Estimation and edge correction in the Renshaw-Särkkä model

Ottmar Cronie

Mathematical Sciences
Chalmers University of Technology & University of Gothenburg

August 20, 2008

Outline

- The process
- Simulations
- Estimation
- Data
- Edge effects
- Future work

- New individuals arrive to the region of interest, $W \subseteq \mathbb{R}^d$, according to a Poisson process with intensity $\nu(W)\alpha$.
- Each individual is assigned a position $\mathbf{x}_i \sim Uni(W)$ together with an initial size (mark) $m_i(t_0^i) = \varepsilon_i$ (t_0^i = arrival time of individual i).
- An individual changes its size, deterministically, according to

$$dm_i(t) = f(m_i(t);\Theta)dt + \sum_{\substack{j \neq i \\ j \in \Omega_t}} h(m_i(t), m_j(t), ||\mathbf{x}_i - \mathbf{x}_j||;\Theta)dt$$

where

Ω_t	index set comprising the individuals alive at time t
$f(\cdot)$	individual growth function
$h(\cdot)$	spatial interaction function ($\|\| \cdot \|\|$ Euclidean distance)
Θ	set of model parameters

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Possible death scenarios in the model:

- Individuals die naturally according to a death process with intensity $\mu(m_i(t))$, $i \in \Omega_t$, i.e.
$\mathbb{P}\left[\text{Individual } i \text{ dies naturally in } (t, t+dt) \middle| m_i(t)\right] = \mu(m_i(t))\,dt + o(dt)$
- Individuals suffer a competitive death if some event, depending on $m_i(t)$ and/or $dm_i(t)$, occurs.

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The process - model modifications

One possibility is to consider interactive death to occur when $m_i(t) < 0$.

Other options:

- $dm_i(t) < 0 \Rightarrow$ individual is considered dead through competition
- $dm_i(t) < 0 \Rightarrow$ individual loses its individual growth and is considered dead through competition when $m_i(t) < 0$.

Some natural suggestions for the natural death:

- $\mu(m_i(t)) \equiv \mu$
- $\mu(m_i(t)) = \mu\frac{m_i(t)}{1+m_i(t)}$
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Individual growth functions:

- Linear growth: $f(m_i(t), \lambda_i, K_i) = \lambda_i \left(1 - \frac{m_i(t)}{K_i}\right)$
- Logistic growth: $f(m_i(t), \lambda_i, K_i) = \lambda_i m_i(t) \left(1 - \frac{m_i(t)}{K_i}\right)$

where λ_i is the growth rate and K_i is the carrying capacity.

The process - interaction functions

Interaction functions:

- Symmetric interaction:

$$h(m_i(t), m_j(t), ||\mathbf{x}_i - \mathbf{x}_j||, \Theta) = -c\,\mathbb{I}\{||\mathbf{x}_i - \mathbf{x}_j|| < r(m_i(t) + m_j(t))\}$$

  where $\Theta = \{r, c\}$, $r > 0$ and $\mathbb{I}\{\cdot\}$ denotes the indicator function.

- Area interaction:

$$h(m_i(t), m_j(t), ||\mathbf{x}_i - \mathbf{x}_j||, \Theta) = -c\frac{\nu\left[B\left(\mathbf{x}_i, rm_i(t)\right) \cap B\left(\mathbf{x}_j, rm_j(t)\right)\right]}{\pi r^2 m_i(t)^2}$$

  where $\Theta = \{r, c\}$, $r > 0$, $B(\mathbf{x}, \epsilon)$ denotes a closed ball (disk in $\mathbb{R}^2$) with center $x \in \mathbb{R}^d$ and radius ϵ, and $\nu(\cdot)$ denotes volume.
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- Parameter choices: $\varepsilon_i \equiv \varepsilon, \lambda_i \equiv \lambda, K_i \equiv K, r, c > 0$
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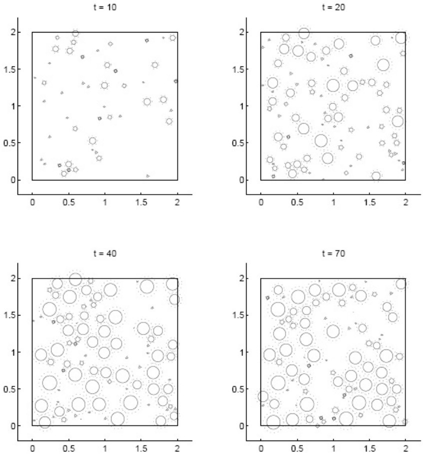
Realisation:

