Lecture 8

Inverse estimate and Interpolation of non-smooth functions 4.5, 4.8

Let K be bounded in $\mathbb{R}^n$. Given v let $\hat{v}(\hat{x}) = v((\operatorname{diam} K)\hat{x})$, $\forall \hat{x} \in \hat{K}$,

$\hat{K} = \{ \frac{1}{\operatorname{diam} K} x : x \in K \}$ (Scaling)

$|\hat{v}|_{W^k_p(\hat{K})} = h_K^{k-\frac{n}{p}} |v|_{W^k_p(K)}$ ($h_K = \operatorname{diam} K$)

$\uparrow$ change of variables

let $\hat{P} = \{ \hat{v} : v \in P \}$

Lemma 4.5.3 Local inverse est.

let $\rho h \le h_K \le h$, $h \in (0,1]$

$P \subset W^l_p(K) \cap W^m_q(K)$, $1 \le p, q \le \infty$, $0 \le m \le l$

There exists $C(\hat{P}, \hat{K}, l, p, q, \rho)$ s.t

$$\|v\|_{W^l_p(K)} \le C h^{m-l+\frac{n}{p}-\frac{n}{q}} \|v\|_{W^m_q(K)}$$

Ex: $\|v\|_{W^1_2(K)} \le C h^{-1} \|v\|_{L^2(K)}$

Proof: We first consider $m=0$

$\dim(P) < \infty \Rightarrow \|\hat{v}\|_{W^j_p(\hat{K})} \le C \|\hat{v}\|_{L_q(\hat{K})}$ $\forall v \in P$.

equivalence of norms.

Also $|\hat{v}|_{W^j_p(\hat{K})} \le C \|\hat{v}\|_{L_q(\hat{K})}$

$$|v|_{W^j_p(K)} = h_K^{-j+\frac{n}{p}} |\hat{v}|_{W^j_p(\hat{K})} \le h_K^{-j+\frac{n}{p}} \|\hat{v}\|_{L_q(\hat{K})}$$

$$= C h_K^{-j+\frac{n}{p}} h_K^{-\frac{n}{p}} \|v\|_{L_q(K)} = C \underbrace{h^{l-j}}_{\le 1} h_K^{-l+\frac{n}{p}-\frac{n}{q}} \|v\|_{L_q(K)}$$

Therefore $\|v\|_{W^l(K)} \le C h^{-l+\frac{n}{p}-\frac{n}{q}} \|v\|_{L_q(K)}$

$0 < m \le l$ follows by applying this to $D^\alpha v$. ∎

Theorem 4.5.4 Global bound

Let $\{T^h\}$ be quasi-uniform, all assumptions from Lemma 4.5.3 and $V_h = \{v : v|_T \in P_T \ \forall T \in T^h\}$

Then $C = c(l, p, q, \rho)$

$$\left(\sum_T \|v\|^p_{W^l_p(T)} \right)^{1/p} \le C h^{m-l+\min(0, \frac{n}{p} - \frac{n}{q})} \left(\sum_{T \in T_h} \|v\|^q_{W^m_q(T)} \right)^{1/q}$$

$\forall v \in V_h$.

Proof: Quasi-uniform $\Rightarrow$

$$\|v\|_{W^l_p(T)} \le C(\hat{P}_T, \hat{T}, l, p, q, \rho) h^{m-l+\frac{n}{p} - \frac{n}{q}} \|v\|_{W^m_q(T)} \quad \forall T \in T_h$$

The affine equivalence $\Rightarrow$ uniform bound of C

We can raise to p and sum

$$\left(\sum_T \|v\|^p_{W^l_p(T)} \right)^{1/p} \le C h^{m-l+\frac{n}{p} - \frac{n}{q}} \left(\sum \|v\|^p_{W^m_q(T)} \right)^{1/p}$$

$p \ge q \Rightarrow \left(\sum \|v\|^p_{W^m_q(T)} \right)^{1/p} \le \left(\sum \|v\|^q_{W^m_q(T)} \right)^{1/q}$

