# Stochastic Fields with Shallow Water Flow Dynamics

Jörg Wegener

Mathematical Statistics, Lund University, Sweden

Smögen, 20$^{th}$ August 2010

# Outline

- Deterministic Model
  - Physical motivation
  - Deterministic flow
- Stochastic Model
  - Covariance structures
  - 'Static' flow
- Stochastic-Deterministic Model
  - Embedding deterministic flow
  - Dynamic flow

# Outline

'static' stochastic flow:
Covariance structures in space & time

deterministic velocities:
numerical solution of SWEs

dynamic flow:
stochastic fields with subordinated SWE dynamics

# EXISTENCE AND SMOOTHNESS OF THE NAVIER–STOKES EQUATION

CHARLES L. FEFFERMAN

The Euler and Navier–Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or $3$). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1\le i\le n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \ge 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The *Navier–Stokes* equations are then given by

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n, t \ge 0), \tag{1}$$

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n, t \ge 0) \tag{2}$$

with initial conditions

$$u(x,0) = u^\circ(x) \qquad (x \in \mathbb{R}^n). \tag{3}$$

Here, $u^\circ(x)$ is a given, $C^\infty$ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given, externally applied force (e.g. gravity), $\nu$ is a positive coefficient (the viscosity), and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables. The *Euler equations* are equations (1), (2), (3) with $\nu$ set equal to zero.

**(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$.**

**(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$.**

**(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$.**

**(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$.**

Fluids are important and hard to understand. There are many fascinating problems and conjectures about the behavior of solutions of the Euler and Navier–Stokes equations. (See, for instance, Bertozzi–Majda [1] or Constantin [3].) Since we don't even know whether these solutions exist, our understanding is at a very primitive level. Standard methods from PDE appear inadequate to settle the problem. Instead, we probably need some deep, new ideas.

# Equations of motion

Conservation of momentum

- Fluids:
  *Navier-Stokes* equation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \rho \nabla \Phi + \nabla \cdot \mathbb{T}$$

- Solid bodies:
  *Newton's* second law

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F}$$

Note:

- all thermodynamic processes excluded

# Shallow Water Approximation

$$w = -h\nabla \cdot \mathbf{v}_h \ ,$$

where

height of surface
$v_h$ horizontal velocity
$w$ vertical velocity

# Shallow Water Equations

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + fv$$
$$\frac{\partial v}{\partial t} = -g\frac{\partial h}{\partial y} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - fu$$
$$\frac{\partial h}{\partial t} = -\frac{\partial (hu)}{\partial x} - \frac{\partial (hv)}{\partial y}$$

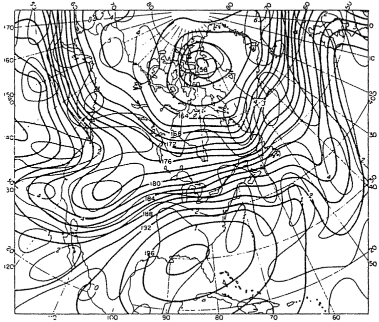
- Used to model: Tsunamis, flows in rivers, internal waves, Jupiter's Atmopshere

Reprinted from *Tellus*, Vol. 2, 1950, pp 237–254

# Numerical Integration of the Barotropic Vorticity Equation

By J. G. CHARNEY, R. FJÖRTOFT[1], J. von NEUMANN
The Institute for Advanced Study, Princeton, New Jersey[2]

the Eniac. It may be of interest to remark that the computation time for a 24-hour forecast was about 24 hours, that is, we were just able to keep pace with the weather. However, much



# Linear Shallow Water Equations

After linearization and reduction to one dimension.

- PDE with constant coefficients:

$$\frac{\partial u}{\partial t} = -u_0 \frac{\partial u}{\partial x} - g \frac{\partial h}{\partial x}$$
$$\frac{\partial h}{\partial t} = -u_0 \frac{\partial h}{\partial x} - h_0 \frac{\partial u}{\partial x} \ ,$$

where $u$ denotes horizontal velocity (positive towards East), $h$ surface elevation, $u_0, h_0, g$ constants.
One wave equation in $u$.

# Theoretical Solution

- Solution in $u$:

$$u(x,t) = \pm A\sqrt{\frac{g}{h_0}}\cos(k(x-ct)) \ ,$$

where $A$ denotes amplitude, $k$ wave number and $c$ phase velocity:

$$c = u_0 \pm \sqrt{gh_0} \ .$$

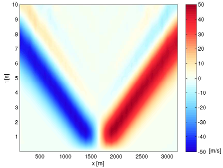



Figure: Positive (negative) velocities denote wavefronts travelling to the right (left). Notice: constant velocities along the characteristic lines $x = x_0 + (u_0 \pm \sqrt{gh_0})t$.

