Error estimation and adaptivity for multiscale methods

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Outline and Papers

Outline

- Model problem
- Motivation and previous work
- Derivation of proposed method with examples
- A posteriori error estimates
- Adaptive algorithm
- Application to oil recovery problem
- Conclusions and future work

Papers

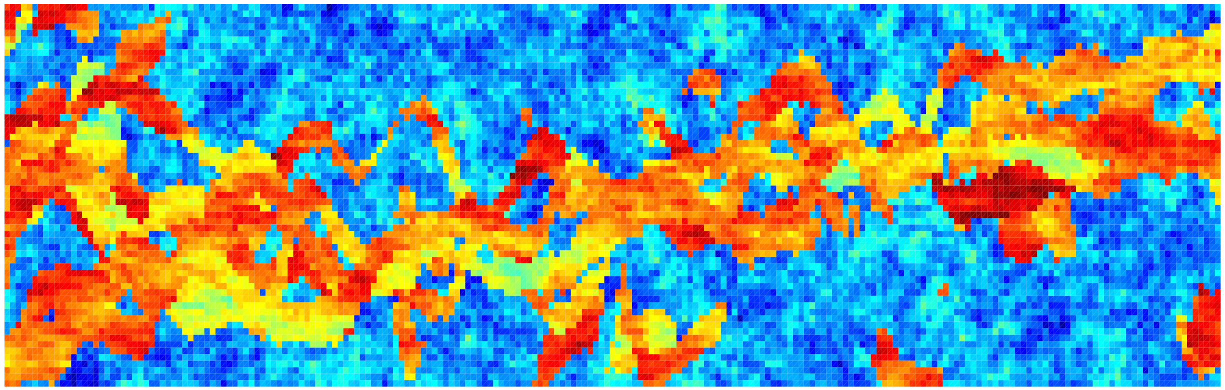
- M.G. Larson and A. Målqvist, *Adaptive Variational Multiscale Methods Based on A Posteriori Error Estimation: Energy Norm Estimates for Elliptic Problems*, CMAME 2007
- A. Målqvist, *Multiscale methods for elliptic problems* (in review MMS)

Model Problem

Elliptic equation: Find u such that

$$\mathcal{L}u = -\nabla \cdot a\nabla u + \nabla \cdot (bu) + cu = f \quad \text{in } \Omega$$

where $\mathcal{L}$ is elliptic with multiscale coefficients, f is a given function, and $\Omega \subset \mathbf{R}^d$ is a domain.



Weak form: Let $a(v, w) = (\mathcal{L}v, w)$ and $l(w) = (f, w)$ for all $v, w \in \mathcal{V}$, for an appropriate function space $\mathcal{V}$. Find $u \in \mathcal{V}$ such that,

$$a(u, v) = l(v), \quad \text{for all } v \in \mathcal{V}.$$

Motivation: Why Resolve the Coefficients?

Simple periodic example: Consider the Poisson equation

$$\mathcal{L}u = -\nabla \cdot a \nabla u = f,$$

with periodic coefficient $a = a(x/\epsilon)$ solved using the finite element method on a mesh of size H, we have (Hou-Wu-Cai),

$$\|\sqrt{a}\nabla(u - u_h)\|_{L^2(\Omega)} \leq C\frac{H}{\epsilon}\|f\|_{L^2(\Omega)},$$

- $\epsilon < H$ will give unreliable results even with exact quadrature.
- $\epsilon > H$ will be to computationally expensive to solve on a single mesh.

From now on we assume nothing on the coefficients, more then what is needed to guarantee existence and uniqueness.

Some Previous Works and Related Methods

- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson-Målqvist 05, Nolen et al. 08, Nordbotten 09
- Multiscale finite element method: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06
- Multiscale finite volume method: Jenny et al. 03
- Heterogeneous multiscale method: Engquist-E 03, E-Ming-Zhang 04
- Equation free: Kevrekidis et al. 05
- ...

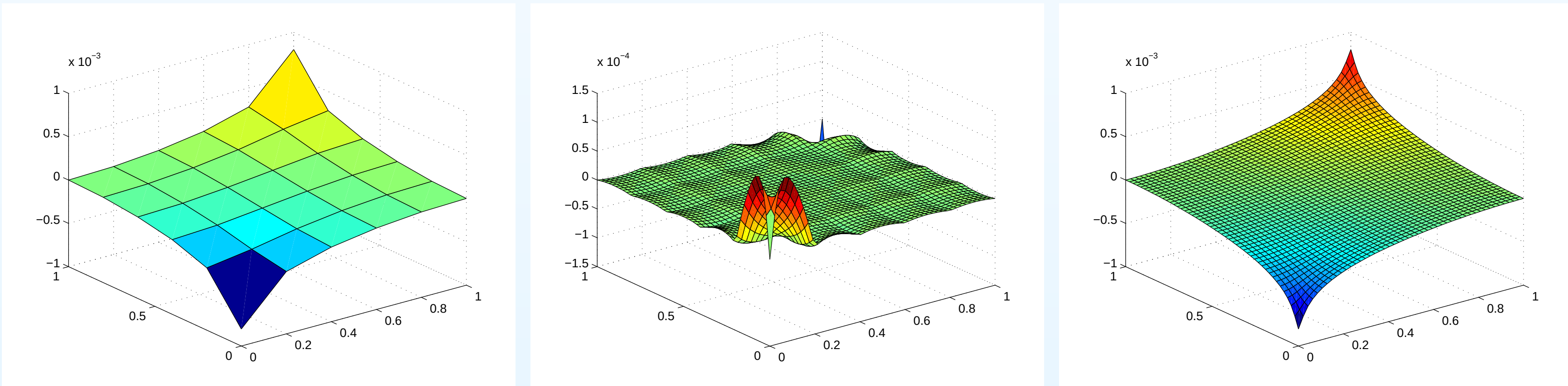
Local approximations (in parallel) on a fine scale are used to modified the coarse scale equation.

The Variational Multiscale Method (VMS)

The weak form reads: find $u \in \mathcal{V}$ such that,

$$a(u, v) = l(v), \quad \text{for all } v \in \mathcal{V}.$$

Now let $\boxed{\mathcal{V}_c \oplus \mathcal{V}_f = \mathcal{V}}$.



- $\mathcal{V}_c$ is a finite dimensional approximation of $\mathcal{V}$. (FE space)
- $\mathcal{V}_f$ can be chosen as e.g. hierarchical basis, $L^2(\Omega)$-orthogonal to $\mathcal{V}_c$, or wavelet modified hierarchical basis.

a-Orthogonal Split

We want to decouple the coarse and the fine scales. We introduce two multiscale projection operators, let $\mathcal{T}, \mathcal{T}^* : \mathcal{V}_c \to \mathcal{V}_f$ such that

$$a(v_c + \mathcal{T} v_c, v_f) = 0, \quad \text{for all } v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f$$
$$a(v_f, v_c + \mathcal{T}^* v_c) = 0, \quad \text{for all } v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f$$

Let $u = u_c + \mathcal{T} u_c + u_f \in \mathcal{V}_c \oplus \mathcal{V}_f$ and $v = v_c + \mathcal{T}^* v_c + v_f$ in the weak form,

$$a(u_c + \mathcal{T} u_c, v_c + \mathcal{T}^* v_c) = l(v_c + \mathcal{T}^* v_c) \quad \text{for all } v_c \in \mathcal{V}_c,$$
$$a(u_f, v_f) = l(v_f) \quad \text{for all } v_f \in \mathcal{V}_f,$$

since $a(u_c + \mathcal{T} u_c, v_f) = a(u_f, v_c + \mathcal{T}^* v_c) = 0$.

