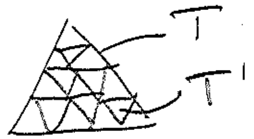
Lecture 10

A convergent adaptive algorithm 9.5

We consider
$$\begin{cases} -\nabla\cdot\alpha\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

With FE approximation $\{u_j\}_{j\geq 1}$ on corresponding triangulations $\{\mathcal{T}_j\}_{j\geq 1}$ where $\mathcal{T}_{j+1}$ is a refinement of $\mathcal{T}_j$, $j \geq 1$. We assume



large enough

$$\gamma_1 h_T \leq h_{T'} \leq h_T$$

if $T \in \mathcal{T}^j$, $T' \in \mathcal{T}^{j+1}$, $T' \subset T$, $0<\gamma_1<1$

We recall the notation

$$R_{A,j}|_T = f + \underbrace{\nabla\cdot\alpha\nabla u_j|_T}_{\in V_j \text{ degree } k}$$

$$J_j|_e = [\alpha n_e \cdot \nabla u_j]_{n_e}$$

We let Q_j be the L^2-projection onto piecewise polynomials of degree $k-1$ w.r.t. $\mathcal{T}_j$ (discontinuous) and

let $\operatorname{osc}_j(T) = h_T \|R_{A,j} - Q_j R_{A,j}\|_{L^2(T)}$, $T \in \mathcal{T}^j$

and $\operatorname{osc}_j(\Omega) = \left(\sum_{T\in\mathcal{T}^j} \operatorname{osc}_j(T)^2\right)^{1/2}$.

and $|||v|||_T = \left(\int_T \alpha|\nabla v|^2\,dx\right)^{1/2}$

Lemma 9.5.3

Suppose $T \in \mathcal{T}^j$ contains a node of $\mathcal{T}^{j+1}$ in the interior. Then we have

$$h_T^2 \|R_{A,j}\|^2_{L^2(T)} \leq C\left(|||u_{j+1} - u_j|||_T^2 + \operatorname{osc}_j(T)^2\right)$$

C depends on shape regularity and α_1 and γ_1.

$(\alpha(x) \geq \alpha_1)$

Proof:

Let p be the node of $\mathcal{T}^{j+1}$ in T and ϕ_p be piecewise linear w.r.t $\mathcal{T}^{j+1}$ equal to 1 in p and 0 in other nodes. Then $v = \phi_p \cdot Q_j R_{A,j}$ belongs to FEspace of $\mathcal{T}^{j+1}$ (since $Q_j R_{A,j}$ is $k-1$ poly in T) and vanishes outside T (since ϕ_p vanishes outside).

In particular $v\big|_{\partial T_j} = 0$.

$$\| Q_j R_{A,j} \|^2_{L^2(T)} \underset{\uparrow}{\le} C \int_\Omega \phi_p (Q_j R_{A,j})^2 \, dx$$

if $k=1$ const, C dep size of $|T^{j+1}|, |T|$
also ok for general polynomials

$$= C \left(\int_\Omega (Q_j R_{A,j} - R_{A,j}) \phi_p Q_j R_{A,j} \, dx + \int_\Omega R_{A,j} v \, dx \right)$$

$$\le C \Big(\| R_{A,j} - Q_j R_{A,j} \|_{L^2(T)} \cdot \| Q_j R_{A,j} \|_{L^2(T)}$$

$$+ a(u - u_j, v) \Big)$$

We also have

V^{j+1} FEspace assoc. $\mathcal{T}^{j+1}$

$$a(u - u_j, v) = a(u_{j+1} - u_j, v)$$

$$\le |||u_{j+1} - u_j|||_T \cdot |||v|||_T$$

$$\le \alpha_1^{1/2} |||u_{j+1} - u_j|||_T \cdot |v|_{H^1(T)}$$

$$\overset{\text{inv.}}{\le} C h_T^{-1} |||u_{j+1} - u_j|||_T \cdot \|v\|_{L^2(T)}$$

$(\phi_p \le 1)$

$$\le C h_T^{-1} |||u_{j+1} - u_j|||_T \cdot \| Q_j R_{A,j} \|_{L^2(T)}$$

$$\therefore \| Q_j R_{A,j} \|_{L^2(T)} \le C \Big(h_T^{-1} |||u_{j+1} - u_j|||_T + \| R_{A,j} - Q_j R_{A,j} \|_{L^2(T)} \Big)$$

$$\Rightarrow h_T^2 \| Q_j R_{A,j} \|^2_{L^2(T)} \le C |||u_{j+1} - u_j|||_T^2 + C \operatorname{osc}_j(T)^2$$

$$h^2 \| R_{A,j} \|^2 \le h^2 \underbrace{\| R_{A,j} - Q_j R_{A,j} \|^2}_{\text{osc}} + \underbrace{\| Q R_{A,j} \|^2 h^2}_{\text{ok}}$$

$$h_T^2 \| R_{A,j} \|^2_{L^2(T)} \le C' |||u_{j+1} - u_j|||_T^2 + C' \operatorname{osc}_j(T)^2$$

