Lecture 6

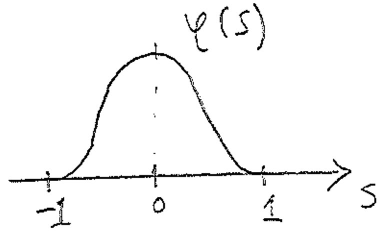
Polynomial approximation theory.

Chapter 4

4.1 Averaged Taylor polynomials (for functions in Sobolev spaces)

Let $B = B(x_0, \rho) = \{ x \in \mathbb{R}^n : |x - x_0| < \rho \}$

Cut-off function:



(i) $\operatorname{supp} \phi = \overline{B}$

(ii) $\int_{\mathbb{R}^n} \phi \, dx = 1$

(iii) $\phi \in C_0^\infty(\mathbb{R}^n)$

Example: $\varphi(s) = \begin{cases} e^{\frac{-1}{1-s^2}}, & |s| < 1 \\ 0, & |s| \geq 1 \end{cases}$

Let $\psi(x) = \varphi\left(\frac{|x - x_0|}{\rho}\right)$ and

$\phi(x) = c^{-1}\psi(x)$ where $c = \int_{\mathbb{R}^n} \psi \, dx$

$$c = \int_{\mathbb{R}^n} \psi(x)\,dx = w_n \int_0^\rho \varphi\left(\frac{r}{\rho}\right) r^{n-1}\,dr =$$

$$= \left\{ s = \frac{r}{\rho},\ r^{n-1}dr = \rho^{n-1} s^{n-1} \cdot \rho\, ds \right\} = w_n \rho^n \int_0^1 \varphi(s) s^{n-1}\,ds$$

$$= c_1 \cdot \rho^n.$$

$$\|\phi\|_{L^\infty} \leq c_1 \rho^{-n}.$$

Let $u \in C^{m-1}(\mathbb{R}^n)$. The Taylor polynomial of u of order m (degree $m-1$) evaluated at y is

$$(T_y^m u)(x) = \sum_{|\alpha| < m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha$$

with multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$

$$x^\alpha = \prod_{i=1}^{m} x_i^{\alpha_i}, \quad \alpha! = \prod_{i=1}^{m} \alpha_i!, \quad |\alpha| = \sum_{i=1}^{m} \alpha_i$$

Definition 4.1.3

Let u have weak derivative of order $m-1$ and $B \subset\subset \Omega$ $(\overline{B} \subset \operatorname{int} \Omega)$

The Taylor polynomial of u averaged over B is

$$Q^m u(x) = \int_B T_y^m u(x)\, \phi(y)\, dy .$$

Proposition 4.1.12

$Q^m u$ is defined for all $u \in L^1(B)$ and

$$Q^m u(x) = \sum_{|\lambda|<m} x^\lambda \int_B \psi_\lambda(y) u(y)\, dy,$$

where $\psi_\lambda(y) = \sum_{\lambda \le \alpha, |\alpha|<m} \frac{(-1)^{|\alpha|}}{\alpha!} a_{(\lambda,\alpha-\lambda)} D^\alpha(y^{\alpha-\lambda}\phi(y))$,

$\psi_\lambda \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \psi_\lambda \subset \overline{B}$.

i.e. $Q^m u$ is a polynomial of degree $< m$ in x.

Proof: typical term in $Q^m u(x)$:

$$\int_B \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \phi(y)\, dy$$

We note that $(x-y)^\alpha = \prod_{i=1}^n (x_i - y_i)^{\alpha_i} =$

$$= \sum_{\gamma+\beta=\alpha} a_{(\gamma,\beta)} x^\gamma \cdot y^\beta \Rightarrow$$

$$\sum_{\gamma+\beta=\alpha} \frac{1}{\alpha!} a_{(\gamma,\beta)} x^\gamma \int_B D^\alpha u(y) y^\beta \phi(y)\, dy =$$

$$= \sum_{\gamma+\beta=\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} a_{(\gamma,\beta)} x^\gamma \int_B \underbrace{u(y) D^\alpha(y^\beta \phi(y))}_{L^1 \text{ function}} dy$$

$$Q^m u(x) = \sum_{|\alpha|<m} \sum_{\gamma+\beta=\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} a_{\gamma,\beta} x^\gamma \int_B u(y) D^\alpha(y^\beta \phi(y))\, dy$$

