Modelling of Spatio-Temporal Processes

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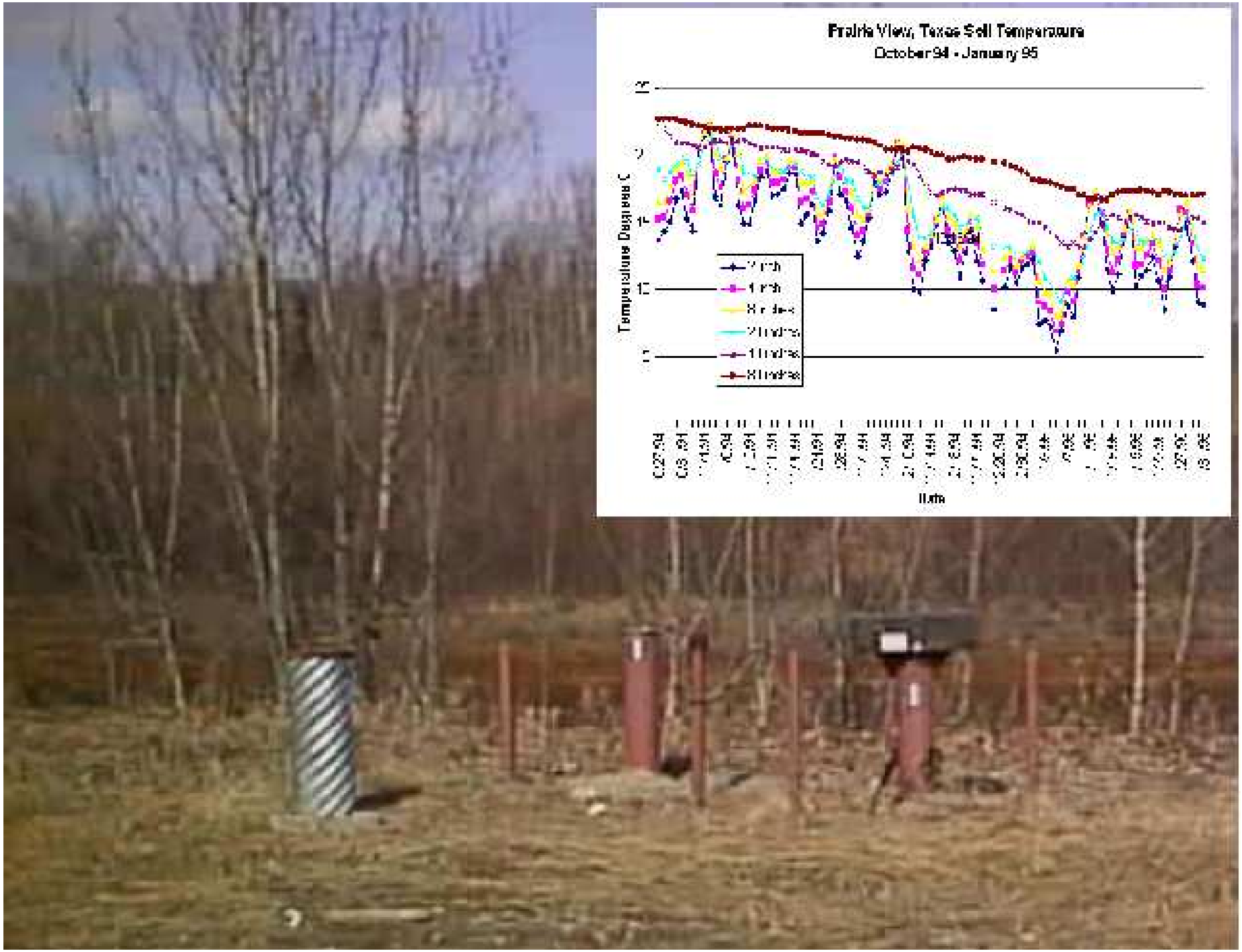
Measurements in soil physics



Geostatistical approach

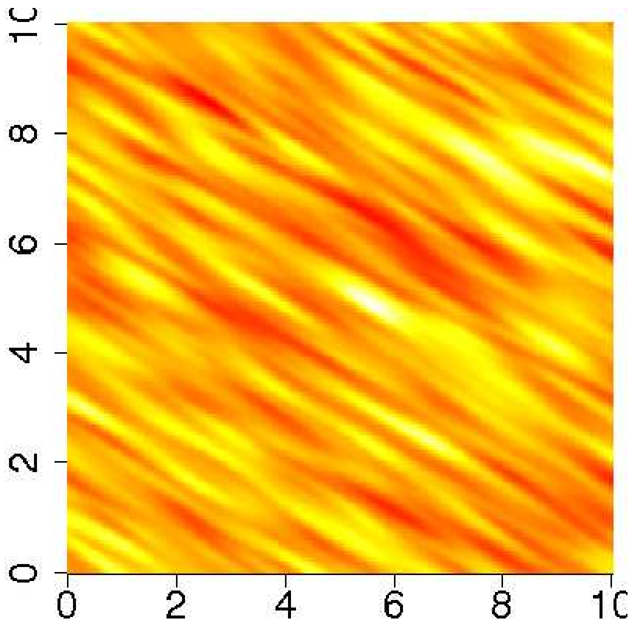
Examples: soil temperature, soil moisture