$\alpha = 1.5$, $\mu = 0.02$, $\lambda = 0.2$, $K = 0.1$, $c = 0.1$, $r = 1.5$, $\varepsilon_i = 0.01$, $dt = 0.01$

The dotted rings are the influence zones $B(\mathbf{x}_i, r\, m_i(t))$.



Estimation: α and μ

Let $N_t = \#\{\text{individuals observed up until } t\}$ and T the final sample point of the process. Then

$$\hat{\alpha} = \frac{N_T}{\nu(W)\,T}$$

- If $\mu(m_i(t)) \equiv \mu$ then we use the ML estimator

$$\hat{\mu} = n_T \Bigg/ \left(\sum_{i=1}^{n_T} t_i + \sum_{j=1}^{m_T} s_j \right)$$

where $t_1, \ldots, t_{n_T}$ and $s_1, \ldots, s_{m_T}$ denote, respectively, the recorded lifetimes of the n_T individuals who died from natural causes by time T and the m_T individuals still alive at time T.

- If $\mu(m_i(t)) = \mu \frac{1}{1+m_i(t)}$ we numerically maximize

$$\begin{aligned} \log L(\mu) \approx & \sum_{i=1}^{n_t} \log \left\{ \int_0^\infty \mu \frac{1}{1+x} e^{-\frac{\mu}{1+x} t_i} \widehat{f_{m(t_i)}(x)}\, dx \right\} \\ & + \sum_{j=1}^{m_t} \log \left\{ \int_0^\infty e^{-\frac{\mu}{1+x} s_j} \widehat{f_{m(T)}(x)}\, dx \right\} \end{aligned}$$

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Least squares estimation of λ, K, r, and c

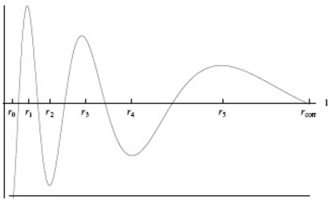
Determine the least squares estimates $\hat{\lambda}$, $\hat{K}$, $\hat{r}$, and $\hat{c}$ by minimizing

$$S(\lambda, K, r, c) := \sum_{t=1}^{T-1} \sum_{i \in \Omega_t} [\tilde{m}_i(t+1; \lambda, K, r, c) - m_i(t+1)]^2$$

with respect to λ, K, r, and c (Naturally the time increments do not need to have size 1). $\tilde{m}_i(t+1; \lambda, K, r, c)$ is the predicted value of $m_i(t+1)$ based on $m_i(t)$ and $dm_i(t)$.

The pair correlation function

The (planar) pair correlation function $g(r) = \frac{K'(r)}{2\pi r}$ has the typical appearance



$r_0 =$ minimum inter-point distance; hard core distance

$r_1 =$ range of most frequent short inter-point distance; distance from typical point to near neighbours

$r_2 =$ distance from typical point to regions with a small number of points beyond the nearest neighbours

$r_3 =$ range of most frequent longer inter-point distance; distance from typical point to regions with further

Bounds and starting values

From $h(m_i(t), m_j(t), ||\mathbf{x}_i - \mathbf{x}_j||, r, c) = -c\frac{\nu\left(B(\mathbf{x}_i, rm_i(t)) \cap B\left(\mathbf{x}_j, rm_j(t)\right)\right)}{\pi r^2 m_i(t)^2}$ one can see that r and c are correlated which causes biased estimates.

For some fixed t a typical influence zone radius is given by $r\mathbb{E}[m_i(t)]$. So

$$2r\mathbb{E}[m_i(t)] + \mathbb{E}[m_i(t)] \leq r_3 \quad \Longrightarrow r \leq \frac{r_3 - \mathbb{E}[m_i(t)]}{2\mathbb{E}[m_i(t)]}$$

Since two trees do not intersect we get that $1 < r \leq \frac{r_3 - \mathbb{E}[m_i(t)]}{2\mathbb{E}[m_i(t)]}$.