Modified Coarse Scale Equations

Find $u_c \in \mathcal{V}_c$ such that

$$a(u_c + \mathcal{T} u_c, v_c + \mathcal{T}^* v_c) = l(v_c + \mathcal{T}^* v_c) - a(u_f, v_c + \mathcal{T}^* v_c)$$

for all $v_c \in \mathcal{V}_c$ (note that $a(u_f, v_f) = l(v_f)$ for all $v_f \in \mathcal{V}_f$).

- If a is symmetric we get $\mathcal{T}^* v_c = \mathcal{T} v_c$ i.e. a symmetric formulation
- In standard VMS $\mathcal{T}^* v_c := 0$ and u_f is included in the coarse scale. The computation of $\mathcal{T} u_c + u_f \in \mathcal{V}_f$ is decoupled and done analytically on each coarse element using homogeneous Dirichlet boundary conditions.
- In MsFEM $u_f = 0$ and therefore not present in the coarse scale equations. Here the computation of $v_c + \mathcal{T} v_c$ is decoupled and solved numerically on each element (or larger domains) using approximate boundary conditions.

Approximation of $\mathcal{T}$, $\mathcal{T}^*$, and u_f

Let $u_c = \sum_i u_c^i \phi_i$ with $\{\phi_i\}$ a basis in $\mathcal{V}_c$.

We have $\mathcal{T} u_c = \sum_i u_c^i \mathcal{T}\phi_i$, $\mathcal{T}^* v_c = \sum_i v_c^i \mathcal{T}^*\phi_i$, and $u_f = \sum_i u_{f,i}$ where,

$$a(\mathcal{T}\phi_i, v_f) = -a(\phi_i, v_f) \quad \text{for all } v_f \in \mathcal{V}_f,$$

$$a(v_f, \mathcal{T}^*\phi_i) = -a(v_f, \phi_i) \quad \text{for all } v_f \in \mathcal{V}_f,$$

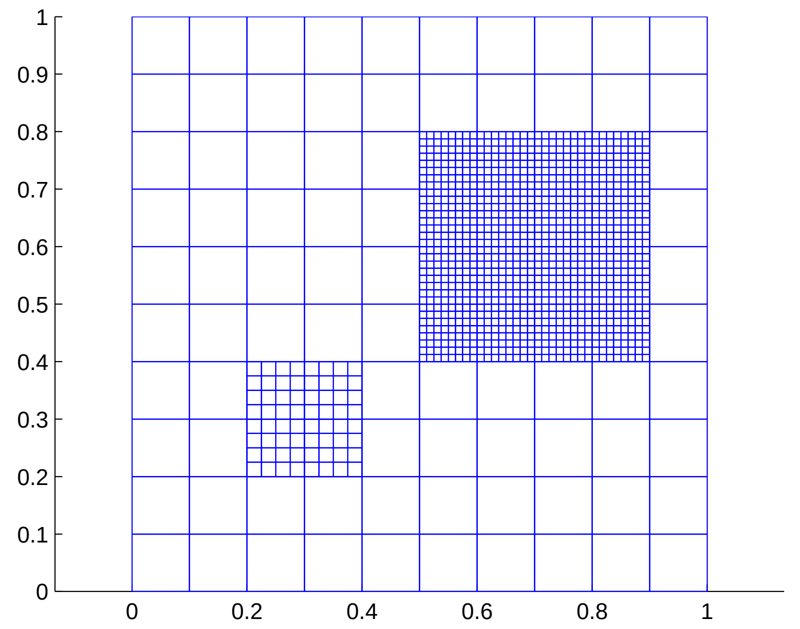
$$a(u_{f,i}, v_f) = (f\phi_i, v_f) \quad \text{for all } v_f \in \mathcal{V}_f.$$

We compte approximations $\tilde{\mathcal{T}}\phi_i, \tilde{\mathcal{T}}^*\phi_i$, and $U_{f,i}$ by

- restricting to a localized patches $\mathrm{supp}(\phi_i) \subset \omega_i$,
- discretizing with a fine subgrid on ω_i,
- and using homogeneous boundary conditions on the patch.

Refinements and Layers

We let H be coarse scale mesh size and h be fine scale mesh size. Further we let L denote the number of layers of coarse elements in the patch.



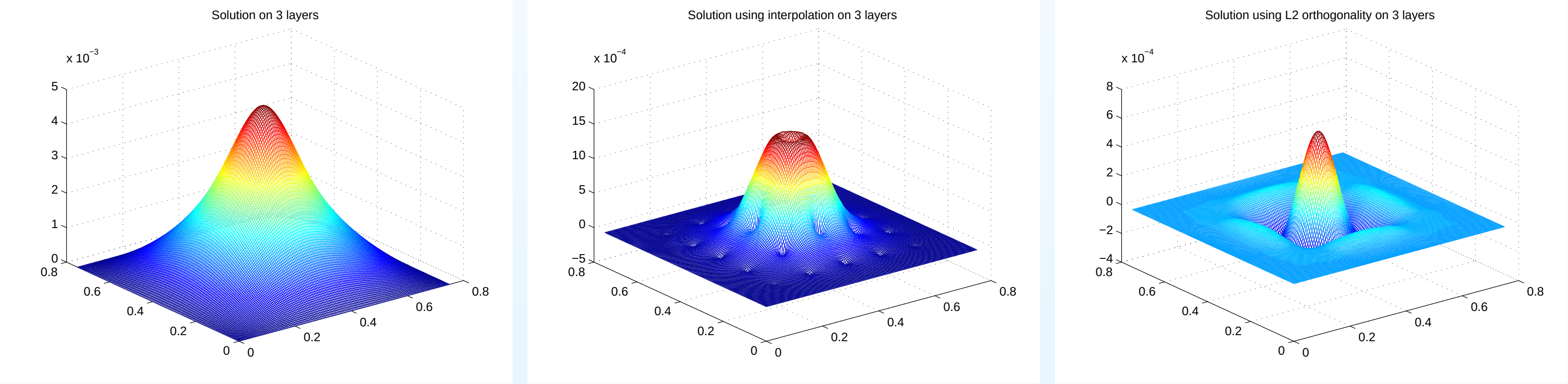
Since we exclude $\mathcal{V}_c$ from the fine scale the condition number is only $\sim \left(\frac{H}{h}\right)^2 \log(H/h)$ using an hierarchical split (Marion-Xu 95).

Simple Observation About Decay in $\mathcal{V}_f$ (Fourier)

Consider the Laplace equation,

$$-\Delta u = \varphi_i \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where φ_i has local support in Ω. The weak form reads: find $u \in \mathcal{W}$ s.t., $(\nabla u, \nabla v) = (\varphi_i, v)$ for all $v \in \mathcal{W}$.