Random field Z:
$Z(x)$ real valued random quantity for all $x \in \mathbb{R}^d$
(based on the same probability space)

Gaussian Random field Z in $\mathbb{R}^d$:
Z is called Gaussian if the distribution of $(Z(x_1), \ldots Z(x_n))$ is multivariate Gaussian for all $x_1, \ldots, x_n \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Stationary random fields

$\mathbb{E}\, Z(x)$ and $\mathrm{Cov}(Z(h+x), Z(x))$ independent of x for all $h \in \mathbb{R}^d$



stationary Gaussian random field completely characterised by

- the expectation $\mathbb{E}\, Z(0)$ and
- $C(h) = \mathrm{Cov}(Z(h+x), Z(x))$

Covariance function

Covariance function of a stationary random field

$$C(x) = \mathrm{Cov}(Z(x), Z(0)), \qquad x \in \mathbb{R}^d$$

A symmetric function C is called positive definite if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i C(x_i - x_j) a_j \geq 0$$

for all $a_i \in \mathbb{R}$, $x_i \in \mathbb{R}^d$, $n \in \mathbb{N}$

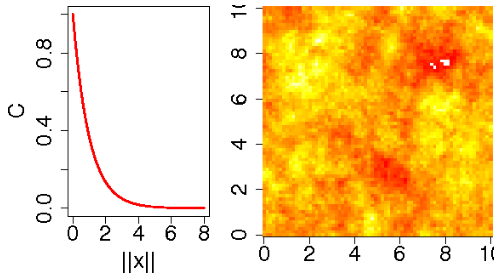
A function $C : \mathbb{R}^d \to \mathbb{R}$ is a covariance function if and only if C is positive definite.

Whittle-Matern class

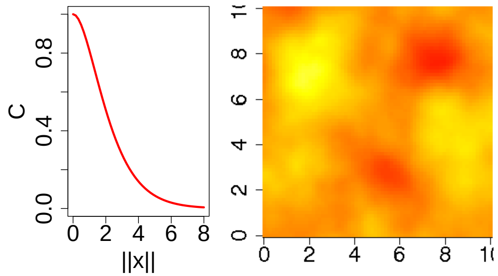
$$W_\nu(x) = \frac{2^{1-\nu}}{\Gamma(\nu)} \|x\|^\nu K_\nu(\|x\|)$$

