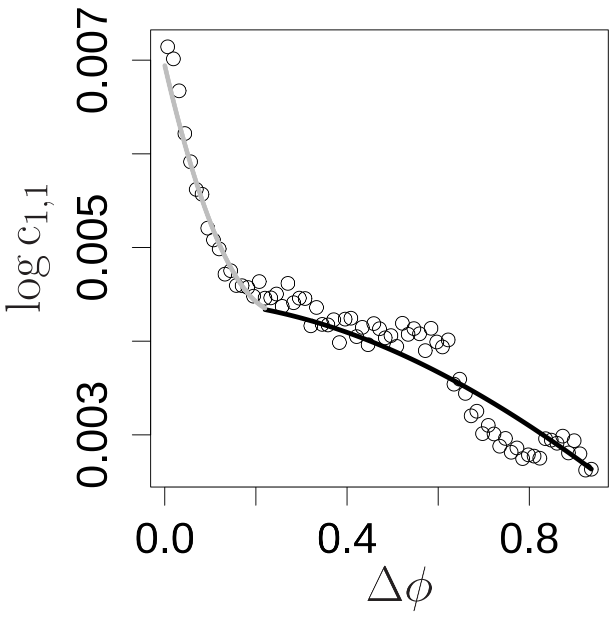
log normal model: simulation

$$g_t(s) = \frac{\phi_0(t)}{2} - \frac{\phi_0(t)}{2t_0(t)}s, \quad s \in [0, t_0(t)].$$



log normal model: simulation