Lecture 3

Sobolev spaces and inequalities

Chapter 1

Inequalities in FE analysis

Minkowski's inequality.

For $1 \le p \le \infty$, $f, g \in L^p(\Omega)$ we have

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$$

Hölder's inequality.

For $1 \le p, q \le \infty$ such that $1 = \frac{1}{p} + \frac{1}{q}$.

If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then

$f \cdot g \in L^1(\Omega)$ and

$$\|f \cdot g\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^q(\Omega)}$$

Schwarz' inequality is Hölder when $p = q = 2$,

$$\|f \cdot g\|_{L^1(\Omega)} \le \|f\|_{L^2(\Omega)} \cdot \|g\|_{L^2(\Omega)}$$

FIRST

Young's inequality, $a, b \in \mathbb{R}^+$

$\frac{1}{p} + \frac{1}{q} = 1$, $\quad ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

Special case $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ or

$$ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$

Schwarz' inequality for sums

$$\sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum_{i=1}^n b_i^2\right)^{1/2}$$

Example:

$$\|e'\|^2 \le \sum_{i=1}^n (R_i(u_h), e - I_h e)_{I_i} \le$$

$$\le \sum_{i=1}^n \|R_i(u_h)\|_{I_i} \cdot \|e - I_h e\|_{I_i} \le C \sum_{i=1}^n \|R_i(u_h)\|_{I_i} \cdot h_i \|e'\|_{I_i}$$

$$\le \left(\sum_{i=1}^{n} c^2 h_i^2 \|R_i(u_h)\|_{I_i}^2\right)^{1/2} \left(\sum_{i=1}^{n} \|e'\|_{I_i}^2\right)^{1/2}$$

$$= \left(\sum_{i=1}^{n} c^2 h_i^2 \|R_i(u_h)\|_{I_i}^2\right)^{1/2} \|e'\|$$

$$\le \frac{1}{2} \sum_{i=1}^{n} c^2 h_i^2 \|R_i(u_h)\|_{I_i}^2 + \frac{1}{2}\|e'\|^2 .$$

Weak derivative and Sobolev spaces

Definition 1.2.3. Given Ω the set of Locally integrable functions is denoted by

$$L^1_{loc}(\Omega) = \{ f : f \in L^1(K) \ \forall \text{compact } K \subset \Omega \}$$

Definition 1.2.4

A function $f \in L^1_{loc}(\Omega)$ has a weak derivative $D_w^\alpha f$ if there exists a $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x) dx \quad \forall \phi \in C_0^\infty(\Omega)$$

Here α is a multi-index

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$

$$D^\alpha \phi = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \phi .$$

We have $D_w^\alpha \phi = D^\alpha \phi$ if $\phi \in C^{|\alpha|}(\Omega)$.

Sobolev spaces:

Let $\|f\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \le k} \|D_w^\alpha f\|_{L^p(\Omega)}^p\right)^{1/p}$,

$1 \le p < \infty$ and

$$\|f\|_{W_\infty^k(\Omega)} = \max_{|\alpha| \le k} \|D^\alpha f\|_{L^\infty(\Omega)}$$

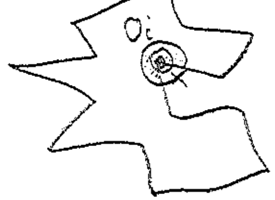
We define the Sobolev space

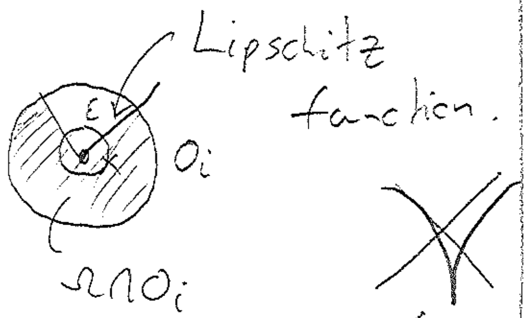
$$W_p^k(\Omega) = \{ f \in L^1_{loc}(\Omega) : \|f\|_{W_p^k(\Omega)} < \infty \}$$

which is a Banach space as $L^p(\Omega)$.

Definition 1.4.4

$\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary $\partial\Omega$ if there is a collection of open sets O_i, a positive parameter ε, an integer N and a number M such that for all $x \in \partial\Omega$ the ball of radius ε entered at x is contained in some O_i, no more than N sets O_i intersects, and each domain $O_i \cap \Omega = O_i \cap \Omega_i$, where Ω_i is a domain whose boundary is a graph of a Lipschitz function ϕ_i.





M is the Lipschitz constant of ϕ_i

Proposition 1.4.1 - 1.4.2

For any domain Ω $k \le m$, $1 \le p \le q \le \infty$

$$W_p^m(\Omega) \subset W_p^k(\Omega), \quad W_q^k(\Omega) \subset W_p^k(\Omega)$$

Theorem 1.4.6 Sobolev's inequality

Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipschitz boundary, k positive integer and $1 \le p \le \infty$ such that

$k \geq n$ when $p = 1$

$k > n/p$ when $p > 1$

Then $\|u\|_{L^\infty(\Omega)} \le C \|u\|_{W_p^k(\Omega)}$

Moreover there is a continuous function in the L^∞ equivalence class of u.

