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ADAPTIVE VARIATIONAL MULTISCALE METHODS

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Summary. This note provides a condensed introduction to the adaptive variational method for an elliptic model problem. Key features of the method include a novel and systematic technique for approximating the fine scales using decoupled localized subgrid problems and adaptive algorithms based on a posteriori error estimates.

1 INTRODUCTION

The adaptive variational multiscale [4,5,6,7] (AVMS) method is a novel multiscale method that builds on the combination of:

- The variational multiscale framework [2].
- A systematic technique for numerical approximation of the fine scale part of the solution based on solving localized subgrid problems on patches.
- A posteriori error estimates and adaptive algorithms that provide control of numerical error as well as automatic tuning of critical discretization parameters.

In this note we give a condensed presentation of the AVMS method and the energy norm a posteriori error estimates for an elliptic model problem with multiscale features in the conductivity, see [6] for further details and numerical examples. A mixed version of the method is presented in [5] and transport dominated problems are considered in [7].

2 THE VARIATIONAL MULTISCALE METHOD

We shall study the following simple model problem: find u such that

$$-\nabla \cdot a\nabla u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma, \tag{1}$$

where Ω is a polygonal domain in $\mathbf{R}^d$, $d = 1, 2$, or 3 with boundary Γ, $f \in L^2(\Omega)$, and $a \in L^\infty(\Omega)$ satisfies $a(x) \geq a_0 > 0$ for all $x \in \Omega$ is a conductivity coefficient with multiscale features.

The variational form of (1) reads: find $u \in \mathcal{V} = H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \text{for all } v \in \mathcal{V}, \tag{2}$$

with the bilinear form $a(u,v) = (a\nabla u, \nabla v)$, for all $u, v \in \mathcal{V}$.

The Variational Multiscale method [1,2] (VMS) is an important framework for constructing multiscale methods. The idea is to decompose the solution into fine $u_f \in \mathcal{V}_f$ and coarse $u_c \in \mathcal{V}_c$ scale contributions as follows

$$\begin{aligned} a(u_c,v_c) + a(u_f,v_c) &= (f,v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(u_f,v_f) = (f,v_f) - a(u_c,v_f) &=: (R(u_c),v_f) \quad \text{for all } v_f \in \mathcal{V}_f. \end{aligned} \tag{3}$$

The fine scale equation are solved in terms of the coarse scale residual $R(u_c)$, and finally eliminate the fine scale solution from the coarse scale equation. This procedure leads to the modified coarse scale equation (4) where the modification accounts for the effect of fine scale behavior on the coarse scales.

$$a(u_c,v_c) + a(\mathcal{T}R(u_c),v_c) = (f,v_c) \quad \text{for all } v_c \in \mathcal{V}_c. \tag{4}$$

Here $\mathcal{T}$ represents an approximate solution operator of the fine scale problem. In several works various ways of analytical modeling of $\mathcal{T}$ are investigated often based on bubbles or element Green's functions [1].

3 APPROXIMATION OF FINE SCALES

In the adaptive variational multiscale method [4,5,6,7] the fine scale equations of equation (3) are decoupled and solved numerically on patches. The idea is to decouple the fine scale equations by including a partition of unity in the right hand side of the fine scale part of equation (3) and then to solve the resulting problems on patches.

We introduce a partition $\mathcal{K} = \{K\}$ of the domain Ω into coarse shape regular elements K of diameter H and we let $\mathcal{N}$ be the set of coarse nodes. Further we let $\mathcal{V}_c$ be the space of continuous piecewise polynomials of one defined on $\mathcal{K}$. We let $u_f = \sum_{i\in\mathcal{N}} u_{f,i}$ where

$$a(u_{f,i},v_f) = (\varphi_i R(u_c),v_f) \quad \text{for all } v_f \in \mathcal{V}_f, \tag{5}$$

and $\{\varphi_i\}_{i\in\mathcal{N}}$ is a partition of unity e. g. the set of Lagrange basis functions in $\mathcal{V}_c$, be the solution to the decoupled fine scale equations.

We introduce this expansion of u_f in the right hand side of the fine scale equation (3) and get: find $u_c \in \mathcal{V}_c$ and $u_f = \sum_{i\in\mathcal{N}} u_{f,i} \in \mathcal{V}_f$ such that

$$\begin{aligned} a(u_c,v_c) + a(u_f,v_c) &= (f,v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(u_{f,i},v_f) &= (\varphi_i R(u_c),v_f) \quad \text{for all } v_f \in \mathcal{V}_f \text{ and } i \in \mathcal{N}. \end{aligned} \tag{6}$$

The next step is to solve the fine scale equations approximately. For each element in the partition of unity we associate a domain ω_i on which we solve Dirichlet problems. We often use coarse mesh stars of many layers as local domains. By adding a layer we mean adding all coarse elements bordering the star. The local domain ω_i contains the support of the element in the partition of unity and is large enough to give a good approximate solution. The quality of the solution is controlled by error estimates. We now define the local finite element space $\mathcal{V}_f^h(\omega_i)$ associated with node i. We refine the coarse mesh on the patch ω_i and let $\mathcal{V}_f^h(\omega_i)$ be the fine part of the hierarchical basis on this mesh.