Lemma 9.5.7

Let e be interior face of $\mathcal{T}^j$ and $T_e = T_e^+ \cup T_e^-$. Assume T_e^+, T_e^- contain interior nodes belonging to $\mathcal{T}^{j+1}$. Then

$$h_e \| J_j \|^2_{L^2(e)} \le C\Big(\sum_{T \in T_e} |||u_{j+1} - u_j|||_T^2 + osc_j(T)^2 \Big)$$

where C depends on shape regularity of $\{\mathcal{T}^j\}_{j\ge 1}$, θ_1, θ_0 and γ_1

Proof: Similar techniques to 9.5.3 ∎
See book

Corollary 9.5.4

$$\mathcal{E}_e(u_j)^2 \le C\Big(\sum_{T \in T_e} |||u_{j+1} - u_j|||_T^2 + osc_j(T)^2 \Big)$$

Proof: Follows by combining the Lemmas.

Note that using G.O.

$$|||u - u_j|||^2 - |||u - u_{j+1}|||^2 = -2a(\overset{\substack{u_{j+1}\\\downarrow}}{u}, u_j) + |||u_j|||^2 + 2a(u, u_{j+1}) - |||u_{j+1}|||^2 = |||u_{j+1}|||^2 - |||u_j|||^2$$

where $a(u, u_j) = a(u_j, u_j)$ and $a(u, u_{j+1}) = a(u_{j+1}, u_{j+1})$

$$= a(u_{j+1} - u_j, u_{j+1} + u_j) = |||u_{j+1} - u_j|||^2$$

$a(u_{j+1} - u_j, v_j) = 0$

i.e. $|||u_{j+1} - u_j|||$ measures error reduction.

If $osc_j(T)$ are small (smooth data)

The error estimator $\mathcal{E}_e(u_j)$ is a local estimate of the reduction of the error.

if 9.5... for all

$|||u - u_j|||^2 \le C\, |||u_{j+1} - u_j|||^2 = C|||u - u_j|||^2 - C|||u - u_{j+1}|||^2$

$C|||u - u_{j+1}|||^2 = (C-1)|||u - u_j|||^2 \Rightarrow |||u - u_{j+1}|||^2 \le \frac{C-1}{C}|||u - u_j|||^2$ if $(osc = 0)$.

One can show by counterexample that without the interior node constraint $\|u_{j+1} - u_j\| = 0$, $osc_j(T) = 0$ but $\mathcal{E}_e(u_j) \ne 0$.

Lemma 9.5.14

Let $T \in \mathcal{T}^j$ be divided into elements in $\mathcal{T}^{j+1}$ with $h_{T'} \le \gamma_2 h_T$ $T' \subset T^{j+1}$

$\leftarrow$ smallest/any $T' \subset T$.

Then for any $0 < \delta < 1$

$$\sum_{T' \in \mathcal{T}_{T,j+1}} osc_{j+1}(T')^2 \le (1+\delta)\gamma_2^2 osc_j(T)^2 + C\delta^{-1} |||u_{j+1} - u_j|||_T^2$$

where $T' \in \mathcal{T}_{T,j+1} = \{T' \in \mathcal{T}^{j+1} : T' \subset T\}$

C depends on $\alpha_0, \alpha_1, \|\nabla \alpha\|_{L^\infty(\Omega)}$ shape regularity of $\{\mathcal{T}^j\}_{j \ge 1}$.

Proof: Straight forward, see Lect.

Convergent adaptive Strategy

Let $\mathcal{T}^1$ be the initial triangulation of Ω and $0 < \theta_1, \theta_2 < 1$

SOLVE $\to$ ESTIMATE $\to$ MARK $\to$ REFINE

For $j = 1, 2, ..$

- Compute $u_j \in V_j$ s.t
  $a(u_j, v) = \int_\Omega f \cdot v \quad \forall v \in V_j$
- Compute $\mathcal{E}_e(u_j)$ for all interior edges e.
  and $osc_j(T)$ for all elements
- Let $\mathcal{F}^j$ be the set of edges
  Choose $\hat{\mathcal{F}}^j \subset \mathcal{F}^j$ s.t
  $$\theta_1^2 \sum_{e \in \mathcal{F}^j} \mathcal{E}_e(u_j)^2 \le \sum_{e \in \hat{\mathcal{F}}^j} \mathcal{E}_e(u_j)^2$$
  and $\hat{\mathcal{T}}^j \subset \mathcal{T}^j$ so that
  $$\theta_2^2\, osc_j(\Omega)^2 \le \sum_{T \in \hat{\mathcal{T}}_j} osc_j(T)$$
- Refine $\mathcal{T}^j$ to get $\mathcal{T}^{j+1}$ so its non-degenerate, each edge contains a new node and each element contains a new node.