$p < q \Rightarrow$ Hölder

$$\left(\sum_T \|v\|^p_{W^m_q(T)} \right)^{1/p} \le \underbrace{\left(\sum_T 1 \right)^{\frac{1}{p} - \frac{1}{q}}}_{C h^{-\frac{n}{p} + \frac{n}{q}}} \cdot \left(\sum_T \|v\|^q_{W^m_q(T)} \right)^{1/q}$$

■

The Scott-Zhang interpolator 4.8

Let $\{T^h\}$ be a family of non-degenerate triangulations of $\Omega \subset \mathbb{R}^n$, $n = 2, 3$

We consider C^0 Lagrange finite element of degree $m-1$.

In order for point evaluations to be ok we need by Sobolev-inequality

$m - \frac{n}{p} > 0 \quad p > 1$
$m - \frac{n}{p} \geq 0 \quad p = 1$ $\Rightarrow W_p^m(\Omega) \subset C^0(K)$

Let $(K, P_K, \mathcal{N}_K)$ be a finite element so that $P_K = P_{m-1}$

$\mathcal{N}_K \subset (C^0(K))' \Rightarrow N_i^K(u) = u(P_i)$ is well defined

Local interpolant

$$(I^K u)\, x = \sum_{i=1}^{k} N_i^K \varphi_i^K(x) = \sum_{N \in \mathcal{N}_K} N^K(u) \varphi_N^K(x)$$

Global interpolant

$$(I^h u)(x) = \sum_z N_z(u) \phi_z(x), \quad u \in C(\Omega)$$

sum over all nodes of T^h.

$N_z(u) = u(z)$, $\phi_z(z') = \begin{cases} 1 & z = z' \\ 0 & z \neq z' \end{cases}$

We have

$$\| I^h u \|_{W_p^1(\Omega)} \leq C \| u \|_{W_p^m(\Omega)}$$

$$\| u - I^h u \|_{W_p^s(\Omega)} \leq C h^{m-s} \| u \|_{W_p^m(\Omega)} \quad s = 0, 1$$

If $p = 2$, $m \geq 2$ (2D, 3D)

Two issues

* $u \notin C^0(\Omega)$
* We need $\| u - I^h u \|_{L_2(\Omega)} \leq$
  $(m = 1)$ $\leq C h \| u \|_{H(\Omega)}$

  A posteriori e.g.

  FE Convergence without assuming $u \in H^2(\Omega)$.

We instead define

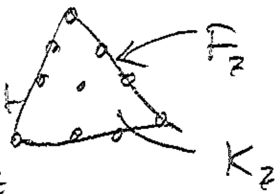
$$(\tilde{I}^h u)(x) = \sum_z \tilde{N}_z(u) \phi_z(x)$$

where $\tilde{N}_z(u)$ is a local average of u. We want

* $\tilde{I}^h v = v \quad \forall v \in V^h$
* $\tilde{I}^h v|_{\partial\Omega} = 0 \quad \text{if } v|_{\partial\Omega} = 0$
* $\|\tilde{I}^h v\|_{W_p^1(\Omega)} \le C \|v\|_{W_p^1(\Omega)}$
* Interpolation error bounds

The region to take mean over can be chosen in different ways. e.g.

For each node z pick element K_z



(i) $z \in \text{int}(K_z)$ let $\tilde{K}_z = K_z$

(ii) $z \in \text{int}(F_z)$, $\tilde{K}_z = F_z$

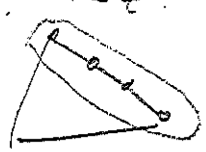
(iii) z vertex of K_z, $\tilde{K}_z = F_z$ or $\tilde{K} = K_z$

(iv) if $z \in \partial\Omega$ $\tilde{K}_z = F_z \subset \partial\Omega$

Given $(K_z, P_z, \mathcal{N}_z)$ define $(\tilde{K}_z, \tilde{P}_z, \tilde{\mathcal{N}}_z)$ by

$\tilde{P}_z = \{ f|_{\tilde{K}_z} : f \in P_z \}$

and assume $\tilde{\mathcal{N}}_z \subset \mathcal{N}_z$ determines $\tilde{P}_z$ i.e. we assume that the restriction of functions to $\tilde{K}_z$ are themselves appropriate functions for an element with a subset of the nodes from K_z.