$\nu > 0$

K_ν : modified Bessel function



$\nu = 0.5$

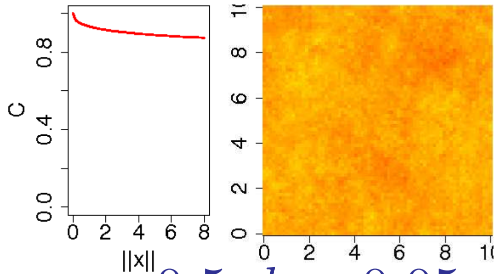


$\nu = 2$

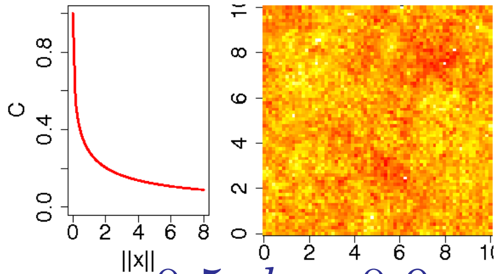
Generalised Cauchy class

$$C(x) = (1 + \|x\|^a)^{-b/a}$$

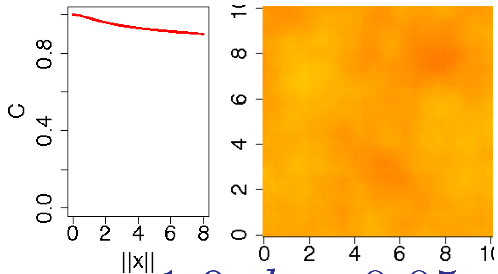
$a \in (0, 2], \quad b > 0$



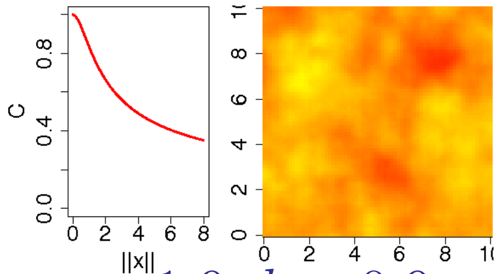
$a = 0.5$, $b = 0.05$



$a = 0.5$, $b = 0.9$



$a = 1.9$, $b = 0.05$

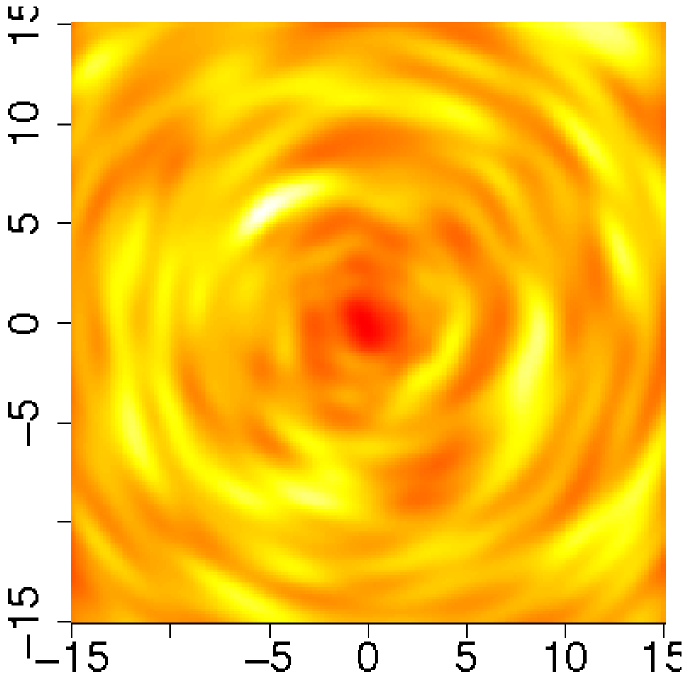


$a = 1.9$, $b = 0.9$

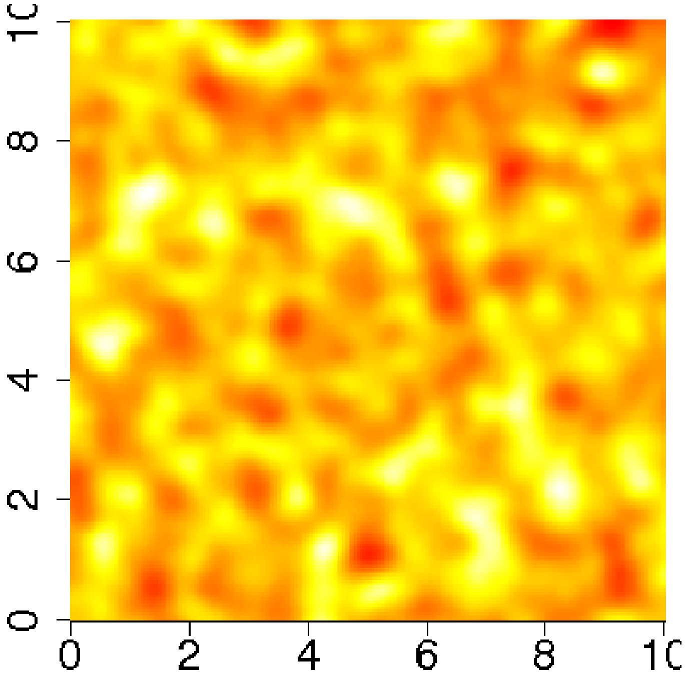
Isotropy

$$\mathrm{Cov}(Z(Ax), Z(Ay)) \;=\; \mathrm{Cov}(Z(x), Z(y))$$

for all $x, y \in \mathbb{R}^d$ and all rotation matrices A



Stationary and isotropic random field



Covariance function C is rotationinvariant, i.e.

$$C(x) = \varphi(\|x\|)$$

for some function $\varphi : [0, \infty) \to \mathbb{R}$.

Kyriakidis and Journel (1999)

Isotropy is well defined in space, while it has no meaning in a space-time context due to the intrinsic ordering and nonreversibility of time. Scales and distance units are different between space and time and cannot be directly compared in a physical sense.

1. Geometric Anisotropy

Let $\varphi(\|\cdot\|)$ be a model valid in $\mathbb{R}^4$ and

$$M : (x,t) \mapsto \begin{pmatrix} A & -v \\ a & \tau \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

Then a valid space-time model is

$$C(x,t) = \varphi(\|M(x,t)^\top\|)$$

Examples in rainfall modelling

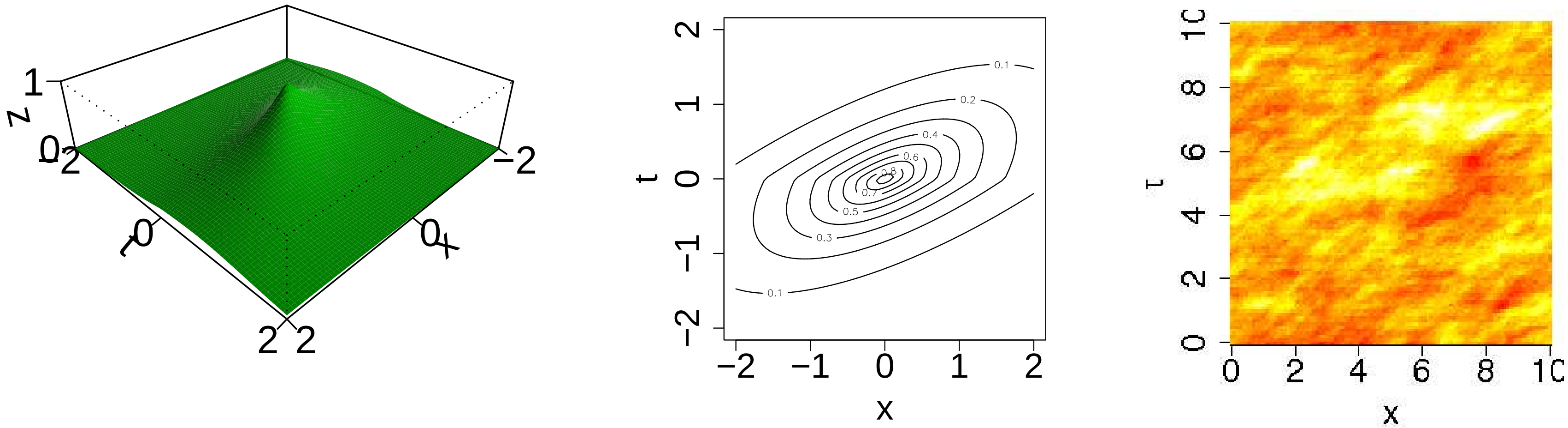
- Armstrong et al. (1993)
  $a = v = 0$, $C(x,t) = \varphi(\|(Ax, \tau t)\|)$
- frozen model (Gupta and Waymire, 1987)
  $A = \mathrm{id}$, $a = \tau = 0$, $C(x,t) = \varphi(\|x - vt\|)$
- Cox and Isham (1988)
  $A = \mathrm{id}$, $a = \tau = 0$, $v = RV$, R and V random variables

