Spectral theorem and ergodicity for a class of second order non-Gaussian processes

Krzysztof Podgórski
Mathematical Statistics
Lund University

August 17, 2010

Outline

Isometry between random variables and functions

Isometry between random variables and functions

- $X(\tau)$ – a weakly stationary stationary process

Isometry between random variables and functions

- $X(\tau)$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Covariance function

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Covariance function $\mapsto$ Bochner theorem

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Covariance function $\mapsto$ Bochner theorem $\mapsto$ Spectral measure

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Covariance function $\mapsto$ Bochner theorem $\mapsto$ Spectral measure

$$R(\boldsymbol{\tau}) = \mathbb{C}ov(X(\boldsymbol{\tau}), X(\mathbf{0})) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\sigma(\boldsymbol{\lambda})$$

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Covariance function $\mapsto$ Bochner theorem $\mapsto$ Spectral measure

$$R(\boldsymbol{\tau}) = \mathbb{C}ov(X(\boldsymbol{\tau}), X(\mathbf{0})) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\sigma(\boldsymbol{\lambda})$$

$$X(\tau) \overset{isometry}{\longleftrightarrow} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})$$

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Covariance function $\mapsto$ Bochner theorem $\mapsto$ Spectral measure

$$R(\boldsymbol{\tau}) = \mathbb{C}ov(X(\boldsymbol{\tau}), X(\mathbf{0})) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\sigma(\boldsymbol{\lambda})$$

$$X(\tau) \overset{isometry}{\longleftrightarrow} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})$$

- Y – random variable defined through $X(\boldsymbol{\tau})$, $\mathbb{E}|Y|^2 < \infty$

Isometry between random variables and functions

- $X(\boldsymbol{\tau})$ – a weakly stationary stationary process
- $\tau \in \mathbb{R}$ – time variable,
- $\boldsymbol{\tau} = (x, y, t) \in \mathbb{R}^3$ – spatio-temporal fields

Covariance function $\mapsto$ Bochner theorem $\mapsto$ Spectral measure

$$R(\boldsymbol{\tau}) = \mathbb{C}ov(X(\boldsymbol{\tau}), X(\mathbf{0})) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\sigma(\boldsymbol{\lambda})$$

$$X(\tau) \overset{isometry}{\longleftrightarrow} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})$$

- Y – random variable defined through $X(\boldsymbol{\tau})$, $\mathbb{E}|Y|^2 < \infty$
- f – function on $\mathbb{R}^d$, $\int |f|^2 d\sigma < \infty$

$$Y \overset{isometry}{\longleftrightarrow} f$$

LUND UNIVERSITY

Spectral theorem

Spectral theorem

- Take $f = \mathbf{1}_A$ and $Y = \zeta(A)$ be corresponding random variable

$$Y = \int \mathbf{1}_A(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda}) \stackrel{isometry}{\longleftrightarrow} \mathbf{1}_A$$

Spectral theorem

- Take $f = \mathbf{1}_A$ and $Y = \zeta(A)$ be corresponding random variable

$$Y = \int \mathbf{1}_A(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda}) \stackrel{isometry}{\longleftrightarrow} \mathbf{1}_A$$

- $\zeta(\boldsymbol{\lambda}) \stackrel{def}{=} \zeta(-\infty, \boldsymbol{\lambda}]$.

Spectral theorem

- Take $f = \mathbf{1}_A$ and $Y = \zeta(A)$ be corresponding random variable

$$Y = \int \mathbf{1}_A(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda}) \stackrel{isometry}{\longleftrightarrow} \mathbf{1}_A$$

- $\zeta(\boldsymbol{\lambda}) \stackrel{def}{=} \zeta(-\infty, \boldsymbol{\lambda}]$.
- $\zeta(\boldsymbol{\lambda})$ – complex, zero mean, orthogonal (uncorrelated) increments,

Spectral theorem

- Take $f = \mathbf{1}_A$ and $Y = \zeta(A)$ be corresponding random variable

$$Y = \int \mathbf{1}_A(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda}) \overset{isometry}{\longleftrightarrow} \mathbf{1}_A$$