$(\tilde{K}_z, \tilde{P}_z, \tilde{\mathcal{N}}_z)$ is a finite element (on K_z or F_z).

We have $\phi_N^{K_z}|_{\tilde{K}_z} = \phi_N^{\tilde{K}_z} \quad \forall N \in \tilde{\mathcal{N}}_z$

$\phi_N^{K_z}|_{\tilde{K}_z} = 0 \quad \forall N \in \mathcal{N}_z \setminus \tilde{\mathcal{N}}_z$

In this case $N_z^{\tilde{K}_z}(u) = u(z)$.

Local basis (on $\tilde{K}_z$) is given by

$\{\phi_N^{\tilde{K}_z}\}_{N\in\tilde{N}_z}$ with $L_2(\tilde{K}_z)$-dual

$\{\psi_N^{\tilde{K}_z}\}_{N\in\tilde{N}_z}$ i.e. $\int_{\tilde{K}_z} \phi_M^{\tilde{K}_z}\psi_M^{\tilde{K}_z} = \delta_{MN}$

with $\psi_N^{\tilde{K}_z} \in \tilde{P}_z$

The existance of $\psi_N^{\tilde{K}_z}$ is guaranteed by Riesz rep. theorem.

We can extend to $\{\psi_N^{\tilde{K}_z}\}_{N\in N_z}$ by letting $\psi_N^{\tilde{K}_z} = 0$ $N \in N_z \setminus \tilde{N}_z$.

Therefore $\int_{\tilde{K}_z} \phi_N^{K_z}(x)\psi_M^{\tilde{K}_z}(x)\,dx = \delta_{M,N}$ $\forall N, M \in N_z$

For any $u \in L^1(\tilde{K}_z)$ for all nodes z,

$$\tilde{I}^h u = \sum_z \left(\int_{\tilde{K}_z} \psi_{N_z}^{\tilde{K}_z}(x)u(x)\,dx\right)\phi_z$$

Theorem 4.8.7

$\tilde{I}^h v = v \quad \forall v \in V^h$

$$\tilde{I}^h v = \sum_z \Big(\underbrace{\int_{\tilde{K}_z} \psi_{N_z}^{\tilde{K}_z}(x)u(x)\,dx}_{u=\sum \alpha_i\phi_i,\ \int\psi_i\phi_j=\delta_{ij}}\Big)\phi_z = \sum_z u(z)\phi_z = v$$

$v|_{\partial\Omega} = 0 \Rightarrow \tilde{K}_z$ chosen so $\tilde{I}^h v|_{\partial\Omega} = 0$.

Theorem 4.8.12

Let T_h be non-degenerate, $v \in W_p^k(\Omega)$

$0 \le k \le m$, $1 \le p \le \infty$

$$\left(\sum_{K\in T_h} h_K^{p(s-k)} \|v - \tilde{I}^h v\|_{W_p^s(K)}^p\right)^{1/p} \le C|v|_{W_p^k(\Omega)}$$

for $0 \le s \le k$.

Corollary 4.8.15 $\left(\sum_{K\in T_h} \|\tilde{I}^h v\|_{W_p^k(K)}^p\right)^{1/p} \le C|v|_{W_p^k}$

and with $h = \max_{K\in T_h} \operatorname{diam} K$

$$\left(\sum \|v - \tilde{I}^h v\|_{W_p^s(K)}^p\right)^{1/p} \le C h^{k-s} |v|_{W_p^k(\Omega)}$$