We use as starting values in our estimation (data sampled at $t_1, \dots, t_n$)

$$\begin{aligned}
\hat{r}_0 &= \left(\frac{r_3 - \mathbb{E}[m_i(t)]}{2\mathbb{E}[m_i(t)]} + 1\right)/2 \\
\hat{K}_0 &= \max_{i \in \Omega_{t_n}} \{m_i(t_n)\} \\
\hat{\lambda}_0 &= \left\{\lambda > 0 : g(t_1, \lambda, \hat{K}_0, \varepsilon) = \max_{i \in \Omega_{t_1}} \{m_i(t_1)\}\right\} \text{ where} \\
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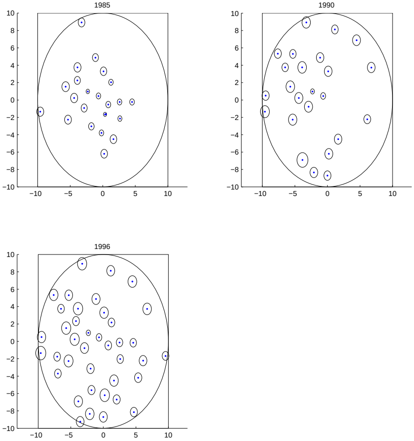
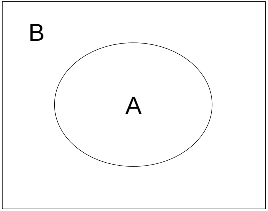
Data: Swedish pines (data set under investigation)



Correcting for edge effects

Data $\mathbb{X} = \left\{ \mathbf{x}_1(t_j), \ldots, \mathbf{x}_{n_{t_j}}(t_j) \right\}_{j=1}^{n}$ is sampled in the circular region A.

$B = A^c$ represents the rest of our region of interest.



Correcting for edge effects: Rotations

1. Estimate parameters from $\mathbb{X}$ to generate the parameter set $\hat{\Theta}_* = \{\hat{\mu}_*, \hat{\alpha}_*, \hat{\lambda}_*, \hat{K}_*, \hat{r}_*, \hat{c}_*\}$.
2. Simulate the process on $W = A \cup B$, based on $\hat{\Theta}_*$ and $t_1, \dots, t_n$ (where W is wrapped onto a torus).
3. For $t_1, \dots, t_n$, remove what has been simulated in A.
4. For the angles $\theta_1, \dots, \theta_k$, $0 \leq \theta_i < 2\pi$, $\forall i = 1, \dots, k$, perform counterclockwise rotations of $\mathbb{X}$ around the center of A to get $\mathbb{X}_{\theta_1}, \dots, \mathbb{X}_{\theta_k}$.
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Correcting for edge effects: Rotations

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Correcting for edge effects: Simultaneous growth

1. Estimate parameters from $\mathbb{X}$ to generate the parameter set $\hat{\Theta}_* = \{\hat{\mu}_*, \hat{\alpha}_*, \hat{\lambda}_*, \hat{K}_*, \hat{r}_*, \hat{c}_*\}$.
2. For each time interval $(t_{j-1}, t_j]$ we get n_j new individuals. Simulate $Uni(t_{j-1}, t_j)$-distributed birth times $b_1^j < \ldots, b_{n_j}^j$ and assign these to the individuals which have arrived in $(t_{j-1}, t_j]$ in such an order that the largest individual gets the smallest time, going upwards until the smallest individual has recieved the largest time.
3. Simulate the process on $W = A \cup B$, based on $\hat{\Theta}_*$ and $t_1, \ldots, t_n$ (where W is wrapped onto a torus) where we let each data individual enter at its birth time and grow linearly until its death time, if it dies before T, or until T, if it is stil alive at T.
4. Remove everything found in A at all the sample times (we are left with what has been simulated in B) and replace it by $\mathbb{X}$.
5. Estimate from $\mathbb{X}$ (i.e. from A) but let the individuals in B and A affect each other.

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Future/ongoing work

1. Estimation/edge correction of multiple (homogeneous) data sets simultaneously
2. Maximum likelihood estimation
3. Individual (stochastic) parameters
4. Incorporation of other individual growth functions