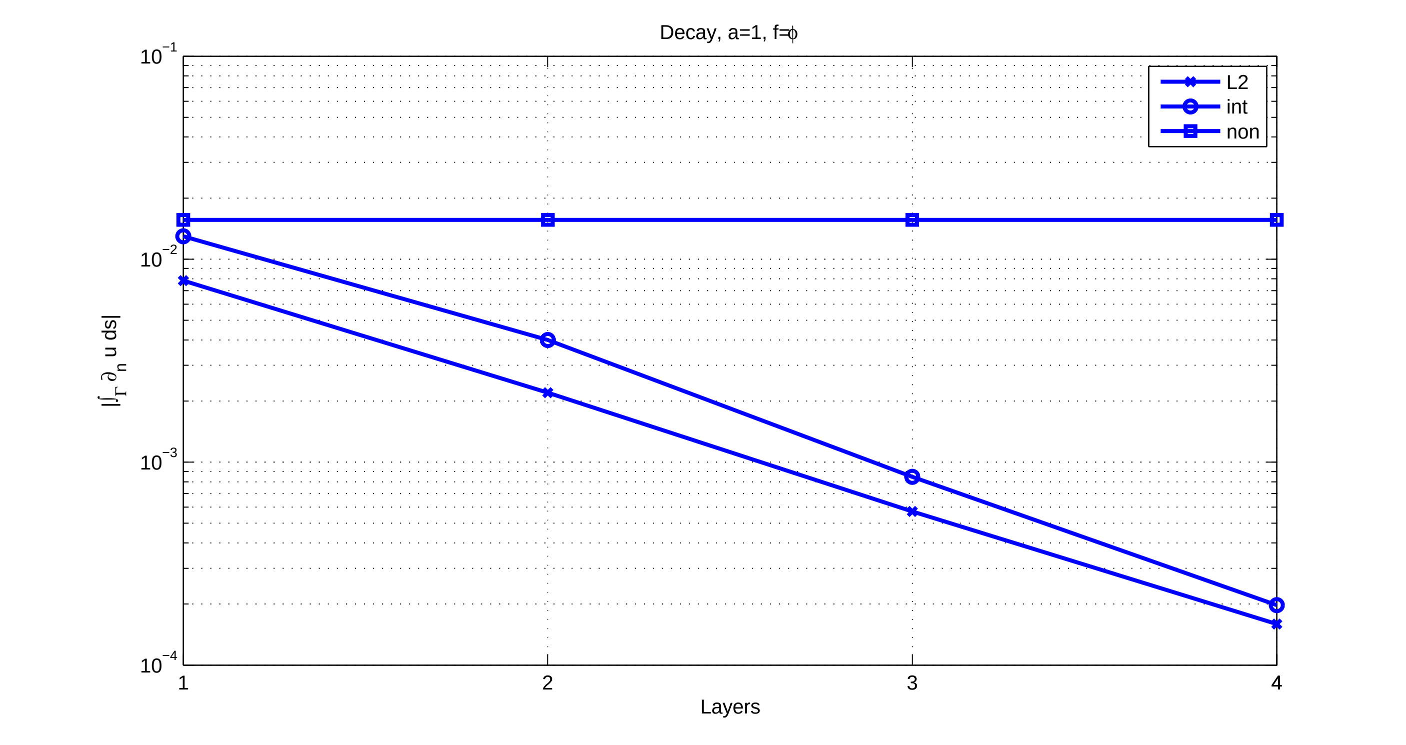


To the left $\mathcal{W} = \mathcal{V}_c \oplus \mathcal{V}_f$, middle $\mathcal{W} = \mathcal{V}_f$ using hierarchical split, and right $\mathcal{W} = \mathcal{V}_f$ using L^2-orthogonal split.

Constraints are realized using Lagrangian multipliers.

Simple Observation About Decay in $\mathcal{V}_f$

Decay of flux integrated over the boundary.



We see exponential decay with respect distance measured in number of coarse elements. This effect gives rapid convergence as the patch size increases.

The Proposed Multiscale Method

- Let $\tilde{\mathcal{T}}, \tilde{\mathcal{T}}^*$ be the computable approximations of $\mathcal{T}, \mathcal{T}^*$
- Let U_f be the computable approximation of u_f

We get: find $U_c \in \mathcal{V}_c$ such that

$$\boxed{a(U_c + \tilde{\mathcal{T}} U_c, v_c + \tilde{\mathcal{T}}^* v_c) = l(v_c + \tilde{\mathcal{T}}^* v_c) - a(U_f, v_c + \tilde{\mathcal{T}}^* v_c)}$$

