Lecture 7

Riesz potentials 4.3

We have a bound of $R^m = I - Q^m$ in terms of Riesz potentials.

Lemma 4.3.1

If $f \in L^p(\Omega)$, $1 < p < \infty$, $m > \frac{n}{p}$

$$\int_\Omega |x-z|^{-n+m} |f(z)| \, dz \leq C_p \, d^{m-\frac{n}{p}} \|f\|_{L^p(\Omega)} \quad \forall x \in \Omega$$

d: diam(Ω)

It also holds for $p=1$ if $m \geq n$.

Proof: Assume $1 < p < \infty$, $m > \frac{n}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_\Omega |x-z|^{-n+m} |f(z)| \, dz \overset{\text{Hölder}}{\leq} \left(\int_\Omega |x-z|^{(-n+m)q} \right)^{1/q} \cdot \|f\|_{L^p}$$

$$\overset{\text{Polar coord.}}{\leq} C \left(\int_0^d r^{(-n+m)q+n-1} \, dr \right)^{1/q} \|f\|_{L^p(\Omega)}$$

$$\leq C' \left(d^{(-n+m)q+n} \right)^{1/q} \cdot \|f\|_{L^p(\Omega)}$$

$$= C \, d^{m-\frac{n}{p}} \|f\|_{L^p(\Omega)} \quad \text{since } \frac{1}{q} - 1 = -\frac{1}{p}$$

If $p = 1$, $m \geq n$

$$\int_\Omega |x-z|^{-n+m} |f(z)| \, dz \leq \| (x-z)^{-n+m} \|_{L^\infty} \cdot \|f\|_{L^1}$$

$$\leq d^{-n+m} \|f\|_{L^1(\Omega)}$$

Proposition 4.3.2 For $u \in W_p^m(\Omega)$

$$\|R^m u\|_{L^\infty(\Omega)} \leq C_{m,n,\gamma,p} \, d^{m-\frac{n}{p}} |u|_{W_p^m(\Omega)}$$

$1 < p < \infty$, $m > \frac{n}{p}$ or $p = 1$, $m \geq n$

Proof: Assume $u \in C^m(\Omega) \cap W_p^m(\Omega)$

$$|R^m u(x)| = m \left| \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x,z) D^\alpha u(z) \, dz \right|$$

$$\leq C'_{m,n,\gamma} \sum_{|\alpha|=m} \int_\Omega |x-z|^{m-n} |D^\gamma u(z)| \, dz$$

$$\leq C_{m,n,\gamma,p}\, d^{m-\frac{n}{p}} |u|_{W_p^m(\Omega)}$$

A density argument is needed to complete the proof. ■

Lemma 4.3.4 Sobolev's Inequality

Suppose Ω has diameter d and is star-shaped w.r.t a ball B. If $u \in W_p^m(\Omega)$ where (i) $1<p<\infty$, $m > \frac{n}{p}$ or (ii) $p=1$, $m \geq n$ then u is continuous on Ω and

$$\|u\|_{L^\infty(\Omega)} \leq C_{m,n,\gamma,d} \|u\|_{W_p^m(\Omega)}$$

Proof: $\|u\|_{L^\infty(\Omega)} \leq \|u - Q^m u\|_{L^\infty} + \|Q^m u\|_{L^\infty}$

$$\leq C_{m,n,\gamma,d} |u|_{W_p^m(\Omega)} + C_{m,n,p} \|u\|_{L^1(\Omega)}$$
$$\leq C_{m,n,\gamma,d} \|u\|_{W_p^m(\Omega)}$$

Continuity remains.

Let $u_j \in C^\infty(\Omega) \cap W_p^m(\Omega)$ s.t

$\|u_j - u\|_{W_p^m(\Omega)} \to 0$ as $j \to \infty$

$\Rightarrow \|u_j - u\|_{L^\infty} \to 0 \Rightarrow$ uniform convergence of continuous functions

$\Rightarrow u$ continuous on Ω. ■

Lemma 4.3.8 Bramble-Hilbert

Let Ω be star-shaped w.r.t. B with radius $\rho > \frac{1}{2}\rho_{max}$. For $p \geq 1$

$$|u - Q^m u|_{W_p^k(\Omega)} \leq C_{m,n,\gamma} d^{m-k} |u|_{W_p^m(\Omega)}$$

$k = 0,1,\ldots,m$.

