A Mixed Adaptive Variational Multiscale Method

Axel Målqvist and Mats G. Larson

axel@math.chalmers.se

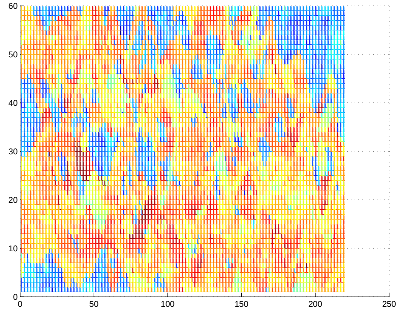
Department of Mathematics, Colorado State University

The Model Problem

Poisson Equation on mixed form:

$$\begin{cases} \frac{1}{a}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ n \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma. \end{cases}$$

Permeability a has multiscale features (SPE).



Weak form

Find $\boldsymbol{\sigma} \in \boldsymbol{V} = \{\boldsymbol{v} \in H(\text{div};\Omega) : \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \Gamma\}$ and $u \in W = L^2(\Omega)/\mathbf{R}$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}, \boldsymbol{v}) + (u, \nabla \cdot \boldsymbol{v}) = 0, \\ -(\nabla \cdot \boldsymbol{\sigma}, w) = (f, w), \end{cases}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Here $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ scalar product for vector and scalar functions.

Coarse and Fine Scales (VMS Hughes)

We introduce spaces $\boldsymbol{V}_c \oplus \boldsymbol{V}_f = \boldsymbol{V}$ and $W_c \oplus W_f = W$.

- $\boldsymbol{V}_c$ is a finite dimensional approximation of $H(\text{div};\Omega)$. We use Raviart-Thomas basis functions.
- W_c is an approximation of $L^2(\Omega)$. We use piecewise constants.
- The degrees of freedom in these spaces should be possible to handle on a single processor.

Coarse and Fine Scales

Find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}(\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f), \boldsymbol{v}_c + \boldsymbol{v}_f) + (u_c + u_f, \nabla \cdot (\boldsymbol{v}_c + \boldsymbol{v}_f)) = 0 \\ -(\nabla \cdot (\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f), w_c + w_f) = (f, w_c + w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

We want to approximate the red terms by solving decoupled local problems.

Global Solutions

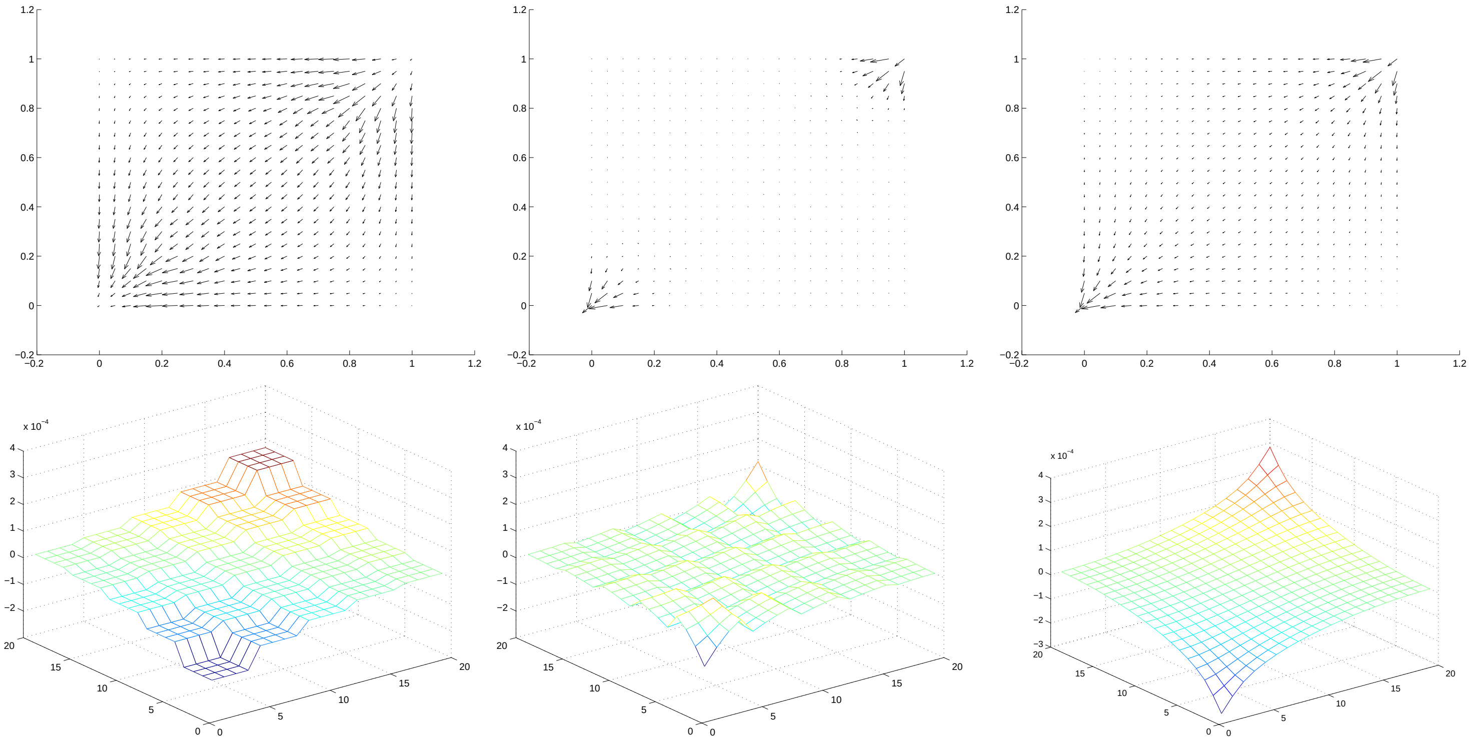


Figure 1: $\boldsymbol{\sigma}$ and u on coarse and fine scale.