- $\zeta(\boldsymbol{\lambda}) \overset{def}{=} \zeta(-\infty, \boldsymbol{\lambda}]$.
- $\zeta(\boldsymbol{\lambda})$ – complex, zero mean, orthogonal (uncorrelated) increments,

$$E(|\zeta(\boldsymbol{\lambda})|^2) = \sigma(-\infty, \boldsymbol{\lambda}]$$

Spectral theorem

- Take $f = \mathbf{1}_A$ and $Y = \zeta(A)$ be corresponding random variable

$$Y = \int \mathbf{1}_A(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda}) \overset{isometry}{\longleftrightarrow} \mathbf{1}_A$$

- $\zeta(\boldsymbol{\lambda}) \overset{def}{=} \zeta(-\infty, \boldsymbol{\lambda}]$.
- $\zeta(\boldsymbol{\lambda})$ – complex, zero mean, orthogonal (uncorrelated) increments,

$$\begin{aligned} E(|\zeta(\boldsymbol{\lambda})|^2) &= \sigma(-\infty, \boldsymbol{\lambda}] \\ \zeta(A) &= \overline{\zeta(-A)} \end{aligned}$$

LUND UNIVERSITY

Spectral theorem

- Take $f = \mathbf{1}_A$ and $Y = \zeta(A)$ be corresponding random variable

$$Y = \int \mathbf{1}_A(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda}) \overset{isometry}{\longleftrightarrow} \mathbf{1}_A$$

- $\zeta(\boldsymbol{\lambda}) \overset{def}{=} \zeta(-\infty, \boldsymbol{\lambda}]$.
- $\zeta(\boldsymbol{\lambda})$ – complex, zero mean, orthogonal (uncorrelated) increments,

$$\begin{aligned} E(|\zeta(\boldsymbol{\lambda})|^2) &= \sigma(-\infty, \boldsymbol{\lambda}] \\ \zeta(A) &= \overline{\zeta(-A)} \end{aligned}$$

- Spectral representation

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}) \overset{isometry}{\longleftrightarrow} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})$$

Spectral theorem through discretization – simulation

Spectral theorem through discretization – simulation

- $$X(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda_+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j),$$

Spectral theorem through discretization – simulation

- $$X(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda_+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j),$$
- $\sigma^2(\boldsymbol{\lambda}_j)$ is proportional to the increments of the spectral measure σ over the grid cell $\Delta\boldsymbol{\lambda}_j$ corresponding to $\boldsymbol{\lambda}_j$,

LUND UNIVERSITY

Spectral theorem through discretization – simulation

- $$X(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda_+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j),$$
- $\sigma^2(\boldsymbol{\lambda}_j)$ is proportional to the increments of the spectral measure σ over the grid cell $\Delta\boldsymbol{\lambda}_j$ corresponding to $\boldsymbol{\lambda}_j$,
- R_j is a zero mean random variable uncorrelated with other R_i's,

$$R_j = \sigma(\Delta\boldsymbol{\lambda}_j)/\sqrt{\mathbb{V}ar(\sigma(\Delta\boldsymbol{\lambda}_j))}.$$

LUND UNIVERSITY

Spectral theorem through discretization – simulation

- $$X(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda_+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j),$$

- $\sigma^2(\boldsymbol{\lambda}_j)$ is proportional to the increments of the spectral measure σ over the grid cell $\Delta\boldsymbol{\lambda}_j$ corresponding to $\boldsymbol{\lambda}_j$,
- R_j is a zero mean random variable uncorrelated with other R_i's,

$$R_j = \sigma(\Delta\boldsymbol{\lambda}_j)/\sqrt{\mathbb{V}ar(\sigma(\Delta\boldsymbol{\lambda}_j))}.$$

- If R_j's are easy to obtain it can serve as a convenient method of simulation over the entire $\mathbb{R}^d$.

Outline

Strictly stationary second order process

Our interest is in the strictly stationary processes, i.e. for each $k \in \mathbb{N}$, $A_j \subset \mathbb{R}$, $\mathbf{t}_j, \mathbf{s} \in \mathbb{R}^d$:

$$\mathbb{P}(X(\mathbf{t}_1) \in A_1, \dots, X(\mathbf{t}_k) \in A_k) = \mathbb{P}(X(\mathbf{t}_1 + \mathbf{s}) \in A_1, \dots, X(\mathbf{t}_k + \mathbf{s}) \in A_k)$$

Strictly stationary second order process

Our interest is in the strictly stationary processes, i.e. for each $k \in \mathbb{N}$, $A_j \subset \mathbb{R}$, $\mathbf{t}_j, \mathbf{s} \in \mathbb{R}^d$:

$$\mathbb{P}(X(\mathbf{t}_1) \in A_1, \ldots, X(\mathbf{t}_k) \in A_k) = \mathbb{P}(X(\mathbf{t}_1 + \mathbf{s}) \in A_1, \ldots, X(\mathbf{t}_k + \mathbf{s}) \in A_k)$$

We are still interested in the second order processes so the covariance function is well defined.

