Existence of weak solutions to the coupled nonlinear Joule heating problem

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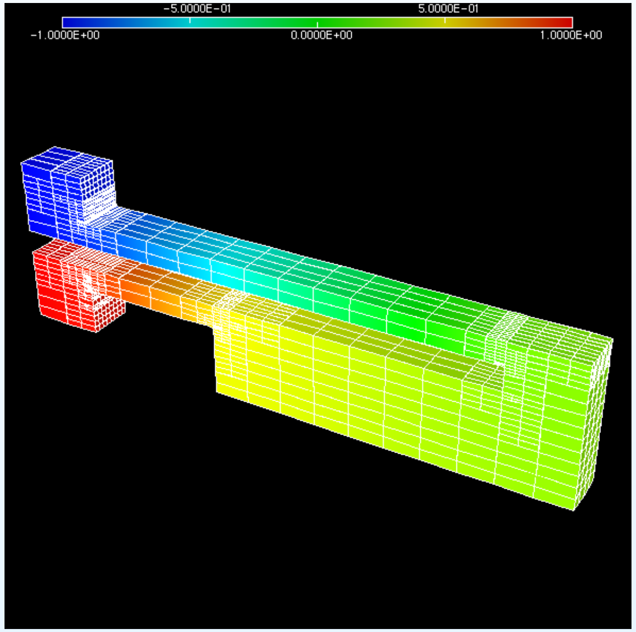
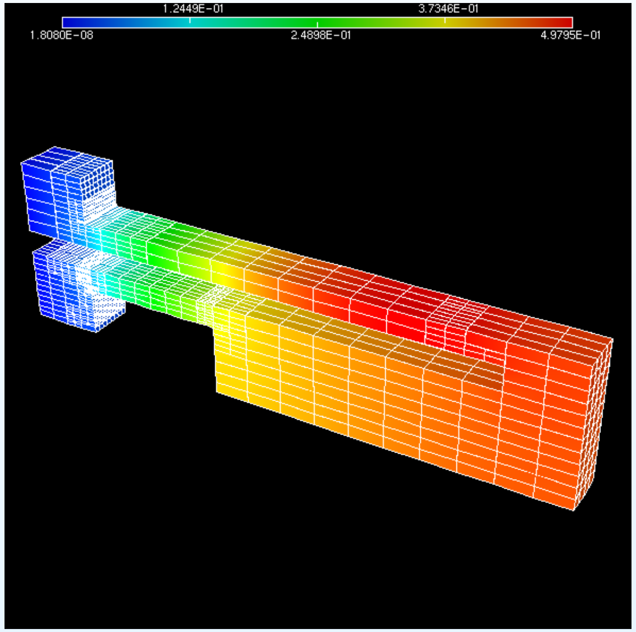
Background

A voltage is applied to the boundary of a device and current flows through it, the current flow produces Joule heating, which induces thermal stresses into the device.





Potential (left) and temperature (right). The pictures comes from M. Larson and F. Bengzon at Umeå University.

Outline

- Problem formulation
- Related work
- Weak form of the potential equation
- Weak form of the temperature equation
- Definition of fixed point map
- The Schauder theorem
- Existence of weak solution in Sobolev spaces
- Comments on numerical implementation
- Future work

Problem formulation

We seek the electric potential Φ satisfying:

$$\begin{cases} -\nabla \cdot (\sigma(u)\nabla\Phi) = 0 & \text{in } \Omega, \\ \Phi = g & \text{on } \partial_D\Omega, \\ \sigma(u)\partial_n\Phi = 0 & \text{on } \partial_N\Omega, \end{cases}$$

where $\sigma = \sigma(x, u)$ is electric conductivity and $g = g(x)$ is a given voltage at the boundary segment $\partial_D\Omega$.
The temperature u satisfies the stationary heat equation:

$$\begin{cases} -\nabla \cdot (\lambda\nabla u) = \sigma(u)|\nabla\Phi|^2 & \text{in } \Omega, \\ -\lambda\partial_n u = \kappa(u - u_\infty) & \text{on } \partial\Omega, \end{cases}$$

where $\lambda = \lambda(x)$ is thermal conductivity, $u_\infty = u_\infty(x)$ is temperature of the ambient media, and $\kappa = \kappa(x)$ is the heat permeability on the boundary.