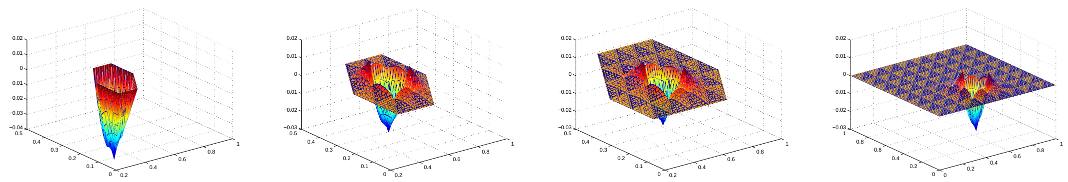

Figure 1: A typical localized solution $U_{f,i}$ of the fine scale equations in a smooth region using one, two, three layer stars, and the entire domain.

The resulting method reads: find $U_c \in \mathcal{V}_c$ and $U_f = \sum_i^n U_{f,i}$ where $U_{f,i} \in \mathcal{V}_f^h(\omega_i)$ such that

$$\begin{aligned} a(U_c, v_c) + a(U_f, v_c) &= (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(U_{f,i}, v_f) &= (\varphi_i R(U_c), v_f) \quad \text{for all } v_f \in \mathcal{V}_f^h(\omega_i) \text{ and } i \in \mathcal{N}. \end{aligned} \tag{7}$$

Since the functions in the local finite element spaces $\mathcal{V}_f^h(\omega_i)$ are equal to zero on $\partial\omega_i$, U_f and therefore U will be continuous. If we just have fine scale features on part of the domain we only solve local problems for these areas. We denote coarse nodes in these areas $\mathcal{F}$ and the rest $\mathcal{C}$. If we write the method in matrix form we would get,

$$(A + T)U_c = b - d, \tag{8}$$

where A and b are the standard finite element stiffness matrix and load vector and the T matrix and d vector arises in analogy with equation (4) since $\mathcal{T}(R(U_c))$ is affine in U_c.

To get an idea of how the localized solution $U_{f,i}$ behaves when the domain ω_i increases we plot different solutions $U_{f,i}$ in Figure 1. Since $U_{f,i}$ is solved in the slice space $\mathcal{V}_f$ and since the right hand side of the fine scale equations of (7) has the same support as φ_i, $U_{f,i}$ will decay rapidly towards the boundary of ω_i, this can also be seen in Figure 1. We can see that one layer stars appears to give bad accuracy while two and more layer stars captures the features of the correct solution.

4 A POSTERIORI ERROR ESTIMATES

In [6] we present the following a posteriori error estimate for the adaptive variational multiscale method in the energy norm $\|e\|_a^2 = a(e, e)$.

Theorem 4.1 *It holds,*

$$\begin{aligned} \|e\|_a^2 \leq{} & C \sum_{i \in \mathcal{C}} \|H\mathcal{R}(U_c)\|_{\omega_i}^2 \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)}^2 \\ & + C \sum_{i \in \mathcal{F}} \left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i \setminus \Gamma}^2 + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2 \right) \|\frac{1}{\sqrt{a}}\|_{L^\infty(\omega_i)}^2, \end{aligned} \tag{9}$$

where

$$(-\Sigma(U_{f,i}), v_f)_{\partial\omega_i} = (\varphi_i R(U_c), v_f)_{\omega_i} - a(U_{f,i}, v_f)_{\omega_i}, \quad \textit{for all } v_f \in V_f^h(\bar{\omega}_i). \tag{10}$$

Here $\mathcal{R}(U_c)$ and $\mathcal{R}_i(U_{f,i})$ are bounds of the coarse and fine scale residual and $\Sigma(U_{f,i})$ is a variational approximation of the $L^2(\partial\omega_i)$ of $\partial_n U_{f,i}$. We can easily understand the contributions to the error. If no fine scale equations are solved we obtain the first term in the estimate; the first part of the second sum measures the effect of restriction to patches; and finally the second part measures the influence of the fine scale mesh parameter h. Using these indicators one may construct an adaptive algorithm for automatic tuning of the size of the patches and the subgrid resolution, see [6] for details.

For the case of periodic oscillations in $a = a(x/\epsilon)$ we get,

Theorem 4.2 *It holds*

$$\|e\|_a^2 \leq C\left(\frac{h}{\epsilon}\right)^2 \|f\|^2 + C\sum_{K\in\mathcal{K}} \|\sqrt{H}\Sigma(U_{f,i})\|^2_{\partial\omega_i\setminus\Gamma} \|\frac{1}{\sqrt{a}}\|^2_{L^\infty(\omega_i)}. \tag{11}$$

Here local problems are solved for all nodes since all areas are equally hard to resolve. Again we see clearly that $\|\Sigma(U_{f,i})\|_{\partial\omega_i}$ which depends on the number of layers and the fine scale mesh size h needs to be balanced. The coefficient a is periodic so we just need to solve a few number of localized problems since the correction matrix for the coarse scale computations will be identical for most patches.

In [4] we also present an error estimate of the adaptive variational multiscale method for a linear function of the error.

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