Strictly stationary second order process

Our interest is in the strictly stationary processes, i.e. for each $k \in \mathbb{N}$, $A_j \subset \mathbb{R}$, $\mathbf{t}_j, \mathbf{s} \in \mathbb{R}^d$:

$$\mathbb{P}(X(\mathbf{t}_1) \in A_1, \dots, X(\mathbf{t}_k) \in A_k) = \mathbb{P}(X(\mathbf{t}_1 + \mathbf{s}) \in A_1, \dots, X(\mathbf{t}_k + \mathbf{s}) \in A_k)$$

We are still interested in the second order processes so the covariance function is well defined.
Are there any interesting processes of this sort?

Building using stochastic integration

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

Building using stochastic integration

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

Building using stochastic integration

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

- By a proper choice of ζ one can get ‘any’ covariance.

LUND UNIVERSITY

Building using stochastic integration

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

- By a proper choice of ζ one can get 'any' covariance.
- Moving average processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \boldsymbol{\lambda})\, d\zeta(\boldsymbol{\lambda}).$$

Building using stochastic integration

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

- By a proper choice of ζ one can get ‘any’ covariance.
- Moving average processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \boldsymbol{\lambda})\, d\zeta(\boldsymbol{\lambda}).$$

- By a proper choice of the kernel f one can get ‘any’ covariance.

Spatial moving avarges

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

Spatial moving avarges

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

Spatial moving avarges

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

- By a proper choice of ζ one can get ‘any’ covariance.

Spatial moving avarges

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

- By a proper choice of ζ one can get 'any' covariance.
- Moving average processes

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \boldsymbol{\lambda})\, d\zeta(\boldsymbol{\lambda}).$$

Spatial moving avarges

Let ζ be a stochastic measure build upon some infinitely divisible distribution with finite second moments.

- Harmonizable processes

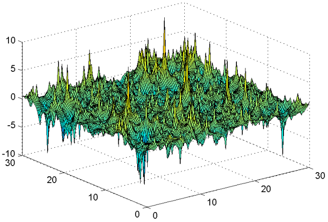
$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\lambda}),$$

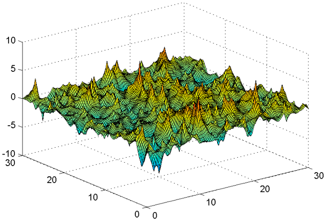
- By a proper choice of ζ one can get 'any' covariance.
- Moving average processes

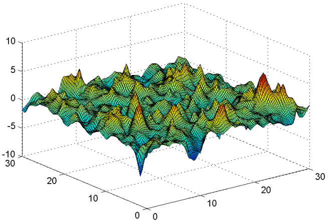
$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \boldsymbol{\lambda})\, d\zeta(\boldsymbol{\lambda}).$$

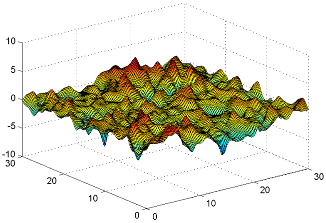
- By a proper choice of the kernel f one can get 'any' covariance.

