Lecture 9

Global and local error bounds

Chapter 9.1–9.3

We consider the Poisson equation on weak form: Find $u \in H^1_0(\Omega)$ s.t

$$a(u,v) = \int_\Omega \alpha \nabla u \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \forall v \in V$$

$$0 < \alpha_0 \le \alpha(x) \le \alpha_1 \quad x \in \Omega,$$

We assume α and f to be piece smooth on a non-degenerate family of simplicial meshes $\{T^h\}_{0<h\le 1}$

and $h(x) = \max_{K \in star(x)} (K)$

star(x)

We have $h(x) \ge diam(K) \quad \forall x \in K$

and $h|_K \le C\, diam(K)$ since the meshes are non-degenerate.

Similarly we assume

$$\max_{x\in K} \alpha(x) \le C \min_{x \in K} \alpha(x) \quad \forall K \in T^h.$$

We let V_h be the space of piecewise polynomials of degree k with interpolant (nodal) I^h. From Thm 4.4.4 assuming sufficient smoothness on u

$$\|u - I^h u\|^2_{H^1(K)} \le \sum_{|\beta|=k} \int_K h^{2k-2} |D^\beta u|^2 dx \quad \forall K \in T^h$$

In energynorm under the assumption on α and optimality of $u_h \in V_h$

$$\|u - u_h\|_E \le C \left(\sum_{|\beta|=k} \int_\Omega \alpha(x) h(x)^{2k-2} |D^\beta u|^2 dx \right)^{1/2}$$

<u>Theorem 9.1.7</u>

We have $\|u-u_h\|_E \le C \| \sqrt{\alpha}\, h^{k-1} |\nabla_k u| \|_{L^2(\Omega)}$

where $|\nabla_k u|(x) = \left(\sum_{|\beta|=k} | D^\beta u(x)|^2 \right)^{1/2}$,

C is independent of α and h

L^2 estimate is given by a duality argument, let $w \in H_0^1(\Omega)$ solve

$a(v, w) = (v, u-u_h) \qquad \forall v \in H_0^1(\Omega)$.

$$\|u-u_h\|^2_{L^2(\Omega)} = a(u-u_h, w - I_h w) =$$

$$= \left(\int_\Omega \alpha^{2-\lambda} (h\nabla(u-u_h))^2 dx \right)^{1/2} \left(\int_\Omega \alpha^\lambda |\nabla w - I_h w| h^{-2} dx \right)^{1/2}$$

$$\le C \left(\int_\Omega \alpha^{2-\lambda} (h \nabla u - u_h)^2 dx \right)^{1/2} \left(\int_\Omega \alpha^\lambda |\nabla_2 w|^2 dx \right)^{1/2},$$

$0 \le \lambda \le 2$ is arbitrary.

A priori information gives (under additional information on α, Ω)

$|\nabla_2 w| \le C \|u-u_h\|$

<u>Theorem 9.1.8</u>

If $|\nabla_2 w| \le C\|u-u_h\|$ it holds

$$\|u-u_h\|_{L^2(\Omega)} \le C \left(\int_\Omega \alpha(x)^2 h(x)^2 |\nabla(u-u_h)|^2 dx \right)^{1/2},$$

C depends on α.

Given the bound in energy norm it holds.

<u>Theorem 9.1.10</u>

There is a constant $K>0$ s.t. if the mesh size variation $\| |\nabla h| \|_{L^\infty(\Omega)} < K$

then $\|u-u_h\|_{L^2(\Omega)} \le C \| \sqrt{\alpha}\, h^2 |\nabla_2 u| \|_{L^2(\Omega)}$

Note that $\| |\nabla h| \|_{L^\infty(\Omega)}$ still allows for strong mesh gradings, e.g. geometric grading

Error estimators 9.2

Lemma 9.2.1

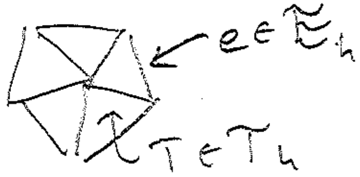
It holds

$$a(u-u_h, v) = R(v), \quad \forall v \in V$$

where $R \in V'$ is the residual

$$R(v) := \sum_{T \in \mathcal{T}^h} \int_T (f + \nabla \cdot (\alpha \nabla u_h)) v \, dx + \sum_{e \in \mathcal{E}^h} \int_e [\alpha n_e \cdot \nabla u_h]_{n_e} v \, ds \quad \forall v \in V$$

where n_e denotes a unit normal to e.



This follows from integration by parts, see Lecture 1.

Remember $[\phi]_n(x) := \lim_{\varepsilon \to 0} \phi(x+\varepsilon n) - \phi(x - \varepsilon n)$

We note that R has two distinct parts:

1) $R_A|_T = f + \nabla \cdot \alpha \nabla u_h |_T$, $R_A \in L_1(\Omega)$

2) $R_J(v) := \sum_e \int_e [\alpha n_e \cdot \nabla u_h] v \, ds \quad \forall v \in V$

the "jump" term.

