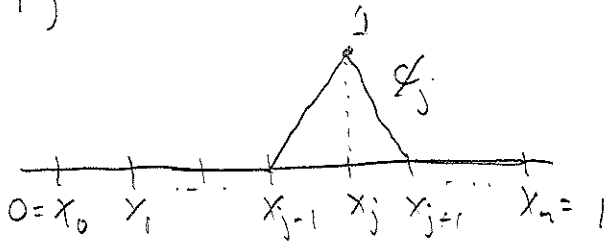
Lecture 2

FEM in 1D

Chapter 0

Interpolation estimate in 1D

$d=1$, $\Omega=(0,1)$



Let $h_j = x_j - x_{j-1}$

$h = \max\limits_{1\le j\le n} h_j$

$I_j = (x_{j-1}, x_j)$, $\dim S_h = n-1$ # interior nodes

$S_h = \{ v \in C(\bar\Omega) : v_{I_j} \in \Pi_1,\ v(0)=v(1)=0 \}$

Lemma 0.4.1

$\{\phi_i\}_{i=1}^{n-1}$ is a basis for S_h.

Proof: (i) $\sum_{i=1}^{n} c_i \phi_i(x) = 0 \Rightarrow c_j = 0$, $j=1,\dots,n-1$

We get this by letting $x = x_j$

$\Rightarrow$ ϕ_i are linearly independent

(ii) $\{\phi_i\}$ span S_h since

For any $v \in S_h$ let $v_I = I_h v =$

$= \sum_{i=1}^{n-1} v(x_i)\phi_i$, the nodal interpolant

Then $v - v_I$ is 0 at nodes

x_i and linear so $v - v_I = 0$. ▣

Theorem 0.4.5

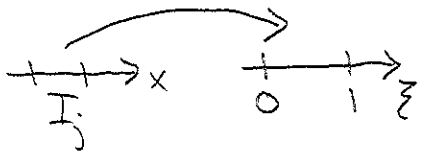
$$|u - I_h u|_{H^1(I)} \le C h \|u''\|_{L^2(I)}$$

Proof:

Let $e = u - I_h u$. It is sufficient

to show $\int_{I_j} |e'(x)|^2 dx \le C h_j^2 \int_{I_j} |u''(x)|^2 dx \underset{(I_h u)''=0 \text{ on } I_j}{=} C h_j^2 \int_{I_j} |e''|^2 dx$

$$x = x_{j-1} + h_j \xi$$



Let $w(\xi) = e(x_{j-1} + h_j \xi)$

$$\frac{de}{dx} = \frac{dw}{d\xi} \cdot \frac{d\xi}{dx} = w' \cdot h_j^{-1}$$

$$\frac{d^2 e}{dx^2} = w'' \cdot h_j^{-2}$$

This means we need to show

$$\int_0^1 w'(\xi)^2 h_j^{-2} h_j \, d\xi \le C h_j^2 \int_0^1 w''(\xi)^2 h_j^{-4} h_j \, d\xi$$

or $\displaystyle\int_0^1 w'(\xi)^2 d\xi \le C \int_0^1 w''(\xi)^2 d\xi$,

the scaling makes h_j vannish.

$w(0) = w(1) = 0$ since $(u - I_h u)(x_j) = 0$

$\Rightarrow w'(\hat{\xi}) = 0$ for some $\hat{\xi} \in [0,1]$ (Rolle)

We have

$$w'(y) = w'(\hat{\xi}) + \int_{\hat{\xi}}^{y} w''(x)\,dx$$

$$\Rightarrow |w'(y)| = \left| \int_{\hat{\xi}}^{y} 1 \cdot w''(x)\,dx \right| \le \left(\int_{\hat{\xi}}^{y} 1^2 dx \right)^{1/2} \left(\int_{\hat{\xi}}^{y} |w''(x)|^2 dx \right)^{1/2}$$

$$\le |y - \hat{\xi}|^{1/2} \left(\int_0^1 |w''(x)|^2 dx \right)^{1/2}$$

$$\int_0^1 |w'(y)|^2 dy \le \int_0^1 |y - \hat{\xi}|\, dy \left(\int_0^1 |w''(x)|^2 dx \right)$$

$$\le C \int_0^1 |w''(x)|^2 dx, \quad C = \sup_{0 \le \hat{\xi} \le 1} \int_0^1 |y - \hat{\xi}|\, dy = \frac{1}{2}. \quad \blacksquare$$

The finite element method in 1D.

Let $I = (0,1)$

$$\begin{cases} -(au')' = f & \text{in } I \\ u(0) = u(1) = 0 \end{cases}, \quad 0 < a_0 \le a(x) \le a_1, \; x \in I.$$

On weak form: Find $u \in H_0^1(I)$

$$(au', v') = (f, v) \quad \forall v \in H_0^1(I)$$

$$\|v\|_E = \left(\int a |v'|^2 dx \right)^{1/2} = \|\sqrt{a}\, v'\|,$$

$$|v|_{H^1} = \|v'\|_I, \quad |v|_{H^2} = \|v''\|$$

Unique solution $u \in H^1_0(\Omega)$ is given by Riesz. If a is sufficiently smooth we have

$$\|u''\| = \|\tfrac{1}{a} a u''\| = \|\tfrac{1}{a}(au')' - \tfrac{a'}{a} u'\|$$

$$\le \|\tfrac{1}{a} f\| + \tfrac{1}{a_0}\|a'\|_{L^\infty} \cdot \|u'\| \le$$

$$\le \frac{1}{a_0}\left(\|f\| + \|a'\|_{L^\infty} \cdot \|u'\|\right)$$

By letting $v = u$ in the weak form

$$\|u'\|^2 \le \frac{1}{a_0}(au', u') = \frac{1}{a_0}(f, u') \le$$

$$\le \frac{1}{a_0}\|f\|\,\|u'\| \Rightarrow \|u'\| \le \frac{\|f\|}{a_0}$$

$$\therefore \quad \|u''\| \le \frac{1}{a_0}\|f\|\left(1 + \|a'\|_{L^\infty}\right)$$

<u>Finita element metoden</u>

Find $u_h \in S_h = \{v \in C([0,1]) : v|_{I_j} \in \Pi_1,\ v(0) = v(1) = 0\}$ such that $(a u_h', \chi') = (f, \chi)\ \forall \chi \in S_h$.