# Numerical Solution

Consider $\alpha(i\Delta, n\Delta t) \stackrel{def}{=} \alpha_i^n$ on a tempo-spatial grid. Finite difference approximation of

- spatial derivatives

$$\left(\frac{\partial\alpha}{\partial x}\right)_i^n \approx \frac{\alpha_{i+\frac{1}{2}}^n - \alpha_{i-\frac{1}{2}}^n}{\Delta x} \stackrel{def}{=} (\delta_x\alpha)_i^n$$

- temporal derivative

$$\left(\frac{\partial\alpha}{\partial t}\right)_i^n \approx \frac{\alpha_i^{n+1} - \alpha_i^{n-1}}{2\Delta t}$$

# Numerical Solution

Finite difference formulation of SWE

$$u_{i+\frac{1}{2}}^{n+1} = u_{i+\frac{1}{2}}^{n-1} - 2\Delta t \left( u_0 (\overline{\delta_x u})_{i+\frac{1}{2}}^{n} + g(\delta_x h)_{i+\frac{1}{2}}^{n} \right)$$
$$h_{i+1}^{n+1} = h_{i+1}^{n-1} - 2\Delta t \left( u_0 (\overline{\delta_x h})_{i+1}^{n} + h_0(\delta_x u)_{i+1}^{n} \right)$$

Consistency, accuracy and stability can be determined.

# 1-dim SWE simulation

particleWave.mpeg

# Outline

'static' stochastic flow:
Covariance structures in space & time

deterministic velocities:
numerical solution of SWEs

dynamic flow:
stochastic fields with subordinated SWE dynamics

# Outline

'static' stochastic flow:
Covariance structures in space & time

deterministic velocities:
numerical solution of SWEs

dynamic flow:
stochastic fields with subordinated SWE dynamics

# Stochastic Model

- 'Static' Gaussian field:

$$X(x,t) = \int_{-\infty}^{t} f(t-s)\Phi(x;ds)$$

- Temporal dependence
  *Ornstein-Uhlenbeck* kernel function:

$$f(s) \propto e^{\lambda s} 1_{(-\infty,0]}(s)$$

- Spatial dependence
  Correlation function:

$$r_\Phi(x) = e^{-\frac{x^2}{2\sigma^2}}$$

- Space-time stationary covariance

$$r_X(x,t) = e^{-x^2/2\sigma^2} e^{-\lambda|t|}$$

# Discrete version

- Autoregressive formulation of $X^*$

$$\Delta X = (\rho - 1)X + \sqrt{2(1-\rho^2)\lambda}\,\Phi\ ,$$

where

$$\begin{aligned}
\Delta X &= X(x, t + \Delta t) - X(x, t) \\
\rho &= \exp(-\lambda \Delta t) \\
X &= X(x, t) \\
\Phi &= \Phi_t(x) \text{ innovations with } r_\Phi(x) = e^{-\frac{x^2}{2\sigma^2}}
\end{aligned}$$

*Baxevani, A. and Podgórski, K. and Rychlik, I. (2010), Dynamically evolving Gaussian spatial fields, *Extremes*.

- $\Phi_t(x)$ is generated through its 'spectral representation'

$$\Phi(x) = \sqrt{d\omega} \sum_{j=-N}^{N} \sqrt{S(jd\omega)} R_j \cos(xjd\omega + \phi_j) \; ,$$

where

$S(jd\omega)$ is the discretized spectrum, corresponding to $r_\Phi(x)$
$R_j \sim Rayleigh(1)$, iid, and independent of
$\phi_j \sim U[0, 2\pi]$, iid

# 'Static' Fields I

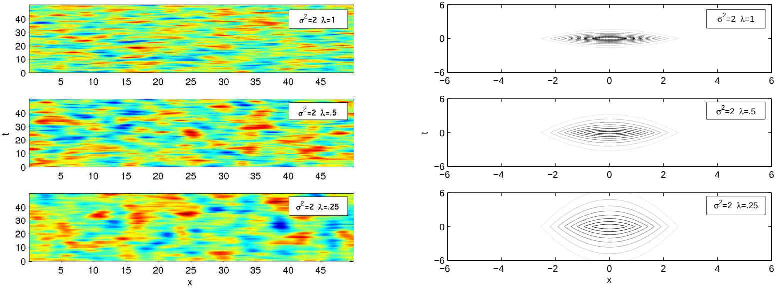



Figure: Spatio-temporal fields $X(x,t)$ for fixed $\sigma$ and $\lambda = .25, .5, 1$ (*left*). Respective tempo-spatial covariance functions (*right*).