Symmetric spatial models - realizations

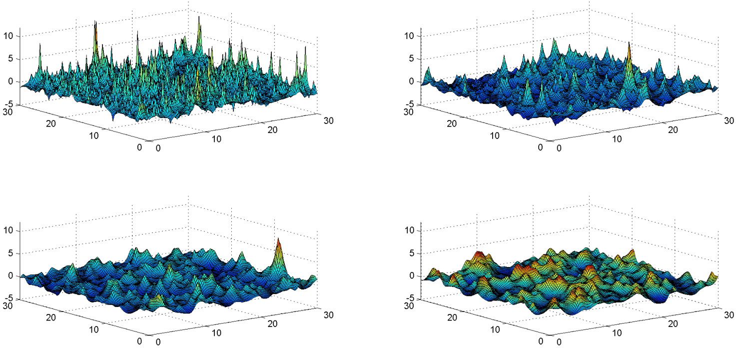




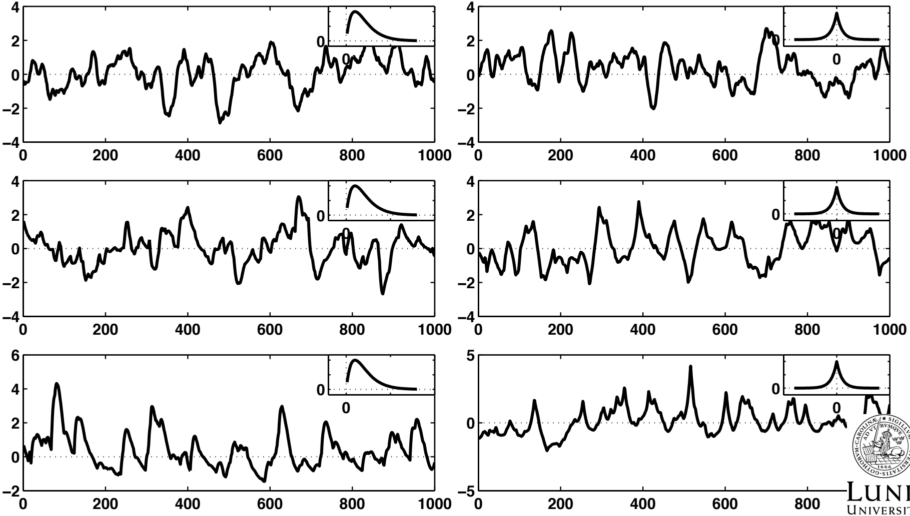




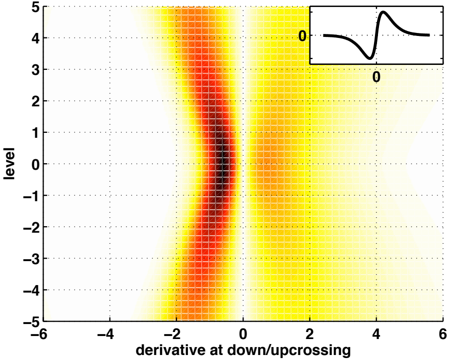
Asymmetric spatial models

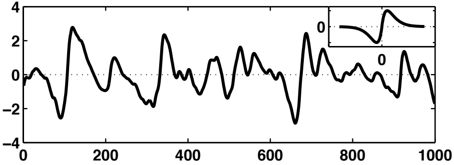


Tilting of trajectories



Another way of tilting

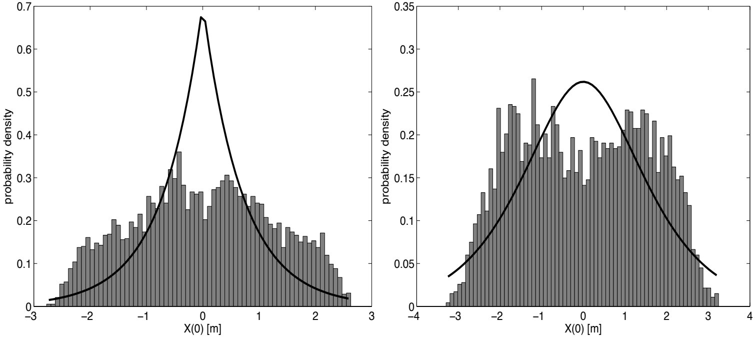




Sampling distributions

Sampling distributions

- The sampling distribution compared with the marginal



Outline

Asymmetric Laplace distribution

Asymmetric Laplace distribution

- An interesting class of processes is obtained by considering an infinitely divisible distribution of

$$\sigma\sqrt{W}Z + \mu W + \delta,$$

where W is a standard exponential independent of a standard normal Z (can be multidimensional).