Example Cox-Isham-Model (Cox and Isham, 1988)

$$C(x,t) = e^{-a|t|}\varphi(\|x - vt\|), \qquad a > 0$$

For instance, in $\mathbb{R}^2$,

$$C(x,t) = e^{-|t|/2} e^{-\|x-(1,1)t\|}$$



2. Separable models

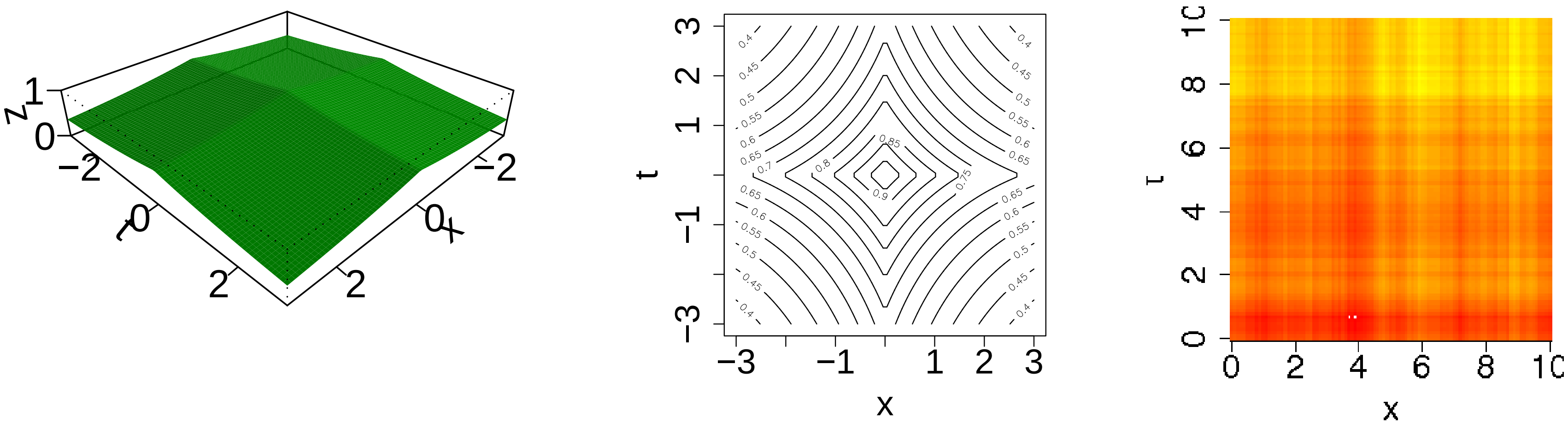
(i) $$C(x,t) = C_x(x) + C_t(t)$$

Construction:

$Z_x(x)$ and $Z_t(t)$ independent random fields

let $Z(x,t) = Z_x(x) + Z_t(t)$

Example: $$C(x,t) = \frac{1}{2}\left[\exp(-\|x\|/3) + \exp(-|t|/3)\right]$$



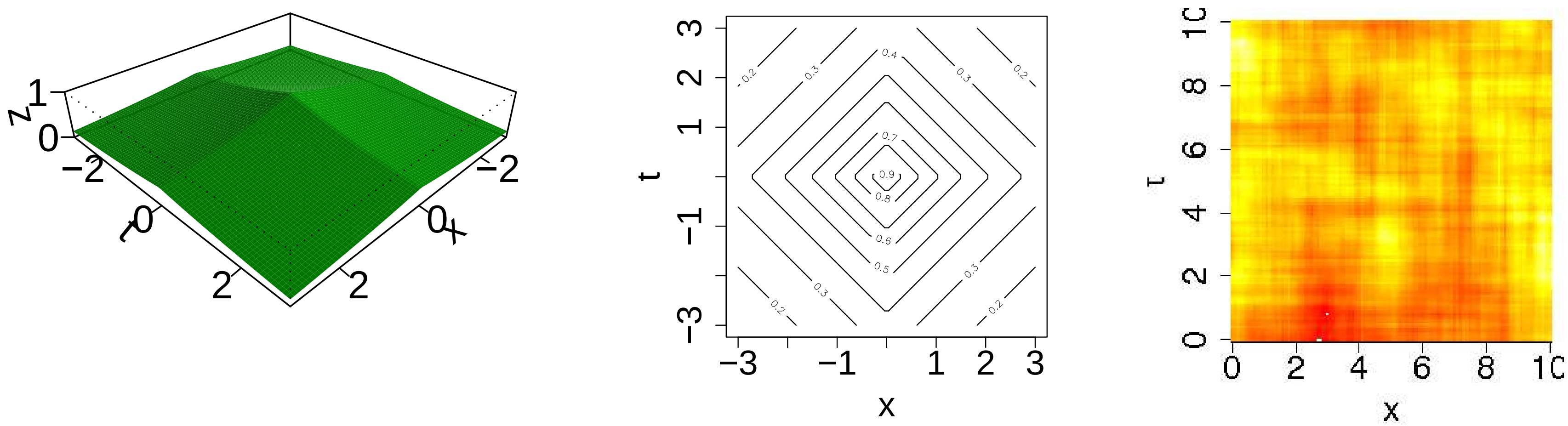
(ii) $$C(x,t) = C_x(x)C_t(t)$$

Construction:

$Z_x(x)$ and $Z_t(t)$ independent random fields

let $Z(x,t) = Z_x(x)Z_t(t)$

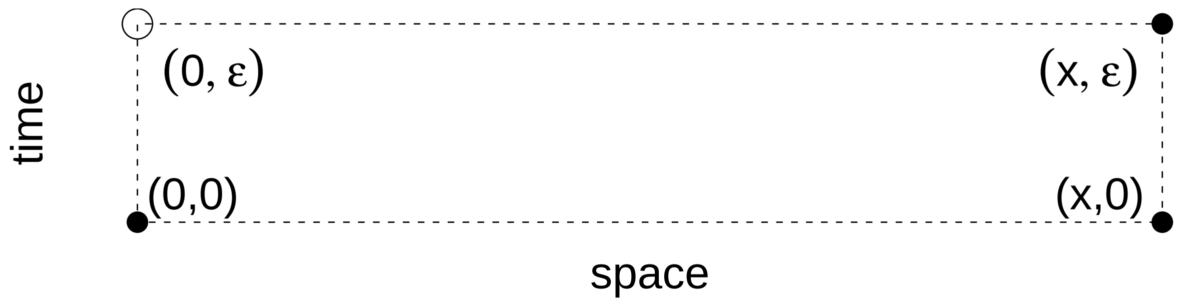
Example: $$C(x,t) = \exp(-\|x\| - |t|)$$



All other models are called non-separable.

Problems with separable models

- lack of interaction of time and space, no guideline how to separate the two component structures (Kyriakidis and Journel, 1999)
- singular kriging systems (Rouhani and Myers, 1990)
- Let $C(x,t) = C(x) + C_0(x)|t|^\alpha + o(|t|^\alpha)$.



The variance of ordinary kriging predictor for $Z(0,\varepsilon)$ is (Stein, 2005b)

$$2C_0(0)\varepsilon^\alpha + o(\varepsilon^\alpha), \quad \text{given } Z(0,0)$$

$$2\left(C_0(0) - \frac{C_0^2(x)}{C_0(0)}\right)\varepsilon^\alpha + o(\varepsilon^\alpha), \quad \text{given } Z(0,0), Z(x,0), Z(x,\varepsilon)$$

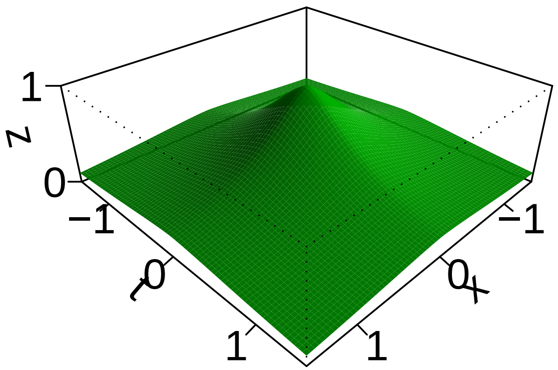
Product-sum models

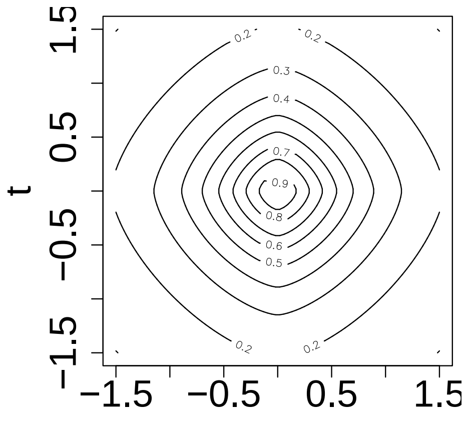
Idea:

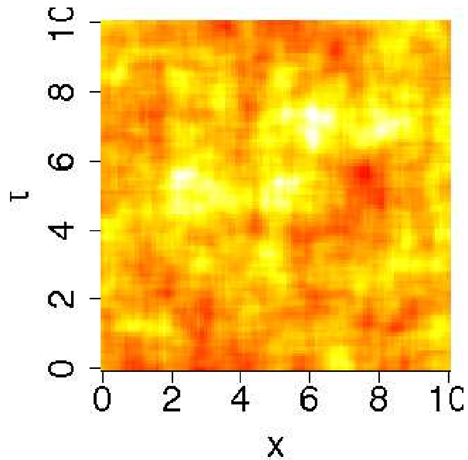
$$C(x,t) = \lim_{n\to\infty} \sum_{i=1}^{n} C_{x,i}(x) C_{t,i}(t), \qquad n \geq 2$$

Example (de Iaco et al., 2002; de Cesare et al., 2002)

$$C(x,t) \;=\; \int_0^\infty C_x(\xi x) C_t(\xi t) \,\mathrm{d}F(\xi)$$







$$C(x,t) = \left(1 + \|x\|^\alpha + |t|^\delta\right)^{-(d+1)/2}, \qquad 1 \leq \alpha, \delta \leq 2$$

3. Fourier-Transform Approach (Stein, 2005b)

Fourier density

$$f(w_1, w_2) = g(w_1, w_2) \left[c_1(a_1^2 + |w_1|^2)^{\alpha_1} + c_2(a_2^2 + |w_2|^2)^{\alpha_2} \right]^{-\nu}$$

$c_i, a_i, \alpha_i \in (0, \infty)$, $d_1/(\alpha_1 \nu) + d_2/(\alpha_2 \nu) < 2$

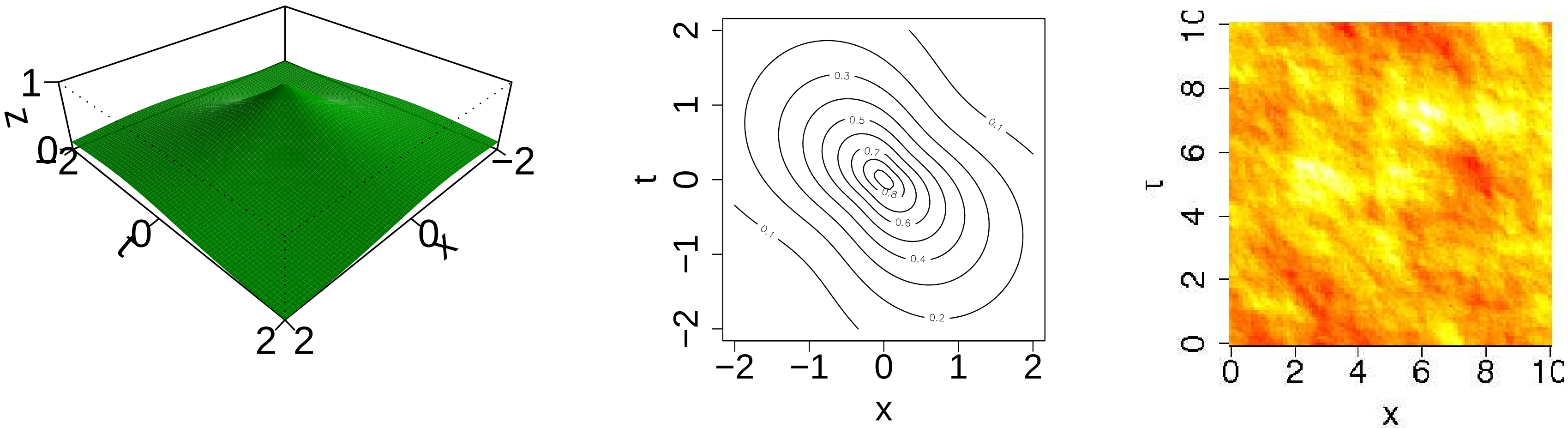
Examples of non-separabel models

- full symmetry if $g(w_1, w_2) \equiv 1$
- lack of full symmetry on $\mathbb{R}^d \times \mathbb{R}$