Main idea

- Construct weak forms individually for both equations in appropriate Sobolev spaces.
- Show that the decoupled problems each have unique weak solution.
- Construct a fixed point map, using a Gauss-Seidel approach, given $u \to \phi \to \bar{u}$ i.e. $\bar{u} = Tu$.
- Show T is compact and continuous from a convex set into itself.
- Use Schauder's fixed point theorem to guarantee existence of at least one fixed point $u^* = Tu^*$.
- Put u^* into first equation to get ϕ^*.

Main difficulty

Under reasonable assumptions we can only expect to have $\sigma(u) \in L^\infty$, furthermore the mixed boundary conditions also forces $\phi \in H^1$, i.e.,

$$\sigma(u)|\nabla\phi|^2 \in L^1 \not\subset H^{-1},$$

for $d = 2, 3$ since, H^1 is continuously imbedded in L^6 ($d = 3$) i.e. $L^{6/5} \subset H^{-1}$ but L^1 is not. In two dimensions we have $L^{1+\epsilon} \subset H^{-1}$ but again $L^1 \not\subset H^{-1}$.

This means we can only use the Lax-Milgram theorem to guarantee unique weak solution $\phi \in H^1$ to the first equation, not to the second equation, $u \notin H^1$.

Related work

- Gallouët and Herbin, 1994, existence in $H^1 \times W^{1,d/(d-1)-\epsilon}$, $d = 2, 3$, for homogenous Dirichlet bc using entropy condition for the temperature equation to get well defined map.
- Elliott and Larsson, 1995, existence in $H^1 \times H^1$ in two dimensions for time dependent version with homogenous Dirichlet bc.
- Gröger, 1989, existence of unique solution to potential equation with mixed bc in $W^{1,2+\epsilon}$ given $u \in L^2$.
- Douglas and Dupont, 1977, Method for finding weak solutions to elliptic problems with rough data (H^{-1} Galerkin), outside H^{-1}. This paper deal with Dirichlet and Neumann bc, not Robin type bc.

Weak form of the potential equation

We have,

$$\begin{cases} -\nabla \cdot (\sigma(u)\nabla\Phi) = 0 & \text{in } \Omega, \\ \Phi = g & \text{on } \partial_D\Omega, \\ \sigma(u)\partial_n\Phi = 0 & \text{on } \partial_N\Omega. \end{cases}$$

Assume that g has enough regularity so that there is a function $\Phi_D \in W^{1,q}(\Omega)$ for some $q > 2$, such that $\mathrm{tr}(\Phi_D) = g$ on $\partial_D\Omega$. We let $\Phi = \Phi_D + \tilde{\Phi}$ where $\tilde{\Phi}$ solves: find $\tilde{\Phi} \in H^1_{0,D}(\Omega)$ such that,

$$(\sigma(u)\nabla\tilde{\Phi}, \nabla v) = -(\sigma(u)\nabla\Phi_D, \nabla v),$$

for all $v \in H^1_{0,D}(\Omega)$. Lax-Milgram theorem guarantees existence of unique weak solution $\Phi \in H^1(\Omega)$.

Gröger's result for mixed boundary conditions

Theorem 0.1 *Given the unique weak solution* $\tilde{\Phi} \in H^1(\Omega)$, $\Phi_D \in W^{1,q}(\Omega)$, *and a regular domain*[a] Ω, *there exists a real number* p, $2 < p \leq q$ *such that* $\Phi \in W^{1,p}(\Omega)$ *and furthermore,*

$$\|\Phi\|_{W^{1,p}(\Omega)} \leq C\|\Phi_D\|_{W^{1,p}(\Omega)} \leq C,$$

where C *is independent of* u.

This means that for some $\epsilon > 0$, $\Phi \in W^{1,2+\epsilon}$, i.e. $\sigma(u)|\nabla\Phi|^2 \in L^{1+\epsilon/2}$.

[a] the boundary segments are separated by a Lipschitz hypersurface

Weak form of the temperature equation

Remember,

$$\begin{cases} -\nabla \cdot (\lambda \nabla \bar{u}) = \sigma(u)|\nabla\Phi|^2 & \text{in } \Omega, \\ -\lambda \partial_n \bar{u} = \kappa(\bar{u} - u_\infty) & \text{on } \partial\Omega, \end{cases}$$

In two dimensions $L^{1+\epsilon/2} \subset H^{-1}(\Omega)$ which means Lax-Milgram framework works and $\bar{u} \in H^1(\Omega)$.

