Preconditioning for Flow in Fractured Porous Media

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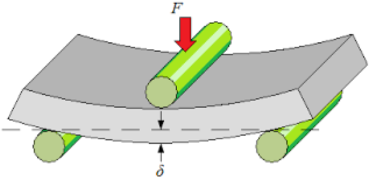
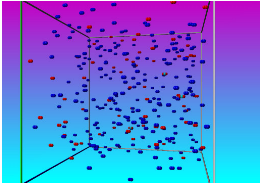
Partial differential equations (PDEs)





Mathematical model of physical processes

1. Complex movement of leaves in the wind. (Navier-Stokes)
2. Temperature distribution in this room. (Heat equation)
3. Someone leaning on a desk. (Linear elasticity)
4. The molecules floating around us. (Schrödinger)
5. Electromagnetic field. (Maxwell)
6. Several coupled physical processes. (Systems of PDEs)

Solving PDEs is crucial in industry, academia, environment, ...

Flow in the earths subsurface

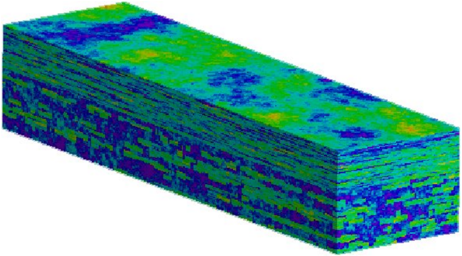
Understanding physical processes in the earths subsurface is very challenging and very important research field.

- Rock formations consist of porous material with varying permeability on multiple scales.
- Apps: ground water flow, CO_2 storage, and oil recovery.

Numerical simulation is important to make correct decisions.

Main challanges

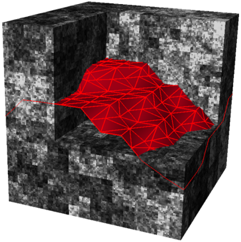
This problem is very difficult!



- Complex geometry with thin layers and fractures.
- Rapidly varying data on multiple scales.
- Non-linear systems of PDEs.
- Data uncertainty due to lack of measurements.

Contains several major challenges in numerical solution of PDEs.

Fractured geometry



Thin structures:

- are very important for the flow.
- are difficult to represent geometrically and to discretize in 3D.
- are not seen on coarse meshes.
- calls for a nonstandard mixed dimensional formulation.

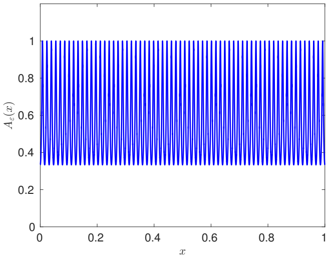
The multiscale problem

Example (periodic coefficient):

We consider

$$-\frac{d}{dx}\left(A(x)\frac{d}{dx}u(x)\right) = 1, \quad u(0) = u(1) = 0,$$

with $A(x) = (2 + \cos(2\pi x/\varepsilon))^{-1}$, where $\varepsilon = 2^{-6}$.



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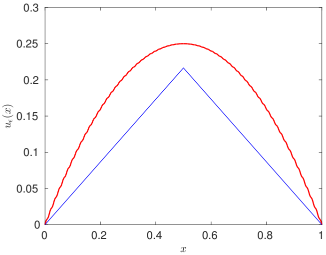

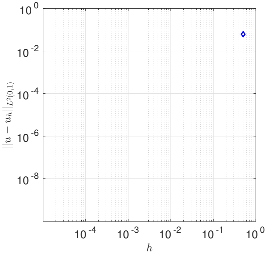

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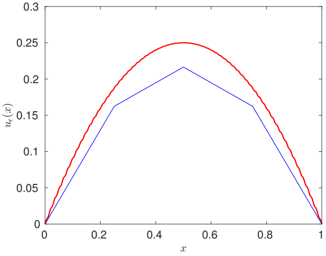
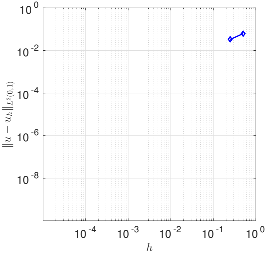


solution and P1-FEM-approximation



$L^2(\Omega)$ – error vs. h

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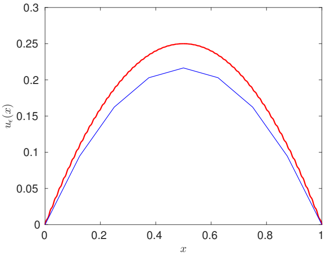