$$g(w_1, w_2) = a + b(w^\top z) w_2 + c_1 |w_1|^2 + c_2 |w_2|^2$$

$a, c_2 \in (0, \infty)$, $b^2 < 4 c_1 c_2$

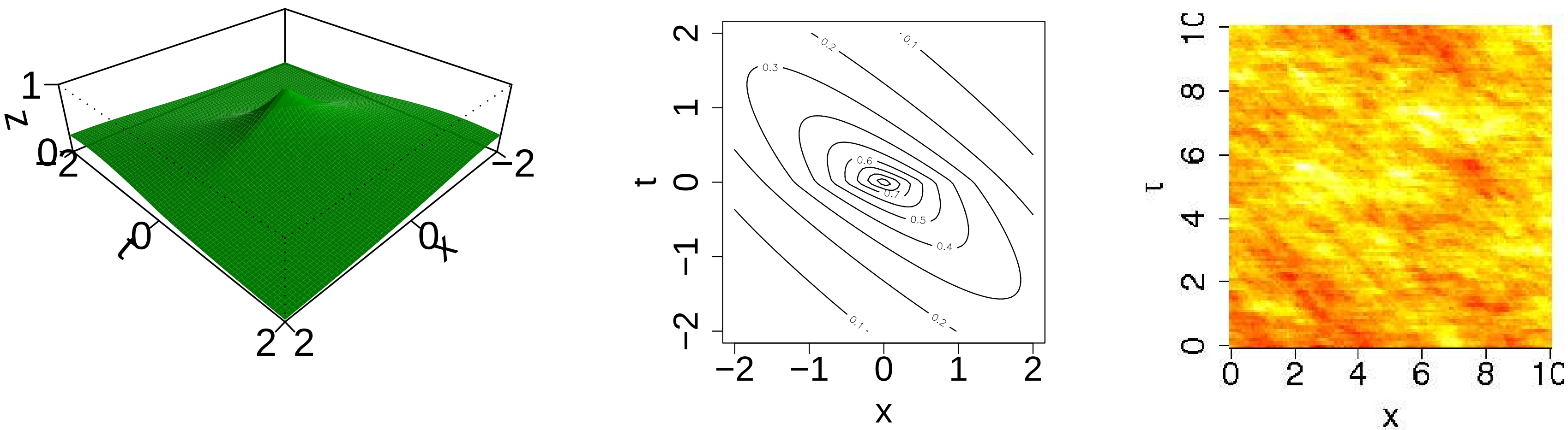
$$\begin{aligned} C(x,t) &= W_\nu(y) - 2(x^\top z)tW_{\nu-1}(y)/[2(\nu-1)(2\nu+3)] \\ y &= \|(x_1,x_2,t)\|, \qquad \nu = 0.5, \quad z = (\sqrt{2},0) \end{aligned}$$



- (Ma, 2003; Stein, 2005b)

$$C(x,t) = \Gamma(\nu+\gamma(t))\,\{\Gamma(\nu+\gamma(t)+d/2)\}^{-1}\,W_{\nu+\gamma(t)}(\|x-Vt\|)$$

$$\nu = 0.5, \qquad \gamma(t) = 1-e^{-|t|}, \qquad V = (-\tfrac{1}{2},-\tfrac{1}{2})$$



4. Non-Separable Normal Mixture Models

C_0: any covariance function
F: any positive finite measure on $[0,\infty)$
Then

$$C(x) = \int_0^\infty C_0(ux)\, F(\mathrm{d}u)$$

is a covariance function.

C: a normal mixture model if $C_0 = e^{-x^2}$

Examples: hyperbolic class (Whittle-Matérn, exponential, Cauchy), stable class

In space-time modelling:

$$C_0(x,t) = \sqrt{|A_{xt}|}\, e^{-h^\top A_{xt} h}, \qquad h^\top \subset (x^\top, t)$$

Fully symmetric models based on completely monotone functions (Gneiting, 2002)

Fully symmetric model:

$$C(x,t) = C(x,-t) \quad (= C(-x,t) = C(-x,-t)), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}$$

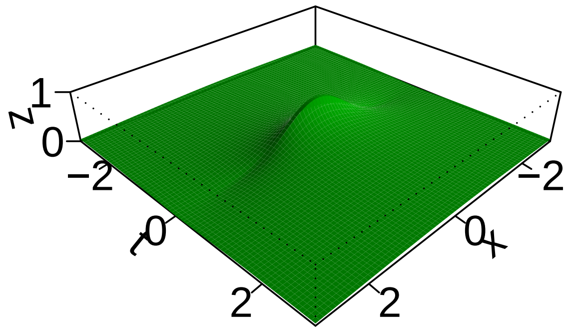
Gneiting (2002):

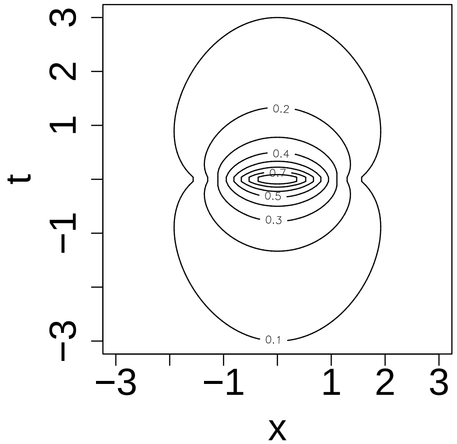
$$A = \frac{1}{\psi(|t|^2)} \cdot 1_{d\times d}, \qquad h = x$$

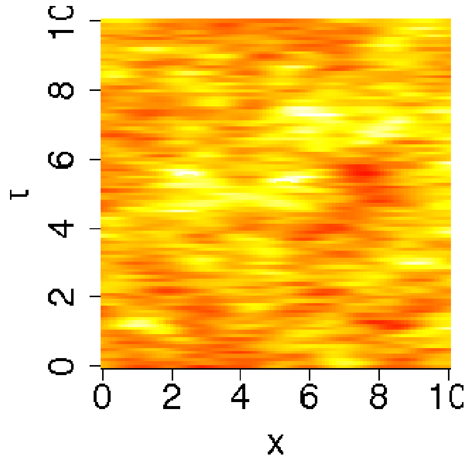
$\psi(t) : [0,\infty) \longrightarrow \mathbb{R}$ positive, completely monotone derivative

