Lecture 4

Finite elements

Chapter 3.1-3.2

Dual space in finite dimensions

Let V be a finite dimensional vector space with basis $\{e_k\}_{k=1}^{d}$.

Every $v = \sum_{k=1}^{d} \alpha_k e_k$.

The corresponding dual space is given by $\{C_k\}_{k=1}^{d}$ where

$C_j(e_k) = \delta_{jk}$.

We get that $C_j(v) = \sum_{k=1}^{d} \alpha_k C_j(e_k) = \alpha_j$

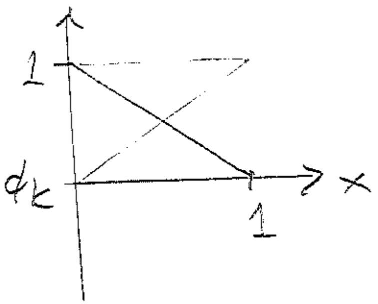
which means that

$v = \sum_{k=1}^{d} C_k(v) e_k$, i.e the dual basis determines the degrees of freedom.

Example: 1D Lagrange element

$K = [0,1]$, $P = P_1 = \{\text{polynomi of deg} \leq 1\}$

$S = \{\phi_1(x), \phi_2(x)\}$, $\phi_1(x) = 1 - x$, $\phi_2(x) = x$



$v = \sum_{k=1}^{2} \underbrace{\alpha_k}_{v(x_k)} \phi_k = \sum_{k=1}^{2} C_k(v) \phi_k$

The dual basis is given by

$C_1(v) = v(0)$, $C_2(v) = v(1)$

The finite element

We use Ciarlet's definition 1978

Definition 3.1.1

Let

(i) $K \subset \mathbb{R}^n$ be bounded closed set with non-empty interior and piecewise smooth boundary (the element domain)

(ii) $\mathcal{P}$ be a finite dimensional space of functions on K (the space of shape functions)

(iii) $\mathcal{N} = \{N_1, N_2, \dots, N_k\}$ be a basis for $\mathcal{P}'$ (the set of nodal variables).

Then $(K, \mathcal{P}, \mathcal{N})$ i called finite element.

The nodal variables lie in some larger function space.

Definition 3.1.2

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. The basis $\{\phi_1, \phi_2, \dots, \phi_k\}$ of $\mathcal{P}$ dual to $\mathcal{N}$ $(N_i(\phi_j) = \delta_{ij})$ is called the nodal basis of $\mathcal{P}$.

Lemma 3.1.4

Let $\mathcal{P}$ be a d-dimensional vector space and let $\{N_1, \dots, N_d\}$ be a subset of the dual space $\mathcal{P}'$. The following statements are equivalent

(a) $\{N_1, \dots, N_d\}$ is a basis for $\mathcal{P}'$

(b) Given $v \in \mathcal{P}$ with $N_i v = 0$, $i = 1, \dots, d$, then $v = 0$.

<u>Proof</u>:

Let $\{\phi_i\}_{i=1}^d$ be a basis for P.

(a) $\Leftrightarrow$ $\forall L \in P'$, $L = \sum_{j=1}^d \alpha_j N_j$

$\Leftrightarrow$ y_i $\exists$ unique $\{\alpha_j\}_{j=1}^d$ s.t.

$y_i = L(\phi_i) = \sum_{j=1}^d \alpha_j N_j(\phi_i)$, $i=1,\dots,d$

$\Leftrightarrow$ $B\alpha = y$ solvable or

B is invertable.

$v \in P$, $v = \sum_{j=1}^d \beta_j \phi_j$.

$N_i(v) = \sum_{j=1}^d \beta_j N_i(\phi_j) = 0 \quad i=1,\dots,d$

(b) $\Leftrightarrow$ $C\beta = 0 \Rightarrow \beta = 0 \Leftrightarrow C$ invertable but $C = B^T$ i.e

(a) $\Leftrightarrow$ (b).

<u>Definition 3.1.8</u>

We say $\mathcal{N}$ determines P if $\psi \in P$ with $N(\psi) = 0 \;\forall N \in \mathcal{N}$ implies $\psi = 0$.

<u>Hyperplane in $\mathbb{R}^n$</u>

Let $L(x) = a \cdot x$, $x \in \mathbb{R}^n$

and $L = \{x : L(x) = 0\}$

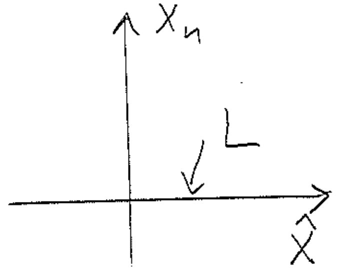
we call L a hyperplane.

<u>Lemma 3.1.10</u>

Let P be a polynomial of degree $d \geq 1$ that vanishes on a hyperplane L. Then we can write $P = LQ$, where Q is a polynomial of degree $d-1$.

