Lecture 5

The interpolant (Definition 3.3.1)

Given a finite element $(K, P, \mathcal{N})$ let $\{\phi_i ; 1 \le i \le k\} \subset P$ be the dual basis of $\mathcal{N}$. Let v be a function for which $N_i(v)$, $N_i \in \mathcal{N}$ are defined, we define the Local interpolant by

$$\mathcal{I}_K v := \sum_{i=1}^{k} N_i(v) \phi_i$$

Proposition 3.3.4 - 3.3.7

- $\mathcal{I}_K$ is a linear operator
- $N_i(\mathcal{I}_K v) = N_i(v)$
- $\mathcal{I}_K(v) = v$ if $v \in P$
- $\mathcal{I}_K = \mathcal{I}_K^2$ proof left out

(Def: 3.3.8)
A subdivision of Ω is a finite collection of element domains $\{K_i\}$ such that

i) $\text{int } K_i \cap \text{int } K_j = \emptyset$ if $i \neq j$

ii) $\bigcup K_i = \overline{\Omega}$

Definition 3.3.9

Let Ω be a domain with a subdivision $\mathcal{T} = \{K_i\}$. Let each element K be associated with shape functions P and nodal variables $\mathcal{N}$ such that $(K, P, \mathcal{N})$ forms a finite element. With m as the highest derivative in the nodal variables N_i for $f \in C^m(\overline{\Omega})$ the global interpolant $\mathcal{I}_{\mathcal{T}} f|_{K_i} = \mathcal{I}_{K_i} f$ for all $K_i \in \mathcal{T}$.

Continuity of $I_\tau f$ is <u>not</u> given without further assumptions on the subdivision.

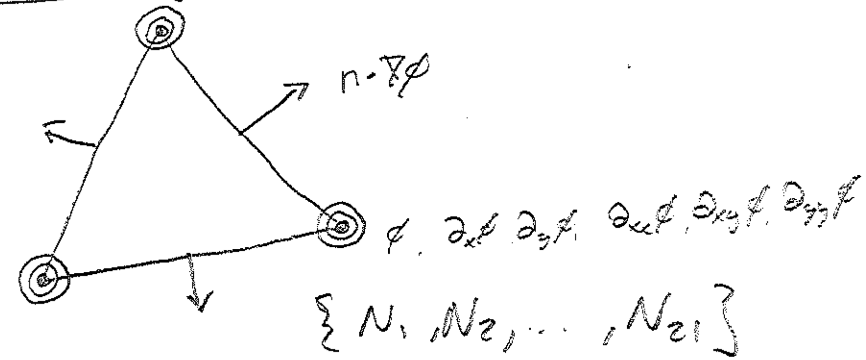
<u>Definition 3.3.11</u>
A triangulation of a polygonal domain Ω is a subdivision consisting of triangles with no vertex lying in the interior of an edge of another triangle.

<u>Definition 3.3.15</u>
If $I_\tau f \in C^r \ \forall f \in C^m(\bar{\Omega})$ then I_τ has continuity order r.

The finite element space
$V_\tau = \{ I_\tau f : f \in C^m(\bar{\Omega})\}$ is a C^r finite element space.

An element (K, P, N) that can produce a C^r FE space is a C^r element.

<u>The Argyris element</u> (C^1)

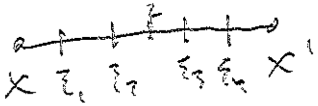


Proposition 3.3.17

The Lagrange and Hermite elements are C^0 elements and Argyris is C^1.

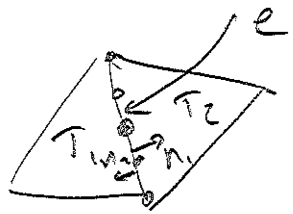
More precisely given a triangulation $\mathcal{T}$ of Ω it is possible to choose edge nodes for the elements $(K, P_K, \mathcal{N})$, $K \in \mathcal{T}$ such that $I_{\mathcal{T}} f \in C^r$ (Lagrange & Hermite $r=0$, Argyris $r=1$) for $f \in C^m$ (Lagrange $m=0$, Hermite $m=1$, Argyris $m=2$).

In particular it's sufficient for each edge $\overline{xx'}$ to have nodes $\xi_i(x'-x)+x$ with $\{\xi_i : i=1,\dots,k+1-2m\}$ symmetric around $\xi=\frac{1}{2}$.



Moreover $I_{\mathcal{T}} f \in W_\infty^{r+1}(\Omega)$.

Proof: Let T_1 and T_2 have a common edge.



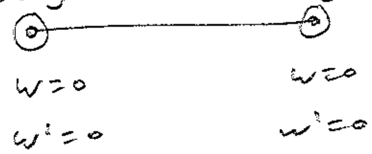
Let $w = I_{T_1} f - I_{T_2} f$ polynomial of degree k (CR rot)

$w|_e = 0$ in the $k+1$ nodes $\Rightarrow w|_e = 0$ (Lagrange)

This means that $I_{\mathcal{T}}$ is continuous.

