Lecture 1

Introduction to FEM

Chapter 5, (see also Larsson, Thomee)

Linear elliptic boundary value problem

$$-\nabla\cdot(a\nabla u) = f \quad \text{in } \Omega \qquad ✡$$
$$u = 0 \quad \text{on } \partial\Omega$$

$a(x) \geq a_0 > 0$, $\forall x \in \Omega$

$\Omega \subset \mathbb{R}^d$, polygonal domain.

Weak form:

We multiply by a test function $\varphi \in C_0^\infty(\Omega)$ and integrate over Ω.

$$\int_\Omega f\cdot\varphi\,dx = \int_\Omega -\nabla\cdot a\nabla u\cdot\varphi\,dx =$$

$$= \{\text{Green's formula}\} = \underbrace{\int_\Omega a\nabla u\cdot\nabla\varphi\,dx}_{a(u,\varphi)}$$

Weak formulation:

Find $u \in H_0^1(\Omega)$ such that

$$a(u,\varphi) = (f,\varphi) \quad \forall\, \varphi \in H_0^1(\Omega)$$

where $(f,g) = \int_\Omega f\cdot g\,dx$, $L^2(\Omega)$-scalar product. $\|f\| = \left(\int_\Omega f^2\,dx\right)^{1/2}$.

We let the H^1 norm and space

$$\|f\|_{H^1(\Omega)} = \left(\|f\|^2 + \|\nabla f\|^2\right)^{1/2}$$

$$H^1(\Omega) = \{ v \in L^2(\Omega) : \nabla v \in (L^2(\Omega))^d \}$$

$$H_0^1(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \} \quad ✡$$

The semi-norm on H^1 is defined as $|v|_{H^1} = \|\nabla v\|$ and is a norm on H_0^1 since ...

The Poincaré inequality gives $\|v\| \leq C\|\nabla v\|, \; \forall v \in H_0^1(\Omega)$ ✡

Therefore $|v|_{H^1}^2 \leq \|v\|_{H^1}^2 =$

$$= \|v\|^2 + \|\nabla v\|^2 \leq (1+C)\|\nabla v\|^2 \leq$$

$$\leq C'|v|_{H^1}^2 \quad \text{i.e} \quad \|\cdot\|_{H^1}$$

and $|\cdot|_{H^1}$ are equivalent on $H_0^1(\Omega)$.

The Finite Element Method

We introduce a family of triangulations $\{T_h\}$✡ with a corresponding family of function spaces

$$S_h = \{v_h \in C(\bar{\Omega}) : v_h|_K \in \Pi_r, \; v_h|_{\partial\Omega} = 0\}$$

where $T_h = \{K\}$ elements and Π_1 is the space of piecewise linear basis functions. ✡

We note that $S_h \subset H_0^1(\Omega)$.

The FEM reads

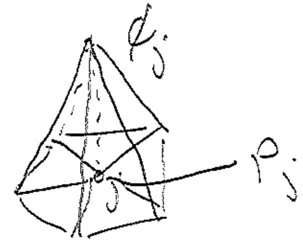
Find $u_h \in S_h$ such that

$$a(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h \quad \text{(Galerkin)}$$

Let $\{\phi_j\}_{j=1}^n$ be a basis for S_h, such that $\phi_j(P_i) = \delta_{ij}$

We get $u_h(x) = \sum_{j=1}^n U_j \phi_j(x)$

$U_j = u_h(P_j)$ and $\chi = \phi_i$



We have

$$\sum_{j=1}^{n} U_j a(\phi_j, \phi_i) = (f, \phi_i), \quad i=1,\dots,n$$

$$KU = F, \qquad (F)_i = (f, \phi_i)$$

$$(K)_{ij} = a(\phi_j, \phi_i)$$

Interpolation

Let $I_h : C(\bar{\Omega}) \to S_h$

be the nodal interpolant

$$(I_h v)(x) = \sum_{j=1}^{n} v(P_j)\phi_j(x) \qquad ✡$$

Interpolation error:

$$|I_h v - v|_{H^1} \le C h^s |v|_{H^{1+s}}, \quad s=1$$

$$|v|_{H^2}^2 = \sum_{i,j=1}^{d} \left\| \frac{\partial^2 v}{\partial x_i \partial x_j} \right\|$$

A priori error bound

We have

$$a(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

$$a(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h$$

Let $\varphi = \chi \in S_h \quad \forall \chi \in S_h$.