<u>Proof</u>: We make a coordinate change so that $L = \{x : x_n = 0\}$

$L(x) = x_n$



$P(x) = P(\hat{x}, x_n) =$

$$= \sum_{j=0}^{d} \sum_{|i| \le d-j} c_{ij} \hat{x}^i x_n^j$$

variables $(x, y, z, \dots)$

total degree $\le d$.

where $\hat{x} = (x_1, \dots, x_{n-1})$, $i = (i_1, \dots, i_{n-1})$

and $P(\hat{x}, 0) = 0$ which means

$$0 = \sum_{|i| \le d} c_{i0} \hat{x}^i \Rightarrow c_{i0} = 0$$

$\Rightarrow$ no x_n^0 terms in P.

Then $$P = x_n \sum_{j=1}^{d} \sum_{|i| \le d-j} c_{ij} \hat{x}^i x_n^{j-1} =$$

$$= L \cdot Q, \quad \deg Q = d-1$$

<u>Triangular finite elements</u>

Let $n = 2$ (dimension)

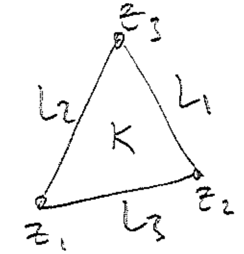
$P_1 = \{a_0 + a_1 x_1 + a_2 x_2\}$

$P_2 = \{a_0 + a_1 x_1 + a_2 x_2 + a_{11} x_1^2 + a_{12} x_1 x_2 + a_{22} x_2^2\}$

k	$\dim(P_k)$
1	3
2	6
3	10
k	$\frac{1}{2}(k+1)(k+2)$

<u>The Lagrange elements</u>

$k=1$) $P = P_1$, $\mathcal{N}_1 = \{N_1, N_2, N_3\}$ the set of nodal variables where $N_i(v) = v(z_i)$, $\{z_i\}_{i=1}^{3}$ are vertices of the element K



Next we show that $\mathcal{N}$ determines P i.e. that $\psi \in P$ with $N(\psi) = 0$ $\forall N \in \mathcal{N}$ implies $\psi = 0$.

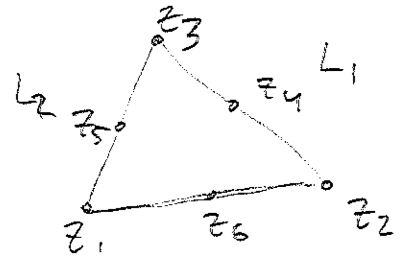
We have $\psi(z_1) = \psi(z_2) = 0 \Rightarrow$

$\psi|_{L_3} = 0$. This means that identically vanishing(?)

$$\underset{\substack{\uparrow \\ \text{degree } 0}}{\psi = L_3 \cdot Q} = c \cdot L_3, \text{ but } \psi(z_3) = c L_3(z_3)$$

$= 0 \Rightarrow c = 0$ i.e $\psi = 0$.

$k=2$) $P = P_2$

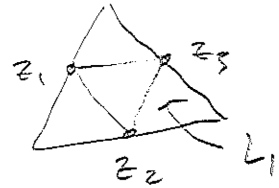


$\psi|_{L_1} = 0 \Rightarrow \psi = L_1 \cdot \underset{\substack{\uparrow \\ \text{degree } 1}}{Q}$, $Q = 0$ on $L_2 \Rightarrow$

$\psi = L_1 \cdot L_2 \cdot \underset{\leftarrow \text{ degree } 0}{Q'} = c L_1 \cdot L_2$

$\psi(z_4) = 0 \Rightarrow c = 0 \Rightarrow \psi = 0$.

$k=1$) Crouzeix - Raviart

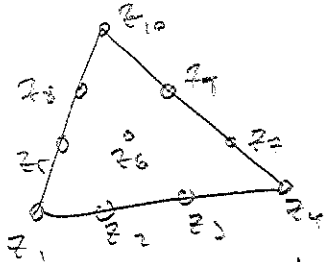


We have $\psi|_{L_1} = 0$

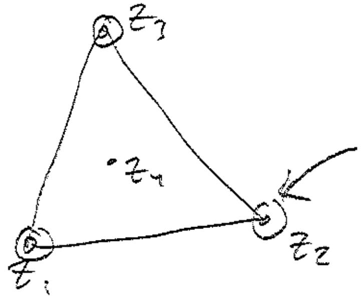
$\Rightarrow \psi = L_1 \cdot Q = L_1 \cdot c$,

but $\psi(z_3) = L_1(z_3) \cdot c = 0 \Rightarrow c = 0$
$\psi = 0$.

$k=3$)



$k=3$) Cubic Hermite



evaluation of gradient, two components $\Rightarrow$ two nodal values.

$\mathcal{N} = \{N_1, \ldots, N_{10}\}$