By letting $v = u - u_h$ we get

$$\alpha_0 |u-u_h|^2_{H^1(\Omega)} \le |R(u-u_h)| \le \|R\|_{H^{-1}(\Omega)} \cdot \|u-u_h\|_{H^1(\Omega)}$$

recall $\|r\|_{H^{-1}} := \sup_{z \in H^1_0(\Omega)} \frac{r(z)}{\|z\|_{H^1(\Omega)}}$.

i.e. $\|u - u_h\|_{H^1(\Omega)} \le C \|R\|_{H^{-1}(\Omega)}$.

Negative norms are difficult to compute.

Theorem: (also from Lecture 1)

$$|u-u_h|_{H^1(\Omega)} \le C \Big(\sum_{T \in \mathcal{T}^h} \mathcal{E}_T(u_h)^2 \Big)^{1/2},$$

$$\mathcal{E}_T(u_h) := h_T^2 \|R_A\|^2_{L^2(T)} + \sum_{e \in \partial T} h_e \| [\alpha n \cdot \nabla u_h]_n \|^2_{L^2(e)}$$

Local error estimate 9.3

We have global bounds of $u-u_h$.

We seek local bounds to make sure the error $u-u_h|_T$ and the error indicator $\mathcal{E}_T(u_h)$ are correlated.

We have $a(u-u_h, v) = (R_A, v)$ $\forall v \in H^1_0(T)$

makes jumps vanish.

Therefore

$$\left|\int_T R_A v\,dx\right| \le \alpha_1 |u-u_h|_{H^1(T)} |v|_{H^1(T)}$$

and $$\alpha_1 |u-u_h|_{H^1(T)} \geq \sup_{0\neq v\in H^1_0(T)} \frac{\left|\int_T v R_A\,dx\right|}{|v|_{H^1(T)}}$$

We assume f and α to be piecewise polynomials. This means that R_A is piecewise polynomial

It holds

$$\sup_{v\in H^1_0(T)} \frac{\int_T v P\,dx}{|v|_{H^1(T)}} \geq C h_T \|P\|_{L^2(T)}$$

where C depends on the degree of P (polynomial) and the chunkiness of the mesh.

With $P = R_A$ we get

$$\alpha_1 |u-u_h|_{H^1(T)} \geq C h_T \|R_A\|_{L^2(T)}$$

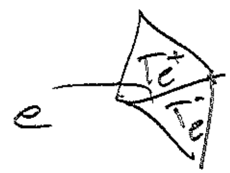
Local Lower bound.

We turn to the jump term.

$$a(e_h, v) = \int_e [\alpha n\cdot\nabla u_h]\, v\,ds \quad \forall v\in V_e$$

where $V_e = \{v\in H^1_0(T_e) : \int_{T_e^+} vP\,dx = \int_{T_e^-} vP\,dx = 0 \ \forall P\in P_r\}$

Polynomials of degree r which R_A is.



Then $$\left|\int_e [\alpha n\cdot\nabla u_h]\, v\,ds\right| \le \alpha_1 |u-u_h|_{H^1(T_e)} |v|_{H^1(T_e)}$$

and $$\alpha |u-u_h|_{H^1(T_e)} \geq \sup_{0\neq v\in V_e} \frac{\left|\int_e [\alpha n\cdot\nabla u_h]\, v\,ds\right|}{|v|_{H^1(T_e)}}$$

Since $[\alpha n \cdot \nabla u]_n$ is a polynomial on the edge e it can be shown

$$\sup_{0 \neq v \in H_0^1(T_e)} \frac{\left|\int_e [\alpha n \cdot \nabla u_h]_n v \, ds\right|}{|v|_{H^1(T_e)}} \geq C h_e^{1/2} \|[\alpha n \cdot \nabla u_h]_n\|_{L^2(e)}$$

Combining these results gives us

<u>Theorem 9.3.6</u> Suppose α and f are piecewise polynomials, then the local lower bound

$$\alpha_1 |u - u_h|_{H^1(T_e)} \geq c\, \mathcal{E}_e(u_h)$$

where $\mathcal{E}_e(u_h)^2 := \sum_{T \subset T_e} h_T^2 \|f + \nabla \cdot \alpha \nabla u_h\|_{L^2(T)}^2 + h_e \|[\alpha n \cdot \nabla u_h]_n\|_{L^2(e)}^2$

Squaring and summing gives a global lower bound

<u>Theorem 9.3.7</u>

$$\alpha_1 |u - u_h|_{H^1(\Omega)} \geq c \left(\sum_{e \in T^h} \mathcal{E}_e(u_h)^2 \right)^{1/2}$$

Note that the reverse of 9.3.6 is not true. The local bound tells us that it is reasonable to refine an element with high error indicator $\mathcal{E}_e(u_h)$.