Theorem 9.5.18

There exists $0<\theta<1$, $\eta>0$ so that

$$|||u-u_{j+1}|||^2 + \eta\, osc_{j+1}(\Omega)^2 \le \theta\left(|||u-u_j|||^2+\eta\, osc_j(\Omega)^2\right)$$

η, θ depend on α, shape regularity, θ_1, θ_2, "refinement amount" γ_1, γ_2.

Proof: $C_1, C_2, C_3, C_4 \geq 1$

$$\theta_1^2 |||u-u_j|||^2 \overset{\text{a post}}{\le} C_1\theta_1^2 \sum_{e\in\mathcal{E}_j} \mathcal{E}_e(u_j)^2 \overset{\text{marking}}{\le} C_1 \sum_{e\in\hat{\mathcal{E}}_j} \mathcal{E}_e(u_j)^2$$

int. node

$$\le C_1C_2\left(|||u_{j+1}-u_j|||^2 + osc_j(\Omega)^2\right)$$

$$\le C_1C_2\left(|||u-u_j|||^2 - |||u-u_{j+1}|||^2\right) + C_1C_2\, osc_j(\Omega)^2$$

$$\Rightarrow |||u-u_{j+1}|||^2 \le \left(1-\frac{\theta_1^2}{C_1C_2}\right)|||u-u_j|||^2 + osc_j(\Omega)^2$$

$$osc_{j+1}(\Omega)^2 = \sum_{T\in\mathcal{T}^j}\ \sum_{T'\in\mathcal{T}_{T,j+1}} osc_{j+1}(T')^2$$

$$\le \sum_{T\in\hat{\mathcal{T}}^j}\ \sum_{T'\in\mathcal{T}_{T,j+1}} osc_{j+1}(T')^2 + \sum_{T\in\mathcal{T}^j\setminus\hat{\mathcal{T}}^j}\ \sum_{T'\in\mathcal{T}_{T,j+1}} osc_{j+1}(T')^2$$

$$\le (1+\delta)\left(\gamma_2^2\sum_{T\in\hat{\mathcal{T}}^j} osc_j(T)^2 + \sum_{T\in\mathcal{T}^j\setminus\hat{\mathcal{T}}^j} osc_j(T)^2\right) + C_3\delta^{-1}\, |||u_{j+1}-u_j|||_T^2$$

and

$$\gamma_2^2 \sum_{T\in\hat{\mathcal{T}}^j} osc_j(T)^2 + \sum_{T\in\mathcal{T}^j\setminus\hat{\mathcal{T}}^j} osc_j(T)^2 = osc_j(\Omega)^2 - (1-\gamma_2^2)\sum_{T\in\hat{\mathcal{T}}^j} osc_j(T)^2$$

$$\overset{\text{marking}}{\le} \left(1-(1-\gamma_2^2)\theta_2^2\right) osc_j(\Omega)^2.$$

$$\Rightarrow osc_{j+1}(\Omega)^2 \le \rho\, osc_j(\Omega)^2 + C\left(|||u-u_j|||^2 - |||u-u_{j+1}|||^2\right)$$

where $\delta>0$ is chosen so $\rho=(1+\delta)(1-(1-\gamma_2^2)\theta_2^2) < 1$.

Pick τ so $\kappa = \rho+\frac{1}{\tau} < 1$

$$|||u-u_{j+1}|||^2 + \tau\, osc_{j+1}(\Omega)^2 \le \left(1-\frac{\theta_1^2}{C_1C_2}\right)|||u-u_j|||^2 + \tau C_4\left(|||u-u_j|||^2 - |||u-u_{j+1}|||^2\right) + (\tau\rho+1)\, osc_j(\Omega)^2$$

$$\Rightarrow (1+\tau C_4)|||u-u_{j+1}|||^2 + \tau\, osc_{j+1}(\Omega)^2 \le \left(1+\tau C_4 - \frac{\theta_1^2}{C_1C_2}\right)|||u-u_j|||^2 + \kappa\tau\, osc_j(\Omega)^2$$

$$\Rightarrow \eta = \frac{\tau}{1+\tau C_4} \qquad \theta = \max\left(1-\frac{\theta_1^2}{C_1C_2(1+\tau C_4)}, \kappa\right)$$

chosen so

Main development: no need for interior node

ok.