Polynomial of degree $m-1$. $(\beta = \alpha - \gamma)$ ∎

Corollary 4.1.15 If Ω is bounded

$$\|Q^m u\|_{W^k_\infty(\Omega)} \le C_{m,n,\beta,\Omega} \|u\|_{L^1(B)}$$

Since $\sup_{y\in B} |\psi_\lambda(y)|$ and $\sup_{x\in\Omega} |D^\alpha x^\lambda|$ are bounded.

Proposition 4.1.17 For any α such that $|\alpha| \leq m-1$

$$D^\alpha Q^m u = Q^{m-|\alpha|} D^\alpha u \quad \text{for all } u \in W_1^{|\alpha|}(B)$$

Proof: If $u \in C^\infty(B)$ then

$$D_x^\alpha Q^m u(x) = \int_B D_x^\alpha T_y^m u(x) \phi(y)\, dy$$

$$= \int_B T_y^{m-|\alpha|} D_x^\alpha u(x) \phi(y)\, dy$$

$$= Q^{m-|\alpha|} D^\alpha u(x).$$

Since $C^\infty(B)$ is dense in $W_1^{|\alpha|}(B)$ it follows for $u \in W_1^{|\alpha|}(B)$ ∎

Ex: $n=1$, $m=3$, $\alpha=1$

$$D_x T_y^3 u(x) = D_x\left(u(y) + Du(y)(x-y) + \tfrac{1}{2} D^2 u(y)(x-y)^2\right)$$

$$= Du(y) + D^2 u(y)(x-y) = (T_y^2 Du)(x)$$

We note that the construction of Q^m is not unique since the cut-off ϕ is not unique.

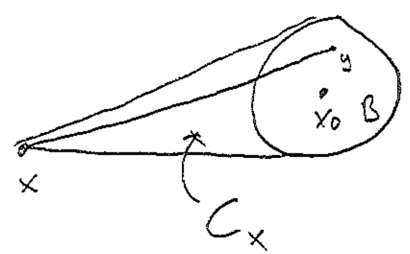
Error representation 4.2

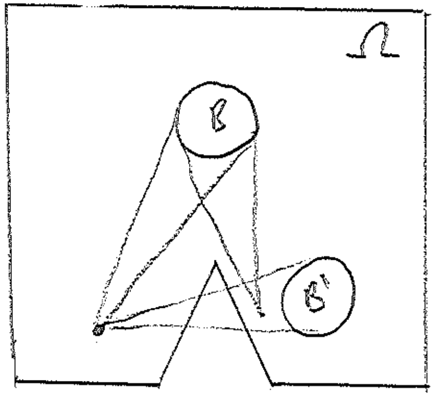
Taylor's theorem in 1D:

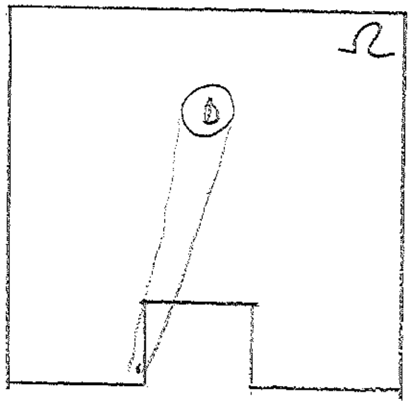
$$f(1) = \sum_{k=0}^{m-1} \frac{1}{k!} f'(0) + m \int_0^1 \frac{1}{m!} s^{m-1} f^{(m)}(1-s)\, ds$$

Definition 4.2.2

Ω is star-shaped with respect to B if for all $x \in \Omega$, the closed convex hull C_x of $\{x\} \cup B$ is contained in Ω.