However, in three dimensions this does not resolve the difficulty.

We need a new weak form that allows $\bar{u}$ to be in a weaker space then $H^1(\Omega)$.

The H^{-1} Galerkin paper extended to Robin bc

Theorem 0.2 *Let $f \in H^{s-2}(\Omega)$ for some $0 \le s \le 3/2$. Further assume $H^2(\Omega)$ regularity for $f \in L^2(\Omega)$. Then there exists a unique pair $(\bar{u}, \psi) \in H^s(\Omega) \times H^{s-1/2}(\partial\Omega)$ such that,*

$$(\bar{u}, -\nabla \cdot \lambda \nabla v) + \langle \psi, \lambda \partial_n v + \kappa v \rangle_{\partial\Omega} = (f, v) + \langle \kappa u_\infty, v \rangle_{\partial\Omega},$$

for all $v \in H^2(\Omega)$. Furthermore, there exists a constant C independent of f such that,

$$\|\bar{u}\|_{H^s(\Omega)} \le C(\|f\|_{H^{s-2}(\Omega)} + \|\kappa u_\infty\|_{H^{s-3/2}(\partial\Omega)}).$$

If $s \ge 1/2$ we identify $\psi =$ tr$(\bar{u}) \in L^2(\partial\Omega)$.

In our case $f \in L^{1+\epsilon/2} \subset H^{-3/2+\delta(\epsilon)}$, $\delta > 0$ i.e. $\bar{u} \in H^{1/2+\delta}(\Omega)$ and tr$(\bar{u})$ is well defined.

Summary so far and definition of fixed point map

Given $u \in L^2(\Omega)$ there is a unique weak solution $\Phi \in W^{1,2+\epsilon}(\Omega)$ to,

$$\begin{cases} -\nabla \cdot (\sigma(u)\nabla\Phi) = 0 & \text{in } \Omega, \\ \Phi = g & \text{on } \partial_D\Omega, \\ \sigma(u)\partial_n\Phi = 0 & \text{on } \partial_N\Omega. \end{cases}$$

These two functions produces a unique weak solution $\bar{u} \in H^{1/2+\delta}(\Omega)$ of,

$$\begin{cases} -\nabla \cdot (\lambda\nabla\bar{u}) = \sigma(u)|\nabla\Phi|^2 & \text{in } \Omega, \\ -\lambda\partial_n\bar{u} = \kappa(\bar{u} - u_\infty) & \text{on } \partial\Omega. \end{cases}$$

Furthermore we have $\|\bar{u}\|_{H^{1/2+\delta}} \leq C\|\Phi\|_{W^{1,2+\epsilon}} \leq C$, where C is independent of u and Φ. Let $\tilde{T} : L^2(\Omega) \to H^{1/2+\delta}(\Omega)$ be defined as $\bar{u} = \tilde{T}u$.

The Schauder fixed point theorem

We note that $H^{1/2+\delta}(\Omega)$ is compactly imbedded in $L^2(\Omega)$ i.e. there is a compact map $E : H^{1/2+\delta}(\Omega) \to L^2(\Omega)$ that identifies the element in $L^2(\Omega)$ corresponding to a given element in $H^{1/2+\delta}$. We also note that $S = \{u \in L^2(\Omega) : \|u\|_{L^2} \le C\}$ is a closed convex set.

We let $T = E \circ \tilde{T} : L^2(\Omega) \to L^2(\Omega)$, in particular we have $T : S \to S$.

Theorem 0.3 *Let S be a closed convex set in a Banach space $X = L^2(\Omega)$ and let T be a continuous and compact mapping of S into itself. Then T has a fixed-point.*

It remains to prove that T is continuous and compact.

Is T compact and continuous?

We have $T = E \circ \tilde{T}$ where, $\|\tilde{T}\|_{H^{1/2+\delta}(\Omega)} \leq C$ i.e. bounded, and E is compact. A map is compact if it maps bounded set into pre-comact[a] sets. Since $\tilde{T}$ maps bounded sets into bounded sets T has to be compact since E is compact.