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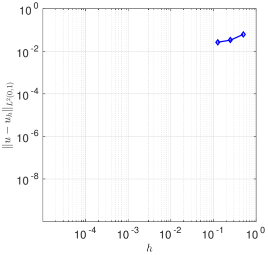

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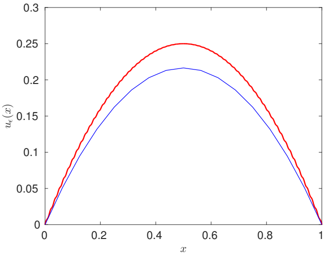

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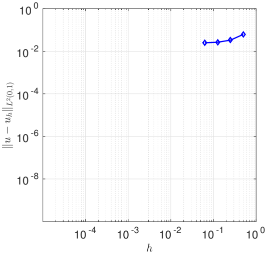

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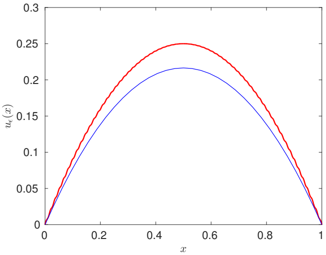

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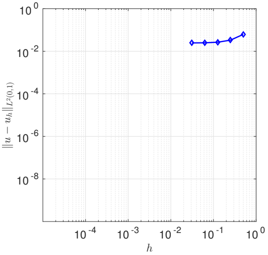

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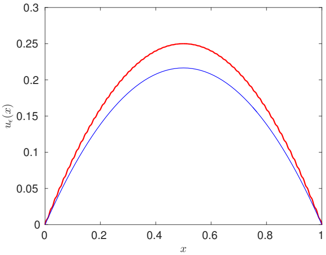

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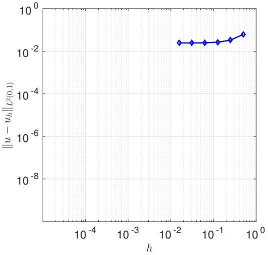

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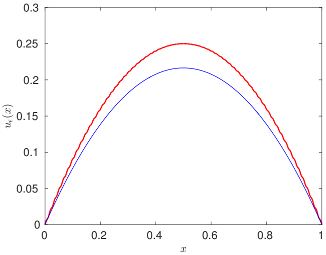

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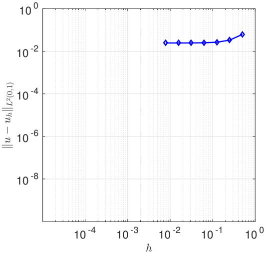

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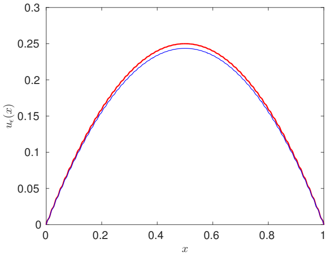

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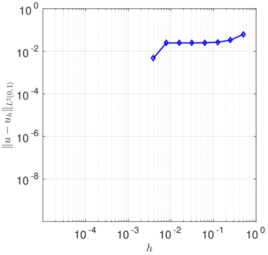

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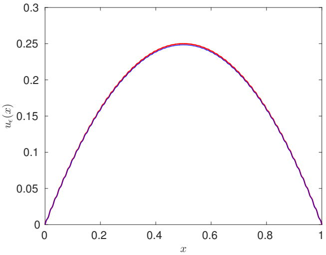

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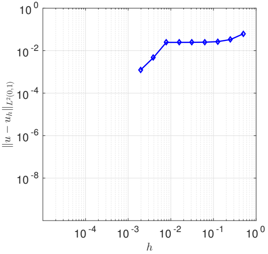

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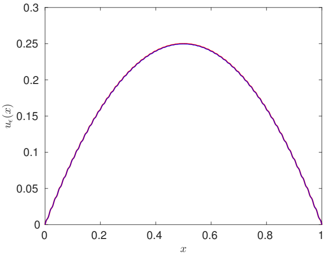