for all $v_c \in \mathcal{V}_c$. On matrix form this leads to a system,

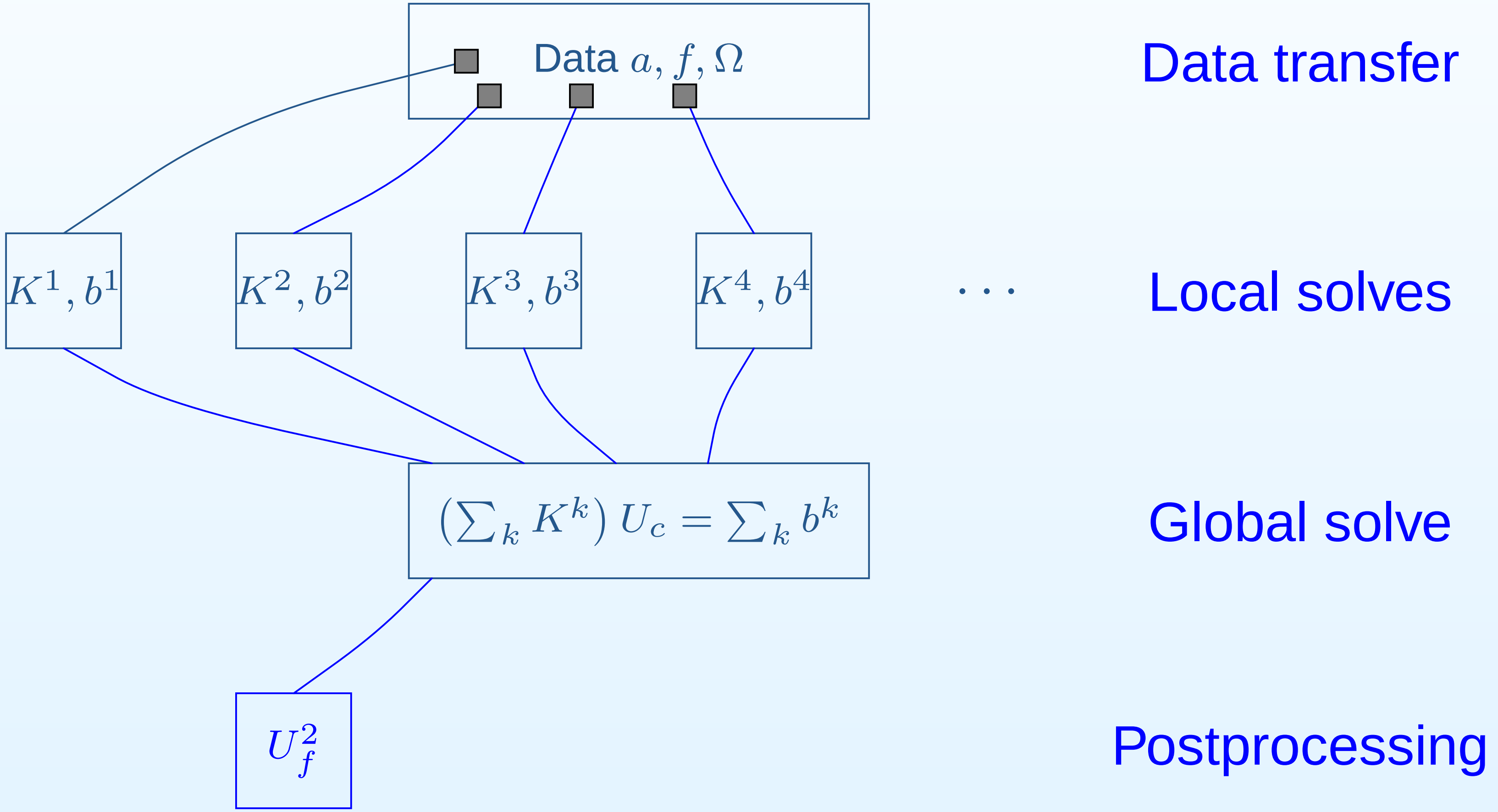
$$\boxed{KU_c = b}$$

Given U_c, U_f, and $\tilde{\mathcal{T}}$, U can be computed.

When $a(U_f, v_c + \tilde{\mathcal{T}}^* v_c)$ is included, the error is orthogonal to $v_c + \tilde{\mathcal{T}}^* v_c$ i.e. $a(u - U_c - \tilde{\mathcal{T}} U_c - U_f, v_c + \tilde{\mathcal{T}}^* v_c) = 0$.

Parallel Structure

One local problem for each coarse dof, minimal communication.



Examples of Applications

Oil recovery: We seek water saturation s (oil is $1 - s$)

$$\sigma - a\lambda(s)\nabla u = 0$$

$$-\nabla \cdot \sigma = q$$

$$\dot{s} + \sigma \cdot \nabla f(s) = 0$$

$f(s)$ is fractional flow function, $\lambda(s)$ is total mobility, a is permeability, and q is a source term.

Organic Semi-Conductor: We seek electric potential ϕ, hole concentration p, and electron concentration n,

$$\dot{n} - \nabla \cdot (D_n \nabla n - \mu_n n \nabla \phi) = 0$$

$$\dot{p} - \nabla \cdot (D_p \nabla p + \mu_p p \nabla \phi) = 0$$

$$-\nabla \cdot \epsilon \nabla \phi = p - n,$$

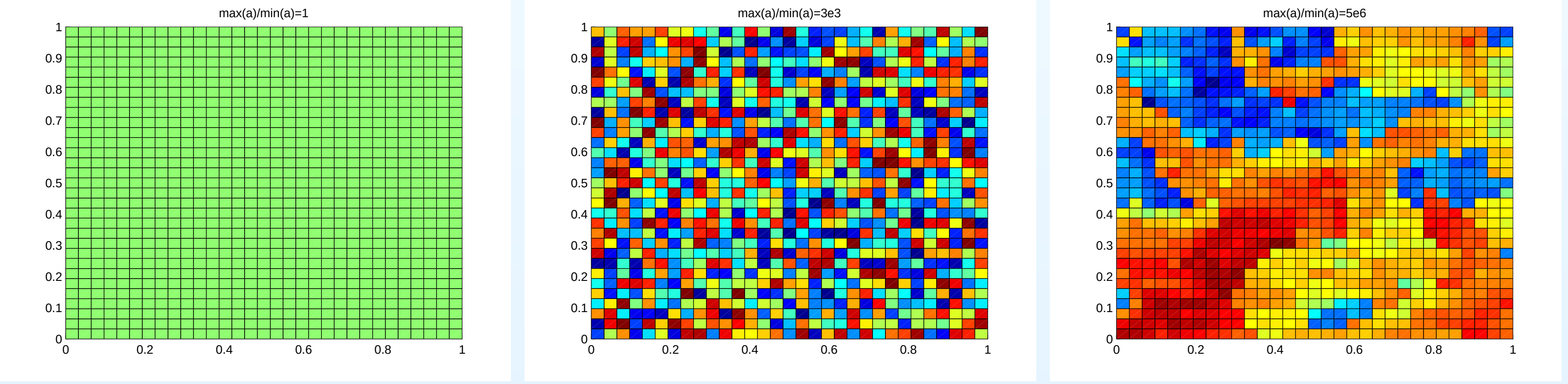
where D_i are diffusions, μ_i mobilities, $i = n, p$, ϵ dielectric func.

The Mixed Problem in Oil Recovery

Poisson equation on mixed form:

$$\begin{cases} \frac{1}{a}\sigma - \nabla u = 0 & \text{in } \Omega \\ -\nabla \cdot \sigma = f & \text{in } \Omega \\ n \cdot \sigma = 0 & \text{on } \Gamma \end{cases}$$

where the permeability a is constant, random, or taken from the SPE data set (upperness in log-scale),



We let $a(v,w) = (\frac{1}{a}v_1, w_1) + (v_2, \nabla \cdot w_1) + (\nabla \cdot v_1, w_2)$ and $l(w) = -(f, w_2)$.

Splitting Based on RT-elements

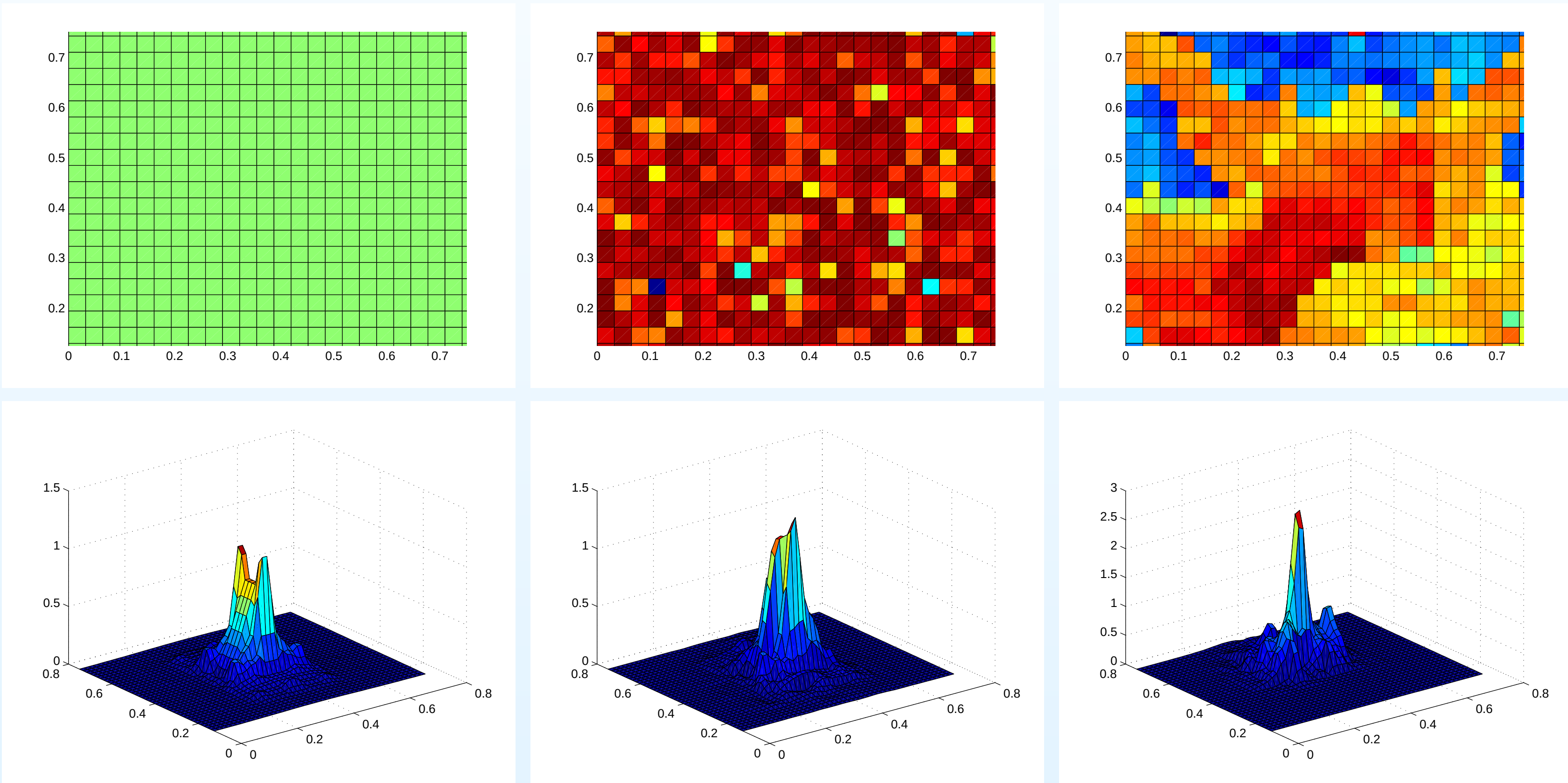
We use lowest order RT basis functions together with piecewise constants.