We have, since $H^{1/2+\delta}(\Omega)$ is compactly imbedded in $L^2(\Omega)$, that weak convergence in $H^{1/2+\delta}$ implies strong convergence in $L^2(\Omega)$.

We would like to show $u_n \to u$ strongly in $L^2(\Omega)$ leads to $Tu_n \to Tu$ strongly in $L^2(\Omega)$. The above result means its enough to prove that $\tilde{T}u_n \to \tilde{T}u$ weakly in $H^{1/2+\delta}(\Omega)$.

[a] closure is compact

Weak continuity of $\tilde{T}$, main ideas

- Since $\|\tilde{T}u_n\|_{H^{1/2+\delta}(\Omega)} \leq C$ we can extract a weakly convergent subsequence, $\tilde{T}u_n \to w \in H^{1/2+\delta}(\Omega)$ weakly.
- We need to show that w solves the equation $\bar{u}$ is unique solution to,

$$(\bar{u}, -\nabla\cdot\lambda\nabla v) + \langle\psi, \lambda\partial_n v + \kappa v\rangle_{\partial\Omega} = (\sigma(u)|\nabla\Phi|^2, v) + \langle\kappa u_\infty, v\rangle_{\partial\Omega}.$$

- Show that the corresponding sequence Φ_n converges weakly in $W^{1,2+\epsilon}(\Omega)$ using the same argument.
- Show that $\Phi_n \to \Phi$ weakly by using the uniqueness of the solution to the potential equation.
- Prove that $\sigma(u_n)|\nabla\Phi_n|^2 \to \sigma(u)|\nabla\Phi|^2$ weakly in $H^{-2}(\Omega)$.
- Given the uniqueness of the solution to the temperature equation we have that $w = \bar{u}$ since w and $\bar{u}$ solves the same equation.

Conclusion of the analysis and main result

We have fulfilled all assumptions of the Schaduer fixed point theorem i.e. the map T has a fixed point in S, $u^* = Tu^*$.

If we plug in this solution u^* into the weak form of the potential equation we get a solution Φ^*.

Theorem 0.4 *Given a domain Ω and data to the temperature equation such that $L^2(\Omega)$ data leads to $H^2(\Omega)$ solution, data $g \in W^{1/2,q}(\partial\Omega)$, for some $q > 2$, $\sigma \in L^\infty(\Omega)$ there exists a weak solution $(\Phi, u) \in W^{1,2+\epsilon}(\Omega) \times H^{1/2+\delta}(\Omega)$ to the stationary Joule heating problem in three dimensions, for some $0 < \epsilon < 1$ and $\delta = \frac{3\epsilon}{2+\epsilon}$.*

Some comments on the main result

- The corresponding result in two dimensions reads: There exist at least one solution $(u, \Phi) \in W^{1,2+\epsilon}(\Omega) \times H^1(\Omega)$ to the stationary Joule heating problem.
- The most limiting assumption in the analysis is that we need H^2-regularity in the solution to the temperature equation given data in L^2. This puts conditions on the coefficients and more importantly the domain.
- The result indicates that using standard H^1 finite element basis functions in the discretization of the temperature equation in three dimensions may lead to problems.

The numerical method proposed by Douglas & Dupont

- The test functions should be in $V_h \subset H^2(\Omega)$, i.e. standard elements does not work. The claim in the paper is that C^1 elements work.
- The problem formulation used would be: find $w_h \in V_h$ such that,
$$(\triangle w_h, \triangle v_h) = (f, v_h), \quad \text{for all } v_h \in V_h,$$
and then $u_h = -\triangle w_h$. This is the H^{-1} Galerkin formulation of the problem $-\triangle u = f$ with homogenous Dirichlet bc.
- A priori error estimates is presented for the error in the $H^2(\Omega)$ norm.

Future work

- Numerical test using H^1 basis functions solving a problem with L^1 data in order to study the behavior of the solution.
- Study convergence of iterative techniques for solving systems of non-linear PDE:s, e.g. the Gauss-Seidel algorithm used in the fixed point mapping or a Newton iteration, see e.g. work by J. W. Jerome.
- Pick a less complicated model problem and analyze the whole process from existence proof to adaptive algorithm.