Star-shaped wrt B Not star-shaped
not B'

Let $u \in C^m(\Omega)$

consider $f(s) = u(y + s(x-y))$

Taylor:

$$u(x) = f(1) = \sum_{k=0}^{m-1} \frac{1}{k!} f^{(k)}(0) + m \int_0^1 \frac{1}{m!} s^{m-1} f^{(m)}(1-s)\, ds$$

$$= \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha + \sum_{|\alpha|=m} (x-y)^\alpha \int_0^1 \frac{m}{\alpha!} s^{m-1} D^\alpha u(x+s(y-x))\, ds$$

$$= (T_y^m u)(x) \quad + \quad \text{——} \;\; \text{"} \;\; \text{——}$$

Definition 4.2.6

The mth-order remainder term

$$R^m u(x) = u(x) - Q^m u(x) =$$

$$= \int_B u(x)\phi(y)\,dy - \int_B T_y^m u(x)\,\phi(y)\,dy$$

$$= \int_B \phi(y)\, m \left(\sum_{|\alpha|=m} (x-y)^\alpha \int_0^1 \frac{s^{m-1}}{\alpha!} D^\alpha u(x+s(y-x))\,ds \right) dy$$

Proposition 4.2.8

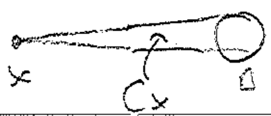
$$R^m u(x) = m \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x,z)\, D^\alpha u(z)\, dz,$$

where $z = x + s(y-x)$, $k_\alpha(x,z) = \frac{1}{\alpha!}(x-z)^\alpha k(x,z)$

$$|k(x,z)| \le C \left(1 + \frac{1}{\rho}|x - x_0|\right)^n |z-x|^{-n}$$

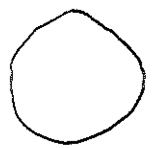
Proof: See book, note that

$y \in B$, $x \notin B \Rightarrow x + s(y-x) \in C_x$



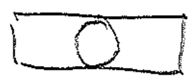
<u>Definition 4.2.16</u> Suppose Ω has diameter d and is star-shaped w.r.t B. Let $\rho_{max} = \sup\{\rho : \Omega$ is starshaped w.r.t ball of radius $\rho\}$. Then the chunkiness parameter of Ω is

$$\gamma = \frac{d}{\rho_{max}} \qquad (\geq 2)$$



$$\Omega = B(0, \tfrac{d}{2}) \Rightarrow \gamma = 2$$

maximal chunky



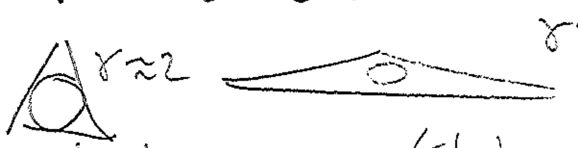
Rectangle: $\Omega = [0,a] \times [0,b]$

$$\gamma = \frac{\sqrt{a^2+b^2}}{b/2} = 2\sqrt{1+\left(\frac{a}{b}\right)^2}$$

$b \approx a \Rightarrow \gamma \approx 2$ chunky

$b \ll a \Rightarrow \gamma \gg 2$ thin

Triangle:



<u>Corollary 4.2.18</u>

B can be chosen so that

$$|k(x,z)| \leq C(\gamma+1)^n |z-x|^{-n}$$

<u>Proof</u>: Choose $B = B_\rho$ s.t $\rho > \frac{1}{2}\rho_{max}$

Then $|k(x,z)| \leq C\left(1 + \frac{|x-x_0|}{\rho}\right)^n |z-x|^{-n}$

$$\leq C\left(1 + \frac{d}{\frac{1}{2}\rho_{max}}\right)^n |z-x|^{-n} \leq$$

$$\leq C'(1+\gamma)^n |z-x|^{-n}$$