Function	Parameters
$\psi(u) = (au^\alpha + 1)^\beta$	$a > 0$, $\alpha \in (0,1]$, $\beta \in (0,1]$
$\psi(u) = (au^\alpha + b)/(au^\alpha + 1)$	$a > 0$, $b \in (0,1]$, $\alpha \in (0,1]$

Example: $C(x,t) = (|3t|+1)^{-1} \exp\left(-\|x\|^2/(|3t|+1)\right)$





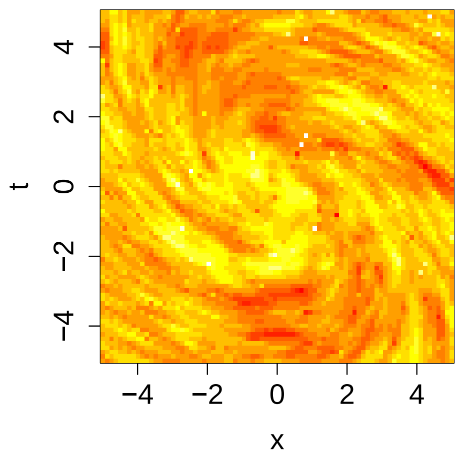


Nonstationary-model of Paciorek (Paciorek, 2003) and Stein (Stein, 2005a)

$$C_0(\xi, \zeta) = \sqrt{|A_{\xi\zeta}|}\, e^{-(\xi-\zeta)^\top A_{\xi\zeta}(\xi-\zeta)}, \qquad \xi, \zeta \in \mathbb{R}^{d+1}$$

$$A_{\xi\zeta} = \left(\Sigma(\xi) + \Sigma(\zeta)\right)^{-1}$$

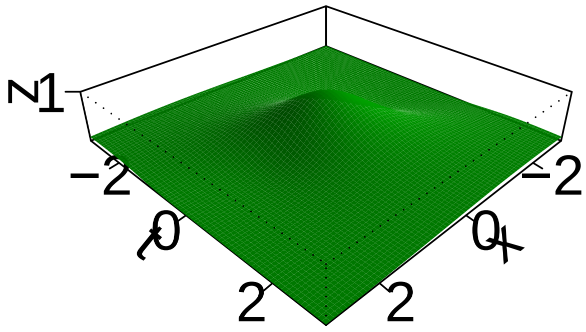
for any choice of strictly positive definite matrices $\Sigma(\xi)$.

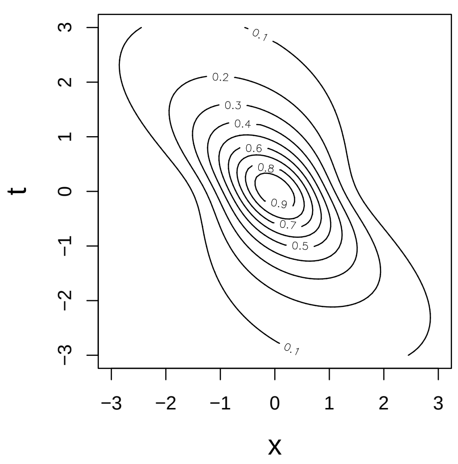


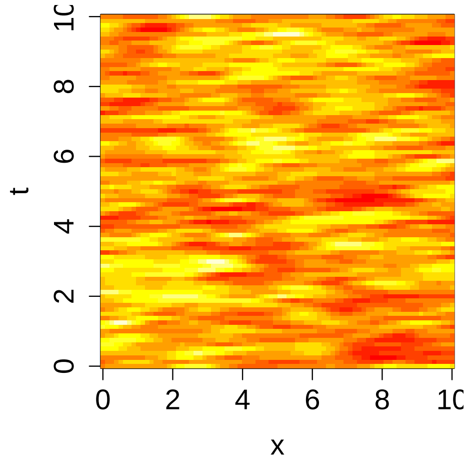
Another model

$$A = M_t/|M_t|, \qquad M_t = \begin{pmatrix} 1+t^2 & -\rho t^2 \\ -\rho t^2 & 1+t^2 \end{pmatrix}, \qquad h = x$$

$\mu = -0.5$, $\rho = 0.5$:







Unconditional Simulation

Methods in the R package `RandomFields` (Schlather, 2001)

method	properties
circulant embedding and variants	grid, 3D
TBM	isotropic, 3D
spectral TBM	isotropic, 2D
Cholesky	up 10.000 pts
random coin	few cov fcts

Unconditional Simulation

Methods in the R package `RandomFields` (Schlather, 2001) V1.4

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Cholesky	up 10.000 pts
random coin	few cov fcts
sequential	imprecise
Gaussian Markov	2D

Unconditional Simulation

Methods in the R package `RandomFields` (Schlather, 2001) V1.4

method	properties	suitable space-time modific.
circulant embedding and variants	grid, 3D	no
TBM	isotropic, 3D	turning layer (fully sym., 4D)
spectral TBM	isotropic, 2D	4D, aniso and layers
Cholesky	up 10.000 pts	no
random coin	few cov fcts	yes (4D)
sequential	imprecise	few spat. pts
Gaussian Markov	2D	no

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