Asymmetric Laplace distribution

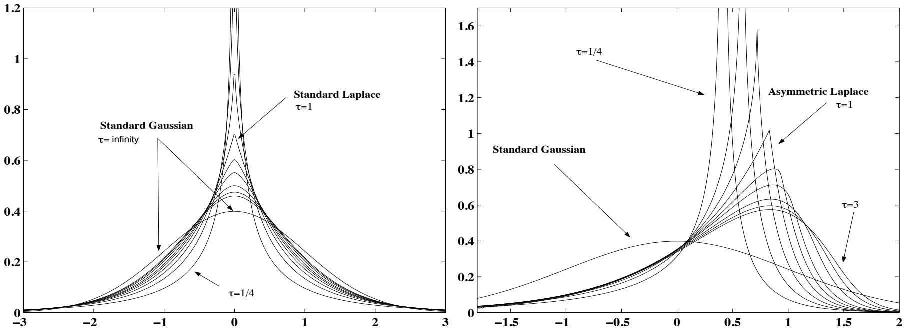
- An interesting class of processes is obtained by considering an infinitely divisible distribution of

$$\sigma\sqrt{W}Z + \mu W + \delta,$$

where W is a standard exponential independent of a standard normal Z (can be multidimensional).

- Complete infinitely divisible convolution group of distributions is obtained by taking for W a gamma random variable.

Examples of the densities



Outline

Laplace moving average

- By the means of stochastic integral we define

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x})\, d\Lambda(\mathbf{x}).$$

Laplace moving average

- By the means of stochastic integral we define

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x}) d\Lambda(\mathbf{x}).$$

- $\Lambda(A)$ has the generalized asymmetric Laplace distribution

$$\phi(t) = \frac{1}{\left(1 - i\mu t + \frac{\sigma^2}{2} t^2\right)^{\lambda(A)}},$$

where λ is the Lebesgue measure in $\mathbb{R}^d$.

Laplace moving average

- By the means of stochastic integral we define

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x})d\Lambda(\mathbf{x}).$$

- $\Lambda(A)$ has the generalized asymmetric Laplace distribution

$$\phi(t) = \frac{1}{\left(1 - i\mu t + \frac{\sigma^2}{2}t^2\right)^{\lambda(A)}},$$

  where λ is the Lebesgue measure in $\mathbb{R}^d$.
- If $d = 1$, then $\Lambda(-\infty, x] = B(\Gamma(x))$, where B is a Brownian motion with drift and Γ is a gamma process.

Laplace moving average

- By the means of stochastic integral we define

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x}) d\Lambda(\mathbf{x}).$$

- $\Lambda(A)$ has the generalized asymmetric Laplace distribution

$$\phi(t) = \frac{1}{\left(1 - i\mu t + \frac{\sigma^2}{2} t^2\right)^{\lambda(A)}},$$

  where λ is the Lebesgue measure in $\mathbb{R}^d$.
- If $d = 1$, then $\Lambda(-\infty, x] = B(\Gamma(x))$, where B is a Brownian motion with drift and Γ is a gamma process.
- Conditionally on Γ the process X_t can be viewed as a non-stationary Gaussian process.

Spectral representation for LMA

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\omega}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\omega})$$

Spectral representation for LMA

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\omega}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x})\, d\Lambda(\mathbf{x})$$

Spectral representation for LMA

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\omega}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x})\, d\Lambda(\mathbf{x})$$

- ζ is a stochastic measure with uncorrelated increments

Spectral representation for LMA

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\omega}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x})\, d\Lambda(\mathbf{x})$$

- ζ is a stochastic measure with uncorrelated increments
- Λ is a stochastic measure with independent increments

Spectral representation for LMA

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\omega}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x})\, d\Lambda(\mathbf{x})$$

- ζ is a stochastic measure with uncorrelated increments
- Λ is a stochastic measure with independent increments
- Problem: Express ζ in the terms of Λ and f and find its properties (distribution, dependence structure, etc.)

Spectral representation for LMA

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^d} \exp(i\boldsymbol{\omega}^T\boldsymbol{\tau})\, d\zeta(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{\tau} - \mathbf{x})\, d\Lambda(\mathbf{x})$$

- ζ is a stochastic measure with uncorrelated increments
- Λ is a stochastic measure with independent increments
- Problem: Express ζ in the terms of Λ and f and find its properties (distribution, dependence structure, etc.)
- One possible application is getting efficiently R_j's in:

$$X(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda_+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j),$$

LUND UNIVERSITY

An uneducated and wild guess

An uneducated and wild guess

- Let $\phi(\omega)$ be an inverse Fourier transform of $f(x)$.

$$
\begin{aligned}
X(\tau) = \int_{-\infty}^{\infty} f(\tau - x)\, d\Lambda(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\tau - x)\omega} \phi(\omega)\, d\omega \\
&= \int_{-\infty}^{\infty} e^{-ix\omega}\, d\Lambda(x) \phi(\omega)\, d\omega.
\end{aligned}
$$