# 'Static' Fields II

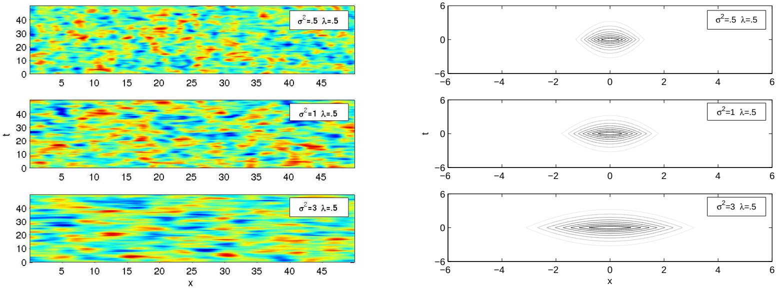



Figure: Spatio-temporal fields $X(x,t)$ for fixed $\lambda$ and $\sigma^2 = .5, 1, 3$ (*left*). Respective tempo-spatial covariance functions (*right*).

# Stochastic Velocities

- Velocity on random surfaces*:

$$v(x,t) = -\frac{X_t(x,t)}{X_x(x,t)} = -\frac{\frac{\partial X(x,t)}{\partial t}}{\frac{\partial X(x,t)}{\partial x}} \approx \frac{\partial x}{\partial t},$$

where

$$v \sim Cauchy(\cdot,\cdot)$$

*Baxevani, A. and Podgórski, K. and Rychlik, I. (2003). Velocities for moving random surfaces. *Probabilistic Engineering Mechanics.*

# Stochastic Velocities

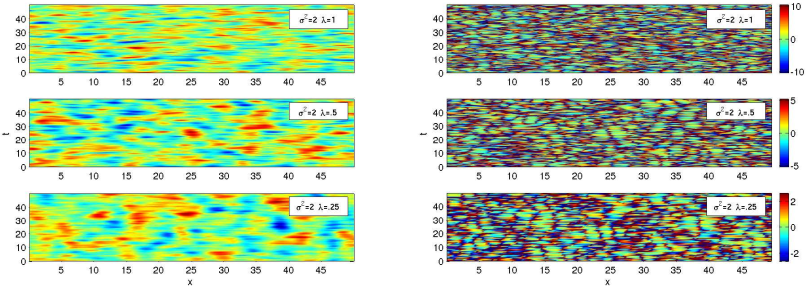



Figure: Spatio-temporal fields $X(x,t)$ (*left*). Respective velocities $v(x,t)$ (*right*). Velocities are truncated at 15% and 85% quantiles.

'No organized motion.'

# Outline

'static' stochastic flow:
Covariance structures in space & time

deterministic velocities:
numerical solution of SWEs

dynamic flow:
stochastic fields with subordinated SWE dynamics

# Outline

'static' stochastic flow:
Covariance structures in space & time

deterministic velocities:
numerical solution of SWEs

dynamic flow:
stochastic fields with subordinated SWE dynamics

# Stochastic-Deterministic Model

Embedding dynamic SWE-flow into random field $X(x, t)$

- Static field:

$$X(x,t) = \int_{-\infty}^{t} f(t-s)\Phi(x; ds)$$

- Deterministic flow $x$:

$$x_{t,h}(x) = x + \int_{t}^{t+h} u(x_{t,s-t}(x), s)ds$$

- Dynamic field:

$$Y(x,t) = \int_{-\infty}^{t} f(t-s)\Phi(x_{t,s-t}(x); ds)$$

# Stochastic-Deterministic Model

- Discretized dynamic field:

$$Y(x, t + \Delta t) \approx \rho Y(x_{t,-\Delta t}(x), t) + \sqrt{2(1-\rho^2)\lambda}\, \Phi_p(x, t)$$

'Static fields are transported according to flow $x$.'

# Deterministic Flow

- Approximate trajectory $x_{t,s-t}(x)$:

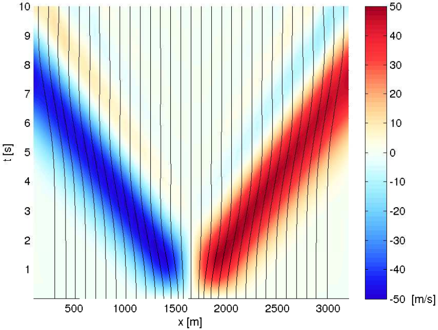
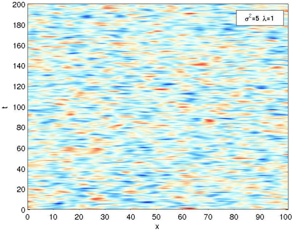



Figure: Approximate deterministic particle trajectories are given in black.