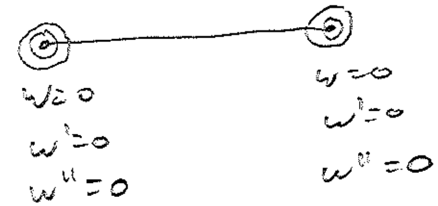
For Hermite (cubic $k=3$) we have

$w|_e = 0$ since polynomial of degree 3 and



For Argyris ($k=5$)

$w|_e = 0$ since

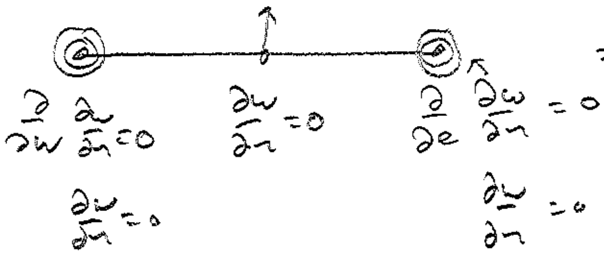


For Argyris we also have to show

$Dw|_e = 0$ i.e. $\frac{\partial w}{\partial e}\Big|_e = 0$ (since $w|_e = 0$)

and $\frac{\partial w}{\partial n}\Big|_e = 0$. $\frac{\partial w}{\partial n}$ is of order 4

and



$\Rightarrow \frac{\partial w}{\partial n}\Big|_e = 0$.

Therefor for any $v \in S_h$ we have

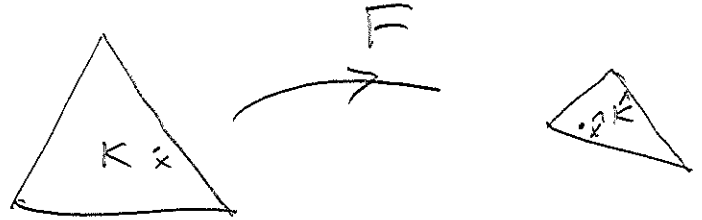
$$v = I_\tau v \begin{cases} \in C^0 & \text{Lagrange, Hermite} \\ \in C^1 & \text{Argyris} \end{cases}$$

We leave $I_\tau f \in W_0^{r+1}(\Omega)$ as an exercise.

Equivalence of elements

Consider two finite elements

$(K, P, \mathcal{N})$ and $(\hat{K}, \hat{P}, \hat{\mathcal{N}})$



and let $F = Ax + b$ be an affine map, A non-singular, $\hat{x} = F(x)$

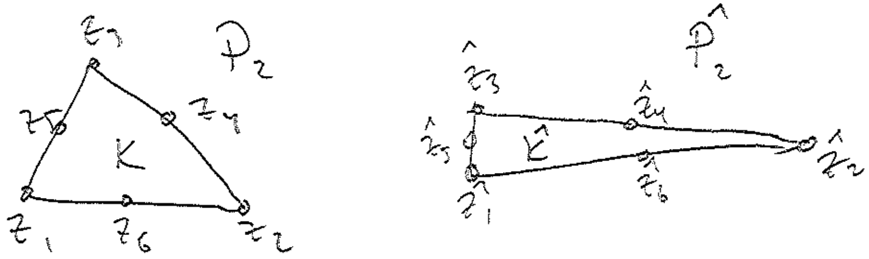
The elements are affine-equivalent if

(i) $F(K) = \hat{K}$

(ii) $\hat{P} \circ F = P$

(iii) $\forall \hat{f} \in \hat{P}$, $\mathcal{N}(\hat{f} \circ F) = \hat{\mathcal{N}}(\hat{f})$

Example, Lagrange $k = 2$



$\hat{x} = Ax + b := F(x)$

$x = A^{-1}\hat{x} - A^{-1}b$

Let $\hat{z}_1 = F(z_1)$, $\hat{z}_2 = F(z_2)$, $\hat{z}_3 = F(z_3)$

which is possible with the six degrees of freedom in A, b. $(n=2)$

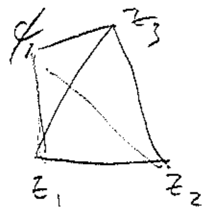
We get $z_5 = \frac{1}{2}(z_1 + z_3)$ but also

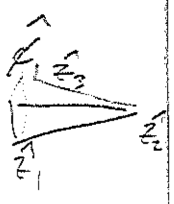
$$\hat{z}_5 = F(z_5) = F(\tfrac{1}{2} z_1 + \tfrac{1}{2} z_3) = \tfrac{1}{2} F(z_1) + \tfrac{1}{2} F(z_3)$$

$$= \tfrac{1}{2}\hat{z}_1 + \tfrac{1}{2}\hat{z}_3 .$$

edges map to edges and interior to in~~terior~~ $F(K) = \hat{K}$.

Now take $\phi_1(x) \in P_1$





$$\hat{\phi}_1(\hat{x}) = \hat{\phi}_1(F(x)) = \{\text{Linear and correct in } z_1, z_2, z_3\}$$

$$= \phi_1(x)$$

Linear combination gives

$\hat{P}_1 \circ F = P_1$. For $k = 2, 3, \ldots$

$\varphi_1 = 2\phi_1^2 - \phi_1, \ldots$

$$\hat{\varphi}_1(\hat{x}) = \hat{\varphi}_1(F(x)) = 2\hat{\phi}_1(F(x))^2 - \hat{\phi}_1(F(x)) =$$

$$\ldots = \varphi_1(x). \Rightarrow \hat{P}_2 \circ F = P_2.$$

Let $\hat{f} \in \hat{P}$, $N_i \in \mathcal{N}$

$$N_i(\hat{f} \circ F) = (\hat{f} \circ F)(z_i) = \hat{f}(\hat{z}_i) = \hat{N}_i(\hat{f}$$

$\forall i, \hat{f}$