- Let Π_c be the RT-interpolant onto the space of lowest order RT functions $\mathcal{V}_c$ and P_c be the L^2-projection onto the space of piecewise constants W_c
- We define $\mathcal{W}_f = (I - P_c)\mathcal{W}$, $\mathcal{W} = L^2(\Omega)$
- We define $\mathcal{V}_f = (I - \Pi_c)\mathcal{V}$, $\mathcal{V} = H(\text{div}; \Omega)$
- This means $\sigma_c = \pi_c\sigma \in \mathcal{V}_c \quad u_c = P_c u \in \mathcal{W}_c$.
- Thus we are using an L^2-orthogonal splitting in the scalar variable.

Hierarchical split for lagrangian elements leads to nodal exactness in the coarse solution $U_c \approx \pi_c u$ while here we get exactness of average values on coarse elements $U_c \approx P_c u$.

Example of Local Solutions $\tilde{\mathcal{T}}_\sigma \phi_i$

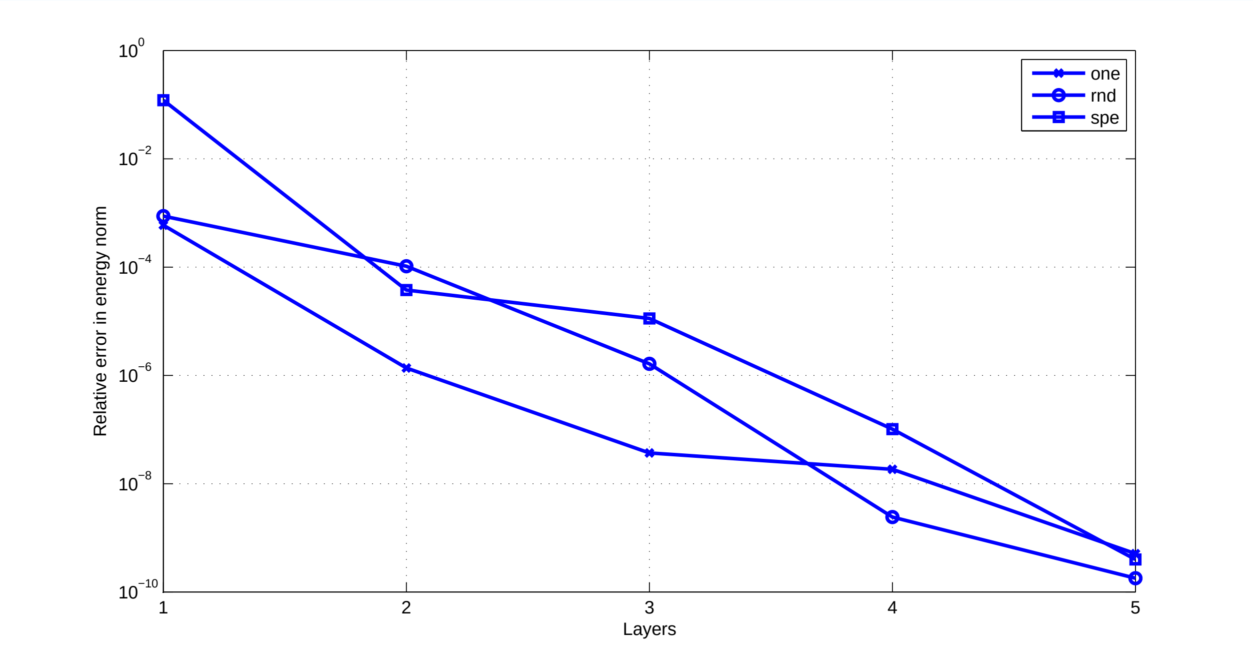
$$
\begin{cases}
(\frac{1}{a}\tilde{\mathcal{T}}_\sigma \phi_i, v_f) + (\tilde{\mathcal{T}}_u \phi_i, \nabla \cdot v_f) = -(\frac{1}{a}\phi_i, v_f) \\
\qquad\qquad\qquad\qquad\qquad -(\nabla \cdot \tilde{\mathcal{T}}_\sigma \phi_i, w_f) = 0.
\end{cases}
$$



We use 3 layer patches and plot absolute value of the flux $|\tilde{\mathcal{T}}_\sigma \phi_i|$.

Example of Convergence

- The reference mesh has 32×32 elements
- The coarse mesh has 8×8 elements.
- We let $f = 1$ lower left corner and $f = -1$ in upper right, otherwise $f = 0$.



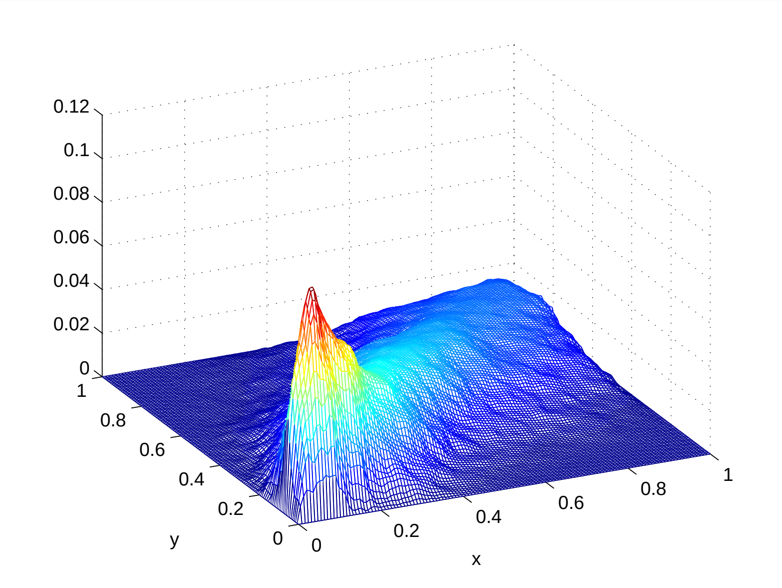
Error compared to reference solution.