Galerkin orthogonality:

$$a(u - u_h, \chi) = 0 \quad \forall \chi \in S_h$$

Best approximation:

Let $\|v\|_E = (a(v,v))^{1/2}$. We have

$$|v|_{H^1}^2 \le \frac{1}{a_0} \int_\Omega a \nabla v \cdot \nabla v \, dx \le \frac{a_1}{a_0} |v|_{H^1}^2$$

where $a(x) \le a_1 \ \forall x \in \Omega$.

We get $\|u - u_h\|_E = \inf_{v_h \in S_h} \|u - v_h\|_E$

Norm equivalence gives

$$|u-u_h|_{H^1} \le \frac{1}{a_0^{1/2}}\|u-u_h\|_E \le \frac{1}{a_0^{1/2}} \inf_{v_h\in S_h} \|u-v_h\|_E$$

$$\le \frac{a_1^{1/2}}{a_0^{1/2}} \inf_{v_h\in S_h} |u-v_h|_{H^1} \le$$

$$\le C\,|u-I_h u|_{H^1} \le C h |u|_{H^2}$$

if $u\in H^2$ (a smooth enough, Ω convex, $f\in L^2$)

<u>Duality argument</u> (Aubin-Nitsche trick)

We introduce the dual problem with data $e=u-u_h$.

Find $w\in H^1_0(\Omega)$ such that

$$a(\varphi, w) = (\varphi, e) \quad \forall \varphi\in H^1_0(\Omega)$$

with $\varphi=e$ we get

$$\|e\|^2 = a(e,w) = a(e, w-I_h w) \le$$

$$\le \|e\|_E \cdot \|w-I_h w\|_E \le$$

$$\le C\|e\|_E\, C\cdot h\,|w|_{H^2} \le C h \|e\|_E$$

$$\le C h^2 |u|_{H^2}$$

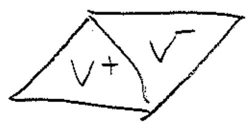
We get $\|u-u_h\| + h|u-u_h|_{H^1} \le C h^2 |u|_{H^2}$

<u>A posteriori error bound</u>

It holds

$$\|u-u_h\|_{H^1} \le C \sum_{K\in T_h} h_K^2 \|f+\nabla\cdot a\nabla u_h\|^2_{L^2(K)} + h_{\partial K} \|[a n\cdot\nabla u_h]\|^2_{L^2(\partial K)}$$

where $[v] = v^+ - v^-$



Adaptive strategy

SOLVE - ESTIMATE - MARK - REFINE

We will show convergence!

Proof. Let $e = u - u_h$

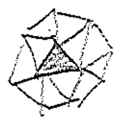
$$\|e\|_E^2 = a(e,e) = a(e, e - \tilde{I}_h e) =$$

$$= (f, e - \tilde{I}_h e) - a(u_h, e - \tilde{I}_h e)$$

$$= \sum_{K \in T_h} (f, e - \tilde{I}_h e)_K - (a\nabla u_h, \nabla e - \tilde{I}_h e)_K$$

$$= \sum_{K \in T_h} (f + \nabla \cdot a \nabla u_h, e - \tilde{I}_h e)_K$$

$$- \sum_{K \in T_h} (n \cdot a \nabla u_h, e - \tilde{I}_h e)_{\partial K} =$$



$$\leq \sum_{K \in T_h} \|f + \nabla \cdot a \nabla u_h\|_K \cdot C h_K \|\nabla e\|_{\omega(K)}$$

$$+ \sum_{K \in T_h} \tfrac{1}{2} | ([n \cdot a \nabla u_h], e - \tilde{I}_h e)|_{\partial K}$$

$$\leq \left\{ \|v - \tilde{I}_h v\|_{\partial K} \leq C h_K^{1/2} \|\nabla v\|_{\omega(K)}^{*} \right\}$$

$$\leq C \sum_{K \in \tilde{T}_h} \left(h_K \|f + \nabla \cdot a \nabla u_h\|_K + h_K^{1/2} \|[n \cdot a \nabla u_h]\|_{\partial K} \right) \|\nabla e\|_{\omega(K)}$$

$$\leq \hat{C} \sum_{K \in T_h} \left(h_K^2 \|f + \nabla \cdot a \nabla u_h\|_K^2 + h_K \|[n \cdot a \nabla u_h]\|_{\partial K}^2 \right)$$

$$\cdot \|e\|_E$$

$\blacksquare$