$d = diam(\Omega)$

Proof:

$k=m$: $|u - Q^m u|_{W_p^m(\Omega)} = |u|_{W_p^m(\Omega)}$ since

$D^\alpha Q^m u = 0$ $|\alpha| = m$

k=0:) $\|u - Q^m u\|_{L^p(\Omega)} = \|R^m u\|_{L^p(\Omega)}$

$$\le m \sum_{|\alpha|=m} \Big\| \int_\Omega K_\alpha(x,z) D^\alpha u(z)\,dz \Big\|_{L^p(\Omega)}$$

(Cor)
$$\le C_{m,n}(1+\gamma)^n \sum_{|\alpha|=m} \Big\| \int |z-x|^{m-n} |D^\alpha u(z)|\,dz \Big\|_{\boxed{L^p(\Omega)}}$$

$$\le \left\{ \begin{array}{l} \text{Lemma 4.3.6, } f \in L^1,\ p,m \ge 1 \\ g(x) = \int_\Omega |x-z|^{m-n} |f(z)|\,dz \Rightarrow \\ \qquad \|g\|_{L^p(\Omega)} \le C_{m,n} d^m \|f\|_{\boxed{L^p(\Omega)}} \end{array} \right\}$$

$$\le C d^m |u|_{W_p^m(\Omega)}$$

$0<k<m$

$$|R^m u|_{W_p^k(\Omega)} = \sum_{|\alpha|=k} \|\underbrace{D^\alpha R^m u}_{= R^{m-k} D^\alpha u}\|_{L^p(\Omega)}$$

$$= \sum_{|\alpha|=k} \|R^{m-k} D^\alpha u\|_{L^p(\Omega)} \underset{\uparrow \text{ prev.}}{\le} C d^{m-k} \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega)} = C d^{m-k} |u|_{W_p^m(\Omega)}$$ ∎

<u>Lemma 4.3.14</u> Friedrichs' inequality

Let Ω be starshaped with respect to B. Then for all $u \in W_p^1(\Omega)$ it holds

$$\|u - \bar{u}\|_{W_p^1(\Omega)} \le C_{n,\gamma} |u|_{W_p^1(\Omega)},$$

$$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x)\,dx .$$

<u>Proof</u>: Hölder gives $\|\bar{\phi}\|_{L^p(\Omega)} \le \|\phi\|_{L^p(\Omega)}$

$\Big($ since $\|\bar{\phi}\|_{L^p}^p = \frac{1}{|\Omega|^p} \int_\Omega \left(\int_\Omega |\phi|\,dx\right)^p \le \underbrace{\frac{|\Omega|}{|\Omega|^p} \cdot \|1\|_{L^q}^p}_{1} \cdot \|\phi\|_{L^p}^p \Big)$

We get $\|u - \bar{u}\|_{L^p(\Omega)} = \|(u-c) - \overline{u-c}\|_{L^p(\Omega)} \le 2\|u-c\|_{L^p}$ (any c)

$$\Rightarrow \|u - \bar{u}\|_{L^p(\Omega)} \le 2 \inf_{c\in\mathbb{R}} \|u-c\|_{L^p(\Omega)} \underset{\uparrow \text{ B.H. } m=1,\ k=0}{\le} C |u|_{W_p^1(\Omega)}$$ ∎

Bounds for interpolation error 4.4

We have estimates for $R^m u$ but they are local. We want to use these results to bound the interpolation error

Lemma 4.4.1

Let $(K, P_m, \mathcal{N})$ be a finite element, up to l derivatives

Then I, $\left(Iu = \sum_{i=1}^{k} N_i(u)\phi_i\right)$ is bounded from $C^l(\bar{K})$ into $W_p^m(\Omega)$, $1 \le p \le \infty$.

Proof:
$$\|Iu\|_{W_p^m(K)} \le \sum_{i=1}^{k} \underbrace{|N_i(u)|}_{\le \|N_i\|_{(C^l)'} \cdot \|u\|_{C^l(K)}} \|\phi_i\|_{W_p^m(K)}$$

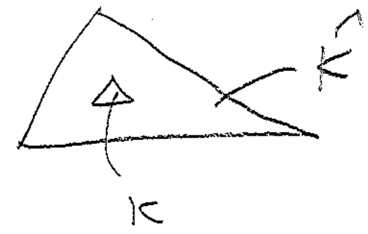
$$\le \left(\sum_{i=1}^{k} \|N_i\|_{C^l(K)'} \cdot \|\phi_i\|_{W_p^m(K)}\right) \cdot \|u\|_{C^l(K)} \quad \blacksquare$$

We denote the operator norm of I

$$\sigma(K) = \|I\|_{C^l(K) \to W_p^m(K)}$$

Definition 4.4.3

Let $\hat{K} = \{ \frac{1}{\operatorname{diam} K} x : x \in K \}$



Theorem 4.4.4

Let $(K, P, \mathcal{N})$ satisfying

(i) K star-shaped and chunk γ.