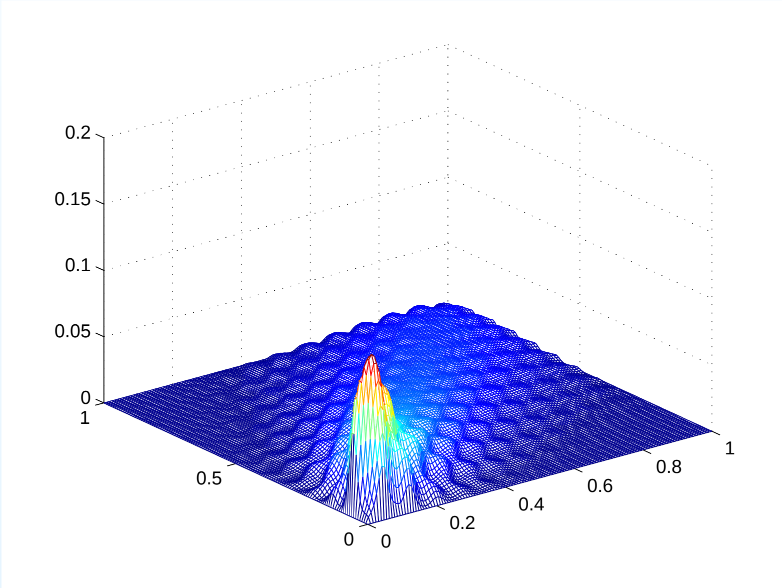
Two Convection Dominated Problems

$$\mathcal{L}u = -\epsilon\triangle u + \nabla\cdot(bu) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

i. Let $b = [N(0,1)+0.25, N(0,1)+0.25]$ and $\epsilon = 0.008$

ii. Let $b = [sin(25\pi y)+0.3, cos(25\pi x)+0.3]$, and $\epsilon = 0.005$.





In both examples $f = 1$ for $0 \le x, y \le 0.1$ and zero otherwise.
We let $a(v,w) = (\epsilon\nabla v, \nabla w) + (\nabla\cdot(bv), w)$ and $l(w) = (f, w)$.

Split Based on Lagrangian Basis Functions

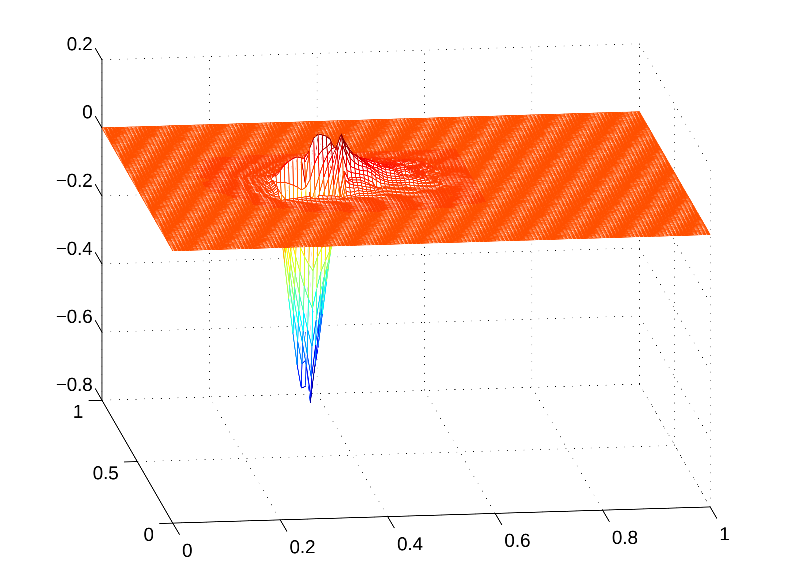
- We let the coarse mesh consist of 16×16 and the reference mesh of 128×128 rectangular elements.
- We let π_c be the interpolant onto bilinear functions of the coarse mesh and let $\mathcal{V}_f = (1 - \pi_c)\mathcal{V}$, i.e. an hierarchical split
- We remember the definition of the fine scale equations,

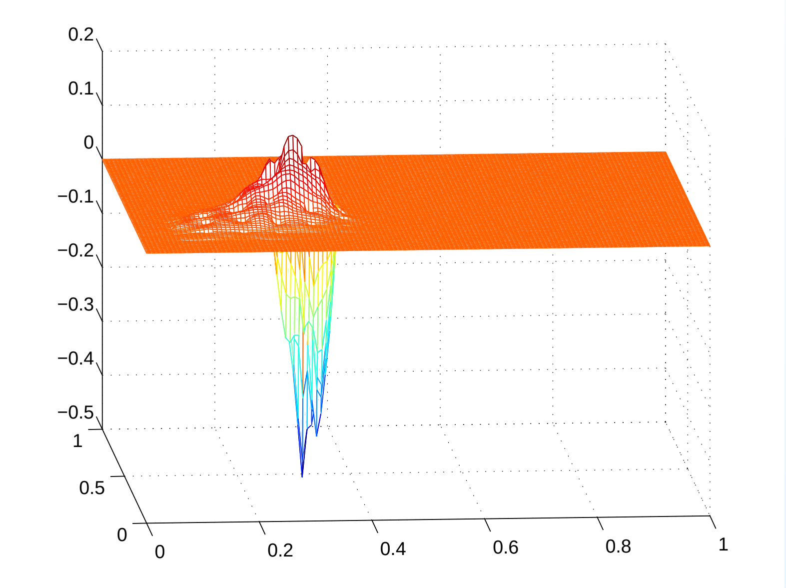
$$a(\mathcal{T}\phi_i, v_f) = -a(\phi_i, v_f),$$
$$a(v_f, \mathcal{T}^*\phi_i) = -a(v_f, \phi_i),$$
$$a(u_{f,i}, v_f) = l(\phi_i v_f),$$

for all $v_f \in \mathcal{V}_f$ and $i \in \mathcal{N}$ (coarse nodes).

Decay of Local Solutions

We use three refinements in all local problems and vary the number of layers. We plot local solutions $\tilde{\mathcal{T}}\phi_i$ and $\tilde{\mathcal{T}}^*\phi_i$ using four layers (random).