Corollary 1.4.7

let $m < k$, $k - m \geq n$ $\quad p = 1$

$k - m > n/p$ $\quad p > 1$

$$\|u\|_{W_\infty^m(\Omega)} \le C \|u\|_{W_p^k(\Omega)}$$

Furthermore

$1 \le p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, then

$W^{1,p}(\Omega) \subset L^q(\Omega)$.

This gives $n=3$, $p=2$, $W^{1,2}(\Omega) \subset L^6(\Omega)$

These results can be extended to non-integer k.

Trace theorem 1.6.6

Suppose Ω has Lipschitz boundary and $1 \le p \le \infty$. Then

$$\|v\|_{L^p(\partial\Omega)} \le C \|v\|_{L^p(\Omega)}^{1-1/p} \cdot \|v\|_{W_p^1(\Omega)}^{1/p} \quad \forall v \in W_p^1(\Omega)$$

Notice

$$\|v - \tilde{I}_h v\|_{L^2(\partial\Omega)} \le C \|v - \tilde{I}_h v\|_{L^2(\Omega)}^{1/2} \|v - \tilde{I}_h v\|_{W_2^1(\Omega)}^{1/2}$$

$$\le C h^{1/2} \|v\|_{W_2^1(\Omega)}$$

Poincare's inequality

We let $v \in H^1(\Omega)$ and

$$\bar{v} = \frac{1}{|\Omega|} \int_\Omega v \, dx .$$

From chapter 4 we have for large class of Ω that

$$\|v - \bar{v}\|_{L^2(\Omega)} \le C_\Omega |v|_{H^1(\Omega)}$$

We get

$$\|v\|_{L^2(\Omega)} \le \|v - \bar{v}\|_{L^2(\Omega)} + \|\bar{v}\|_{L^2(\Omega)} \le$$

$$\le C |v|_{H^1(\Omega)} + \frac{|\Omega|^{1/2}}{|\Gamma|} \cdot \left| \int_\Gamma \bar{v} \, ds \right|$$

$$\le C |v|_{H^1(\Omega)} + \frac{|\Omega|^{1/2}}{|\Gamma|} \left(\left| \int_\Gamma v \, ds \right| + \left| \int_\Gamma v - \bar{v} \, ds \right| \right)$$

Furthermore

$$\left| \int_\Gamma v - \bar{v} \, ds \right| \overset{\text{Schwarz}}{\le} |\Gamma|^{1/2} \|v - \bar{v}\|_{L^2(\partial\Omega)} \le$$

$$\underset{w = v - \bar{v}}{\le} |\Gamma|^{1/2} \left(\|v - \bar{v}\|_{L^2(\Omega)} + |v - \bar{v}|_{H^1(\Omega)} \right)$$

$$\|w\|_{L^2(\partial\Omega)} \le \|w\|_{L^2}^{1/2} \cdot \|w\|_{W_2^1}^{1/2} \le \frac{1}{2}\|w\|_{L^2} + \frac{1}{2}\|w\|_{W_2^1} \le$$

$$\le \frac{1}{2}\|w\|_{L^2} + \frac{1}{2}\left(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right)^{1/2} \le$$

$\leq |\Gamma|^{1/2} C |v|_{H^1(\Omega)}$

We get

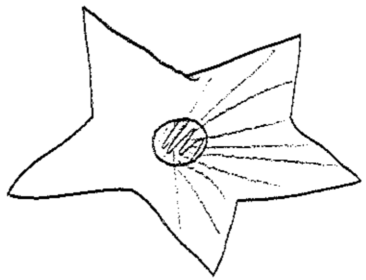
$$\|v\|_{L^2(\Omega)} \leq C\left(\left|\int_\Gamma v\,ds\right| + |v|_{H^1(\Omega)} \right)$$

Poincaré inequality 5.3.5

$$\|v\|_{H^1(\Omega)} \leq C|v|_{H^1(\Omega)} \quad \forall v \in H^1_0(\Omega)$$

or $\|v\|_{L^2(\Omega)} \leq C|v|_{H^1(\Omega)}$.

The requirment on Ω is that its a union of domains that are starshaped with respect to a ball.



The size of the ball affects the constant C.

Negative norms and duality

The dual space B' of a Banach space B is the space of linear functionals on B with norm

$$\|L\|_{B'} = \sup_{0 \neq v \in B} \frac{L(v)}{\|v\|_B}$$

For $1 \leq p < \infty$ it holds $(L^p(\Omega))'$ is isomorphic to $L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$

Definition 1.7.5

Let $1 \leq p \leq \infty$, $k \in N$, $\frac{1}{q} + \frac{1}{p} = 1$, then $W_p^k(\Omega)$ is defined to be the dual to $(W_q^{-k}(\Omega))'$.

In particular $\delta \in W_r^k(\Omega)$ if $k < -n + \frac{n}{p}$.