# Stochastic-Dynamic Fields



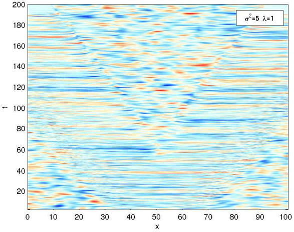




Figure: Stochastic fields for the static case (left) and with subordinated shallow water dynamics (right). Parameters are chosen to $\sigma^2 = 5$ and $\lambda = 1$.

# Stochastic-Dynamic Velocities

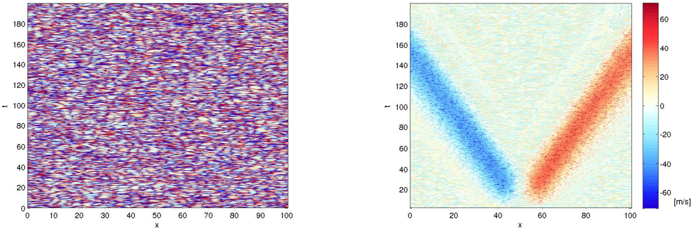



Figure: Corresponding stochastic velocities. Note that dynamic velocities accumulate along the *characteristic curves* $x = x_0 + (u_0 \pm \sqrt{gh_0})t$, whereas the static case lacks any ordered structure. Velocities are truncated.

# Comparison of velocities

- Center of random velocity*, $v_c(x,t)$:

$$\frac{\int_{-\infty}^{0} \left[ u(\mathrm{x}_{t,h}(x), t+h) - \frac{\partial \mathrm{x}_{t,h}(x)}{\partial t} \right] \mathrm{x}'_{t,h}(x) \left( \mathrm{x}^2_{t,h}(x) - \sigma^2 \right) e^{2\lambda h - \frac{\mathrm{x}^2_{t,h}(x)}{2\sigma^2}} dh}{\int_{-\infty}^{0} \mathrm{x}'^2_{t,h}(x) \left( \mathrm{x}^2_{t,h}(x) - \sigma^2 \right) e^{2\lambda h - \frac{\mathrm{x}^2_{t,h}(x)}{2\sigma^2}} dh},$$

where

$$\mathrm{x}'_{t,h}(x) = \frac{\partial \mathrm{x}_{t,h}}{\partial x}$$

*Baxevani, A. and Podgórski, K. and Rychlik, I. (2010), Dynamically evolving Gaussian spatial fields, *Extremes*.

# Theorem

*Assume a spatio-temporal stochastic model with the presented dependece structures. Then, the distribution of random velocity has its center at the deterministic flow velocity.*

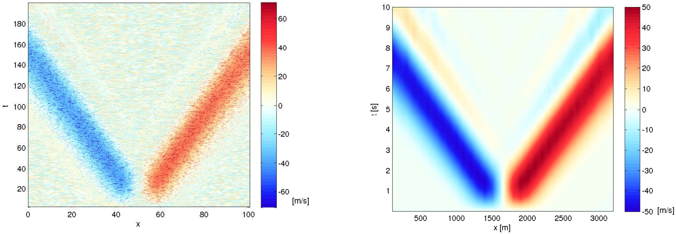



Figure: Comparison of stochastically destorted flow (*left*) to deterministic shallow water flow (*right*).

# Fazit

Remarks

- different covariance structures
- higher dimensions
- non-linear dynamics

Purposes

- prediction, inference
- simple, reducing numerical burden

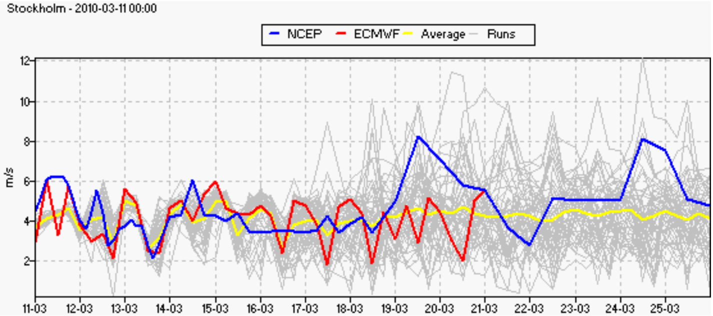



Figure: Ensemble prediction of wind (magnitude).