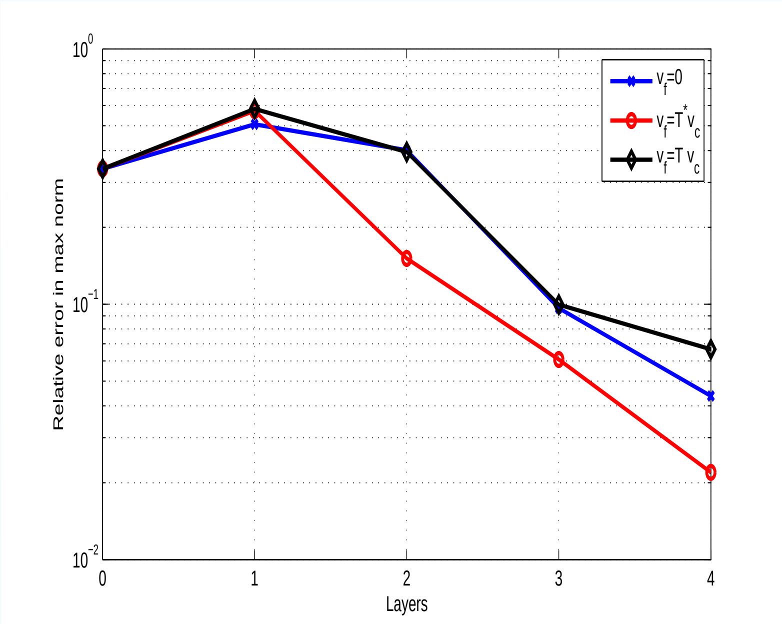


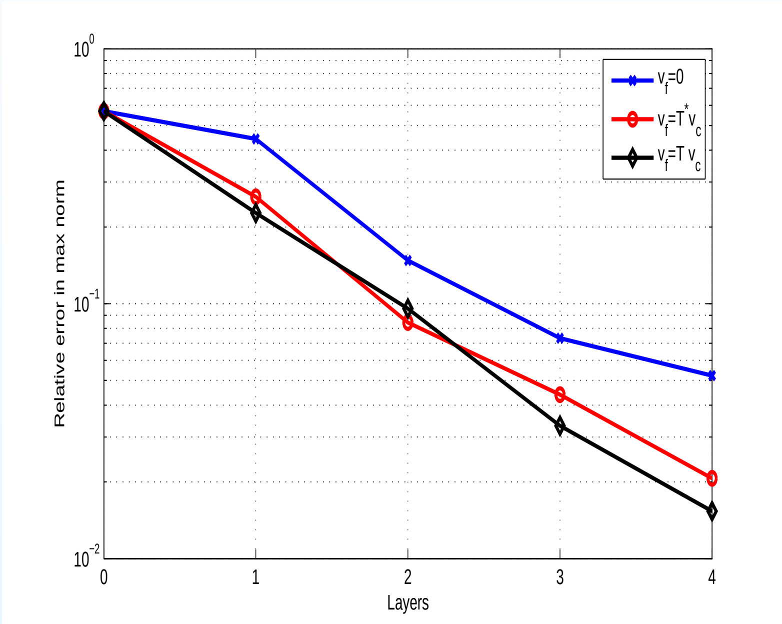


We still get decay but not as quick as in the non-convective problem. Directed patches would reduce the work.

Error in Multiscale Solution

We now plot the max norm of the error $\|u - U\|_{L^\infty(\Omega)}$ with and without $\tilde{\mathcal{T}}^*$ in the right slot.





We observe quick decay (slower) starting from two layers.

The random coefficient appears to be more difficult to resolve.

Adaptive Multiscale Method

The adaptive version of the method (sometimes referred to as AVMS) builds on the following ingredients:

- Error estimation framework
- Adaptive strategy for tuning of critical discretization parameters

The method is designed so that:

$$\boxed{\text{error} \longrightarrow 0 \text{ when } h \longrightarrow 0 \text{ and } L \longrightarrow \infty}$$

- A priori error estimates in progress.
- To circumvent difficulties with choosing discretization parameters h and L we use an adaptive algorithm based on a posteriori error estimates

A Posteriori Error Estimate (Poisson, mixed)

The following energy norm bound holds

$$\|\frac{1}{\sqrt{a}}(\sigma - \Sigma)\|^2 \leq C_a \sum_i \left(R^2_{\omega_i} + R^2_{\partial\omega_i}\right)$$

where

$$R^2_{\omega_i} = \|\frac{1}{a}(\Sigma^i_c(\phi_i + \tilde{\mathcal{T}}_\sigma \phi_i) + \Sigma^i_f) - \nabla U^{i,*}_f\|^2_{\omega_i} +$$

$$\|\frac{h}{a}(f\psi_i + \nabla \cdot (\Sigma^i_c(\phi_i + \tilde{\mathcal{T}}_\sigma \phi_i) + \Sigma^i_f))\|^2_{\omega_i} + \sum_{K \in \omega_i} \|h^{-1/2}[U^{i,*}_f]\|^2_{\partial K}$$

$$R^2_{\partial\omega_i} = \|h^{-1/2} U^{i,*}_f\|^2_{\partial\omega_i \setminus \Gamma}$$

U^* is a post processed version (Lovadina and Stenberg 06) of U, $C_a \sim \|\sqrt{a}\|_{L^\infty(\omega_i)}$.

Adaptive Algorithm

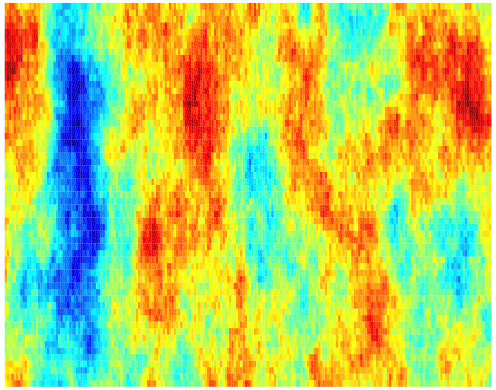
We have the error bound

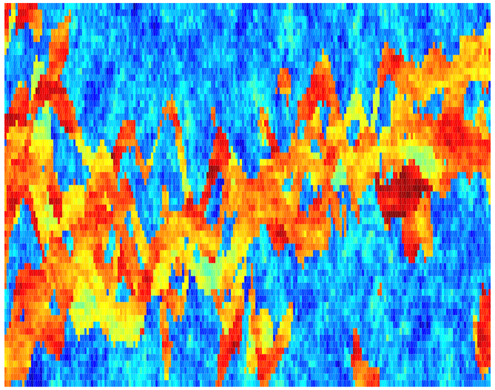
$$\|\sigma - \Sigma\|_a^2 \leq C_a \sum_i \left(R_{\omega_i}^2 + R_{\partial\omega_i}^2\right)$$

Let $h = H/2$ and $L = 1$ for all i.
for $i = 1, \ldots, n$ **do**
    Compute $\tilde{\mathcal{T}}\phi_i$, $\tilde{\mathcal{T}}^*\phi_i$ (if non-symmetric), and $U_{f,i}$.
end for
Compute the solution $\{U, \Sigma\}$.
for $i = 1, \ldots, n$ **do**
    Compute residuals $R_{\omega_i}^2$ and $R_{\partial\omega_i}^2$.
end for
Mark large entries.
For marked entries in $R_{\omega_i}^2$ let $h := h/2$.
For marked entries in $R_{\partial\omega_i}^2$ let $L := L + 1$.
Return to step two or stop if estimators are small enough.