An uneducated and wild guess

- Let $\phi(\omega)$ be an inverse Fourier transform of $f(x)$.

$$X(\tau) = \int_{-\infty}^{\infty} f(\tau - x)\, d\Lambda(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tau-x)\omega}\phi(\omega)\, d\omega$$
$$= \int_{-\infty}^{\infty} e^{-ix\omega}\, d\Lambda(x)\phi(\omega)\, d\omega.$$

- So a good candidate for spectral measure ζ:

$$\zeta(a,b] = \int_a^b Y(\omega)\phi(\omega)\, d\omega,$$

where

$$Y(\omega) = \int_{-\infty}^{\infty} e^{-ix\omega}\, d\Lambda(x).$$

LUND UNIVERSITY

An uneducated and wild guess

- Let $\phi(\omega)$ be an inverse Fourier transform of $f(x)$.

$$X(\tau) = \int_{-\infty}^{\infty} f(\tau - x)\, d\Lambda(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\tau - x)\omega} \phi(\omega)\, d\omega$$
$$= \int_{-\infty}^{\infty} e^{-ix\omega}\, d\Lambda(x) \phi(\omega)\, d\omega.$$

- So a good candidate for spectral measure ζ:

$$\zeta(a, b] = \int_a^b Y(\omega)\phi(\omega)\, d\omega,$$

where

$$Y(\omega) = \int_{-\infty}^{\infty} e^{-ix\omega}\, d\Lambda(x).$$

- But $Y(\omega)$ is not defined in a proper way...

Outline

Ergodic theorem for strictly stationary processes

- It is a classical result of Birkhoff (also Wiener) saying that for each function f defined on a strictly stationary process $X(\tau)$ such that $\mathbb{E}f(X(\tau))$ is finite

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(\tau+s))\,ds = \mathbb{E}f(X(\tau)|\mathcal{J}),$$

where $\mathcal{J}$ is the sigma field of shift invariant sets defined on trajectories of $X(\tau)$.

Ergodic theorem for strictly stationary processes

- It is a classical result of Birkhoff (also Wiener) saying that for each function f defined on a strictly stationary process $X(\tau)$ such that $\mathbb{E}f(X(\tau))$ is finite

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(\tau+s))\,ds = \mathbb{E}f(X(\tau)|\mathcal{J}),$$

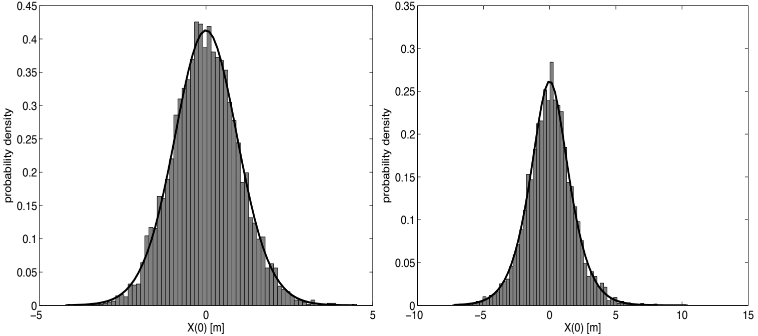
  where $\mathcal{J}$ is the sigma field of shift invariant sets defined on trajectories of $X(\tau)$.
- If $\mathcal{J}$ is made of sets of null or full measures, then process is ergodic, convergence is to the mean, and process is called ergodic.

General ergodic properties of infinitely divisible processes

- For general infinitely divisible processes, their properties were well studied, see for example Cambanis, Podgórski, Weron (1995).

General ergodic properties of infinitely divisible processes

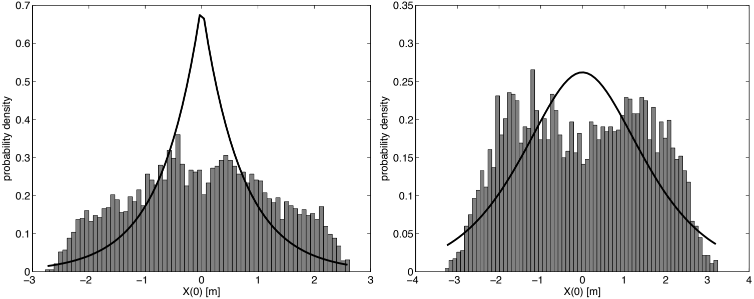
- For general infinitely divisible processes, their properties were well studied, see for example Cambanis, Podgórski, Weron (1995).
- Moving averages are ergodic (they are even mixing), in particular LMA are ergodic:



Invariant sets for harmonizable processes – Open Problem 2

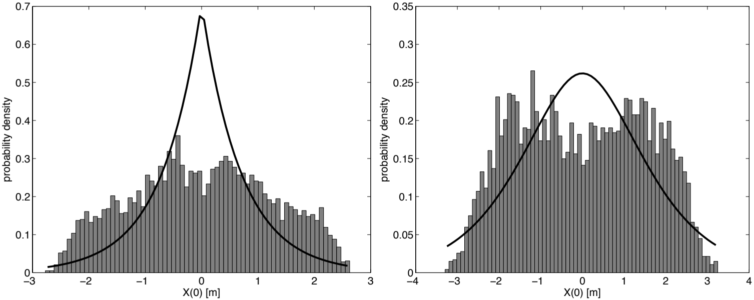
Invariant sets for harmonizable processes – Open Problem 2

- Harmonizable process are not ergodic, in particular Laplace driven harmonizable processes are not ergodic:



Invariant sets for harmonizable processes – Open Problem 2

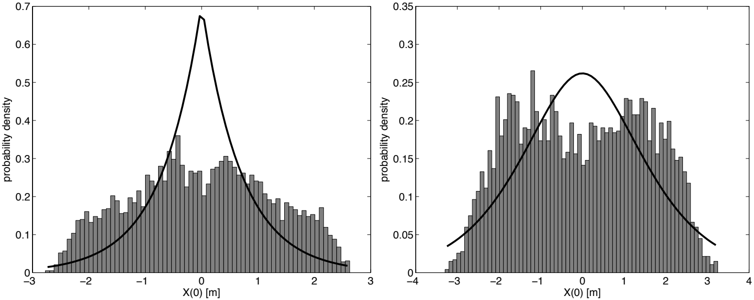
- Harmonizable process are not ergodic, in particular Laplace driven harmonizable processes are not ergodic:



- However due to the ergodic theorem their sample means are convergent to a random variable (conditional expectation).

Invariant sets for harmonizable processes – Open Problem 2

- Harmonizable process are not ergodic, in particular Laplace driven harmonizable processes are not ergodic:



- However due to the ergodic theorem their sample means are convergent to a random variable (conditional expectation).
- Problem: What is the distribution of this random variable or, equivalenty, what is the σ-field of shift invariant sets?

LUND UNIVERSITY

An educated but still wild guess

An educated but still wild guess

- Laplace motion is a pure jump process.

An educated but still wild guess

- Laplace motion is a pure jump process.
- Consider $\mathcal{J}$ made of the events on a Laplace motion that are invariant on the change of order of absolute values of jumps.

An educated but still wild guess

- Laplace motion is a pure jump process.
- Consider $\mathcal{J}$ made of the events on a Laplace motion that are invariant on the change of order of absolute values of jumps.
- This σ-field is shift invariant.

An educated but still wild guess

- Laplace motion is a pure jump process.
- Consider $\mathcal{J}$ made of the events on a Laplace motion that are invariant on the change of order of absolute values of jumps.
- This σ-field is shift invariant.
- Wright (1976) has shown that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X^2(\tau + s)\, ds = \sum_{i=0}^{\infty} J_i^2,$$

An educated but still wild guess

- Laplace motion is a pure jump process.
- Consider $\mathcal{J}$ made of the events on a Laplace motion that are invariant on the change of order of absolute values of jumps.
- This σ-field is shift invariant.
- Wright (1976) has shown that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X^2(\tau+s)\,ds = \sum_{i=0}^{\infty} J_i^2,$$

- Why to care beyond being interesting mathematical problem?

An educated but still wild guess

- Laplace motion is a pure jump process.
- Consider $\mathcal{J}$ made of the events on a Laplace motion that are invariant on the change of order of absolute values of jumps.
- This σ-field is shift invariant.
- Wright (1976) has shown that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X^2(\tau+s)\,ds = \sum_{i=0}^{\infty} J_i^2,$$

- Why to care beyond being interesting mathematical problem? Ans. Statistical inference based on conditional distribution in assessing sample to sample variation.

Why unsolved?

Why unsolved?

“Theorems are fun especially when you are the prover, but then the pleasure fades. What keeps us going are the unsolved problems.”