A priori error bound gives

$$\|u - u_h\|_E = \inf_{v_h \in S_h} \|u - v_h\|_E \le$$

$$\le \|u - I_h u\|_E \le C' h \|u''\|$$

$$\le C h \|f\|$$

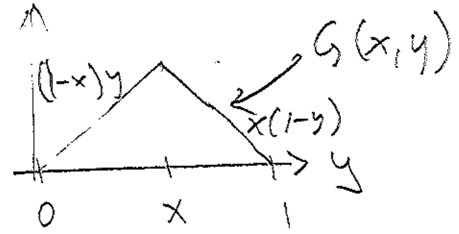
We also have

$$\|u - u_h\|_{H^1} + h\|u - u_h\|_{L^2} \le C h^2 \|f\|$$

<u>The case $a = 1$</u>

$$\begin{cases} -u'' = f & \text{in } I \\ u(0) = u(1) = 0 \end{cases}$$

$$u(x) = \int_0^1 G(x,y) f(y)\,dy = (G(x,\cdot), f)$$



Note that

$G(x_j, \cdot) \in S_h$

$\uparrow$

mesh point

We get $u(x_j) = \int_0^1 G(x_j, y) f(y)\, dy =$

$$= \int_0^1 u_h'(y)\, G(x_j, y)'\, dy =$$

$$= \int_0^{x_j} u_h'(y)(1-x_j)\, dy - \int_x^1 u_h'(y)\, x_j\, dy =$$

$$= (1-x_j)\, u_h(x_j) + x_j u_h(x_j) = u_h(x_j) \qquad j = 0, \dots, n$$

Since $u_h \in S_h \Rightarrow \underline{u_h = I_h u}$

i.e. the FE approximation equals the interpolant.

<u>Adaptive mesh</u> $(a = 1)$

Let $\Delta = \{x_0, x_1, \dots, x_n\}$ be a partition of $[0,1]$. Let S_Δ be the corresponding FE space, $\dim S_\Delta = n-1$.

Assume $\int_0^1 |u'(x)|\, dx = 1$.

Let $\phi(x) = \int_0^x |u'(t)|\, dt$ and let x_i be points so that $\phi(x_i) = \frac{i}{n}$, which is possible since ϕ is increasing and $\phi(0) = 0$, $\phi(1) = 1$.

We can approximate $u(x)$ on $[x_{i-1}, x_i]$ with $c_i = u(x_{i-1})$

$$|u(x) - c_i| = \left| \int_{x_{i-1}}^{x_i} u'(t)\, dt \right| \leq \int_{x_{i-1}}^{x_i} |u'(t)|\, dt$$

$$= \phi(x_i) - \phi(x_{i-1}) = \frac{1}{n}.$$

We have $\|u - u_h\|_{L^\infty} = \|u - I_h u\|_{L^\infty}$

$$\leq \max_{1 \leq i \leq n} \max_{x \in [x_{i-1}, x_i]} |u(x) - u(x_{i-1})| = \frac{1}{n}$$

$$\forall u \;\exists \Delta \text{ s.t. } \|u - u_h\|_{L^\infty} \leq \frac{1}{n}$$

There is always a good mesh to be found.

A posteriori error estimate

$e = u - u_h$,

$$\begin{cases} -(au')' = f & \text{in } I \\ u = 0 & \text{on } \partial I \end{cases}$$

$$\|e\|_E^2 = a(e,e) = a(e, e - I_h e) =$$

$$= (f, e - I_h e) - a(u_h, e - I_h e) =$$

$$= \sum_{i=1}^{n} (f, e - I_h e)_{I_i} - (a u_h', (e - I_h e)')_{I_i}$$

$$= \sum_{i=1}^{n} (f + (a u_h')', e - I_h e)_{I_i} - \underbrace{[a u_h' (e - I_h e)]_{I_i}}_{=0}$$

$$\leq \sum_{i=1}^{n} \|f + (a u_h')'\|_{I_i} \cdot C h_i \|e'\|_{I_i} \leq$$

$$\leq C \left(\sum_{i=1}^{n} h_i^2 \|f + (a u_h')'\|_{I_i}^2 \right)^{1/2} \cdot \|e'\| \leq$$

$$\leq C' \sum_{i=1}^{n} h_i^2 \|f + (a u_h')'\|_{I_i}^2 \cdot \|e\|_E$$

$$\therefore \|u - u_h\|_E \leq C \left(\sum_{i=1}^{n} h_i^2 \|f + (a u_h')'\|_{I_i}^2 \right)^{1/2}.$$

If $a = 1$ $\quad \|u - u_h\|_E \leq C \left(\sum_{i=1}^{n} h_i^2 \|f\|^2 \right)^{1/2}$.

This relation can be used to find a mesh Δ which is adapted to the problem at hand.