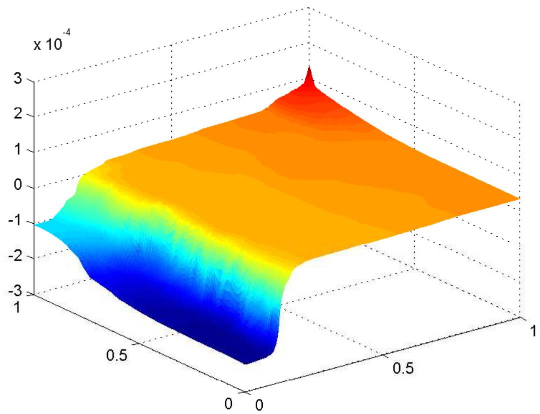
Back to the Oil Recovery Example

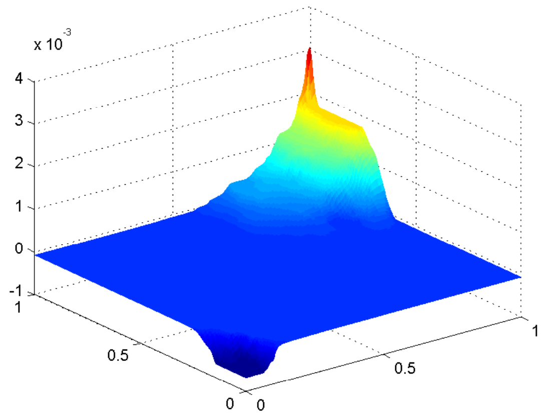
Layer 1 and 50 in the SPE comparative sol. proj. (log scale).





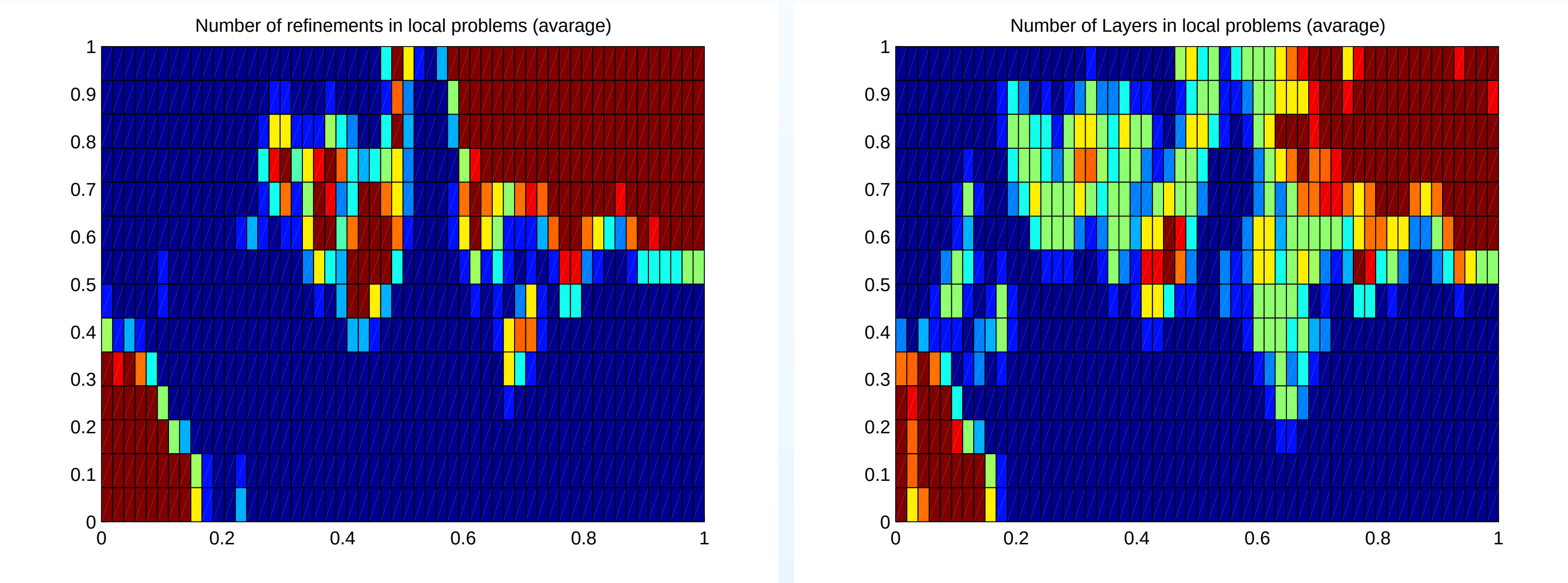
Plot of the sol. (pressure), $q = 1$ upper right $q = -1$ lower left.





Refinements and Layers SPE50

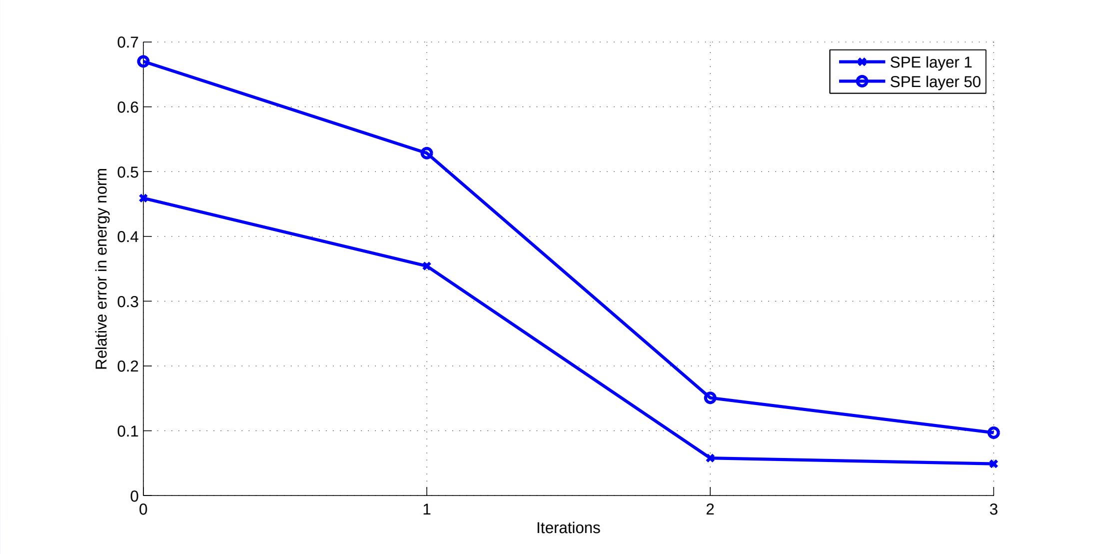
We use 55×15 coarse elements and a reference mesh with 440×120 elements.



We start the adaptive algorithm with one refinement and one layer in all local problems. After three iterations in the algorithm marking 30%.

Convergence of Adaptive Algorithm

We compare error in energy norm with reference solution.



- Critical areas are found
- A majority of the patches uses one layer and one refinement.
- As the water front travels only local problems at the front need to be recomputed.

Summary and Future Work

The adaptive variational multiscale method (AVMS) provides:

- Systematic technique for construction of a computable approximation of the fine scale part of the solution using decoupled localized subgrid problems.
- A posteriori error estimation framework
- Adaptive algorithms for automatic tuning of critical discretization parameters

The decay in $\mathcal{V}_f$ together with the adaptive strategy makes the method efficient.

Future work includes: 3D implementation, a priori error analysis, time-dependent convection dominated problems, other applications