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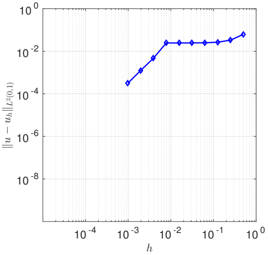

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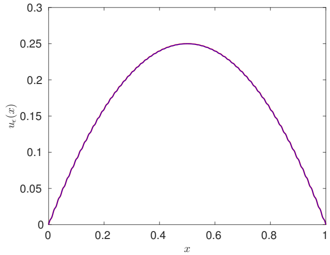

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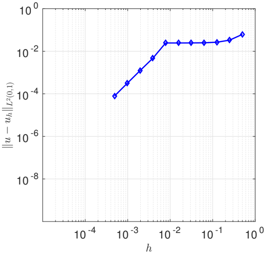

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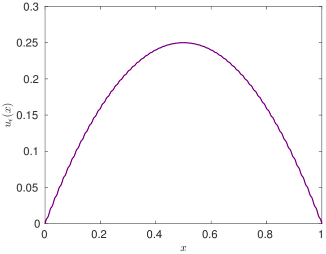

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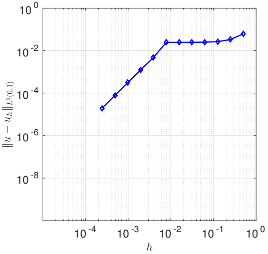

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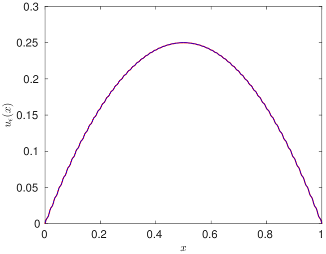

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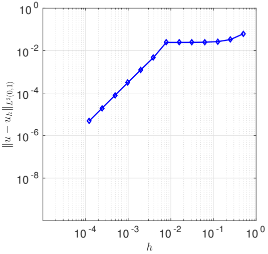

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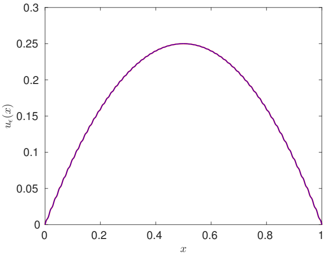

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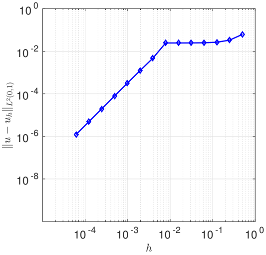

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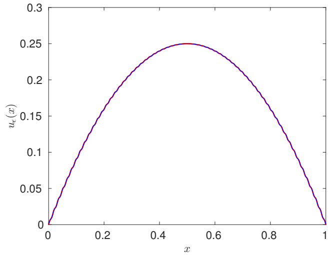

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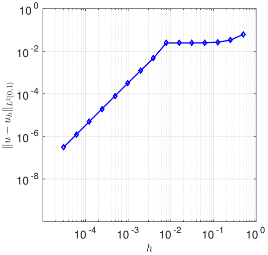


$L^2(\Omega)$ – error vs. h

Efficient numerical linear algebra

After discretization we get vast linear system(s)

$$Ax = b$$

- Iterative methods are needed.
- To conserve mass a mixed formulation is often used. Therefore A has a special structure (saddle point problem).
- The condition number $\kappa(A)$ depends on the data variation and is large.
- Preconditioner B is needed, $BAx = Bb$ with $\kappa(BA) \ll \kappa(A)$.

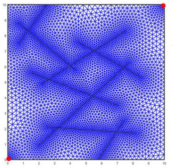
The iteration can be expressed as:

$$x^k = x^{k-1} + B(b - Ax^{k-1}),$$

where B is approximating A^{-1}.

Ana Budiša's contributions

Method development, analysis, implementation, and application!



- Block preconditioning for mixed dimensional saddle point problems
- Preconditioners for mixed dimensional problems in an exterior calculus setting
- Domain decomposition with multiscale flux basis functions for nonlinear fracture models
- Extension to time dependent problems