(ii) $P_{m-1} \subseteq P \subseteq W_\infty^m(\hat{K})$

(iii) $\mathcal{N} \subseteq (C^l(\bar{K}))'$

Assume $m - \frac{n}{p} > l$, $p > 1$

$m - \frac{n}{p} \ge l$, $p = 1$

Then $|v - Iv|_{W_p^i(K)} \le C_{m,n,\gamma,\sigma(\hat{K})} (\operatorname{diam} K)^{m-i} \cdot |v|_{W_p^m(K)}$

$i = 0, 1, \ldots, m$.

Proof:
We start by proving it for $\hat{K}$

We have $IQ^m v = Q^m v$, $Q^m v \in P_{m-1} \subset P$

$$|v - Iv|_{W_p^i(\hat{K})} \le |v - Q^m v|_{W_p^i(\hat{K})} + |Q^m v - Iv|_{W_p^i(\hat{K})}$$

$$= |v - Q^m v|_{W_p^i(\hat{K})} + |I(Q^m v - v)|_{W_p^i(\hat{K})}$$

$$\le |v - Q^m v|_{W_p^i(\hat{K})} + \sigma(\hat{K}) \|Q^m v - v\|_{C^l(\hat{K})}$$

$$\underset{\substack{\text{Sobolev} \\ m - \frac{n}{p} > l \quad p > 1 \\ m - \frac{n}{p} \ge l \quad p = 1}}{\le} \left(1 + \sigma(\hat{K}) C_{m,n,\gamma,l}\right) \|v - Q^m v\|_{W_p^m(\hat{K})}$$

$$\underset{\substack{B.H \\ d = 1 \\ \operatorname{diam} \hat{K} = 1}}{\le} C_{m,n,\gamma,l,\sigma(\hat{K})} |v|_{W_p^m(\hat{K})}$$

General domain

$u \in W_p^m(K)$, let $\hat{u}(y) = u(d \cdot y)$, $\hat{u} \in W_p^m(\hat{K})$. Change of variables

$$|\hat{u}|_{W_p^m(\hat{K})} = d^{m - \frac{n}{p}} |u|_{W_p^m(K)}$$

and $|\hat{u} - \widehat{Q^m u}|_{W_p^i(\hat{K})} = d^{i - \frac{n}{p}} |u - Q^m u|_{W_p^i(K)}$

Local $\rightarrow$ global

To go from local to global bounds we need to uniformly bound $C_{m,n,\gamma,\sigma(\hat{K})}$ when K ranges over a collection of elements. We need to know how $\sigma(\hat{K})$ depends on affine transformation.

Definition: 4.4.13

Let $\{T^h\}$, $0 < h \le 1$ be a family of subdivisions of Ω such that $\max\{\operatorname{diam} T : T \in T^h\} \le h \operatorname{diam}(\Omega)$

$\{T^h\}$ is quasi-uniform if $\exists\, \rho > 0$ s.t

$\min\{\operatorname{diam} B_T : T \in T^h\} \ge \rho\, h \operatorname{diam}$,

where $B_T \subset T$ is the largest ball such that T is star-shaped w.r.t B_T.

$\{T^h\}$ is non-degenerate or regular if $\operatorname{diam} B_T \ge \rho \operatorname{diam} T$ $\forall\, T \in T^h$ some $\rho > 0$ s.t

Theorem 4.4.20

Let $\{T^h\}$ be non-degenerate and Ω polyhedral domain. Let (K, P, N) be a reference element ($P_{m-1} \subset P$, $N \subset (C^{\ell}(\bar{K}))'$, K star-shaped) and (T, P_T, N_T) affine-equivalent to (K, P, N), $T \in T^h$. Then

$$\Big(\sum_{T \in T^h} \| v - I^h v \|^p_{W^s_p(T)} \Big)^{1/p} \le C h^{m-s} |v|_{W^m_p(\Omega)}$$

$\forall v \in W^m_p(\Omega)$, $0 \le s \le m$.

Proof: $\sigma(\hat{T}) \le C$ follows from Prop 4.4.11 and Proof of Thm 4.4.20. The result follows by applying the local estimate in Thm 4.4.4 on each element. $\boxtimes$