Fine Scale Equations

If we let $v_c = 0$ and $w_c = 0$ we get the following relation between the fine and coarse solutions: find $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$ and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) - (u_c, \nabla \cdot \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla \cdot \boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_f$ and $w_f \in W_f$.

This system can be simplified somewhat due to the elements we use.

Orthogonality

We use an hierarchical basis of Raviart-Thomas basis functions on rectangular elements for the flux and the piecewise constants for the pressure.

$$(w_c, \nabla\cdot\boldsymbol{v}_f) = \sum_K (w_c, \nabla\cdot\boldsymbol{v}_f)_K = \sum_K w_c^K \int_{\partial K} \boldsymbol{n}\cdot\boldsymbol{v}_f \, dx = 0$$

where w_c^K is the constant at coarse element K,

$$(w_f, \nabla\cdot\boldsymbol{v}_c) = \sum_K (w_f, \nabla\cdot\boldsymbol{v}_c)_K = \sum_K \nabla\cdot\boldsymbol{v}_c^K \int_K w_f \, dx = 0$$

Decouple Fine Scale Equations

Find $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$ and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla\cdot\boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) - (u_c, \nabla\cdot\boldsymbol{v}_f) \\ -(\nabla\cdot\boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla\cdot\boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_f$ and $w_f \in W_f$.

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_{f,i}, \boldsymbol{v}_f) + (u_{f,i}, \nabla\cdot\boldsymbol{v}_f) = -(\frac{1}{a}\sigma_c^i\boldsymbol{\phi}_i, \boldsymbol{v}_f) \\ -(\nabla\cdot\boldsymbol{\sigma}_{f,i}, w_f) = (f\psi_i, w_f) \end{cases}$$

Here $\boldsymbol{\sigma}_c = \sum_i \sigma_c^i\boldsymbol{\phi}_i$ and $\psi_i = \chi_{\text{supp}\boldsymbol{\phi}_i}/4$.

Two Auxiliary Problems

Find $\boldsymbol{\xi}_i \in \boldsymbol{V}_f$ and $\zeta_i \in W_f$ such that

$$\begin{cases} (\frac{1}{a}\boldsymbol{\xi}_i, \boldsymbol{v}_f) + (\zeta_i, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ \qquad\qquad\qquad -(\nabla \cdot \boldsymbol{\xi}_i, w_f) = 0, \end{cases}$$

and: $\boldsymbol{\beta}_i \in \boldsymbol{V}_f$ and $\rho_i \in W_f$ such that

$$\begin{cases} (\frac{1}{a}\boldsymbol{\beta}_i, \boldsymbol{v}_f) + (\rho_i, \nabla \cdot \boldsymbol{v}_f) = 0, \\ -(\nabla \cdot \boldsymbol{\beta}_i, w_f) = (f\psi_i, w_f), \end{cases}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_f$, and $w_f \in W_f$ and $i \in \mathcal{N}$.

Modified Coarse Scale Equation

We use $\boldsymbol{\sigma}_c = \sum_j \sigma_c^j \boldsymbol{\phi}_j$, $\boldsymbol{\sigma}_f = \sum_j \sigma_c^j \boldsymbol{\xi}_j + \boldsymbol{\beta}_j$, $u_c = \sum_k u_c^k \varphi_k$, $u_f = \sum_j \sigma_c^j \zeta_j + \rho_j$, $v_c = \boldsymbol{\phi}_j$, $w_c = \varphi_k$, $v_f = \boldsymbol{\xi}_j$, and $w_f = \zeta_k$ in the original formulation.

$$\begin{cases} (\frac{1}{a}(\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f), \boldsymbol{\phi}_i + \boldsymbol{\xi}_i) + (u_c, \nabla \cdot \boldsymbol{\phi}_i) + (u_f, \nabla \cdot \boldsymbol{\xi}_j) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, \varphi_k) - (\nabla \cdot \boldsymbol{\sigma}_f, \zeta_k) = (f, \varphi_k) + (f, \zeta_k) \end{cases}$$

The other term vanish due to the orthogonality.

The Method on Matrix Form

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma} \\ \mathbf{U} \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ \mathbf{b} \end{bmatrix},$$

where

$$\mathbf{A}_{ij} = (\tfrac{1}{a}\boldsymbol{\phi}_j + \boldsymbol{\xi}_j, \boldsymbol{\phi}_i + \boldsymbol{\xi}_i),$$
$$\mathbf{B}_{ik} = (\varphi_k, \nabla \cdot \boldsymbol{\phi}_i),$$
$$\mathbf{d}_i = -(\tfrac{1}{a}\boldsymbol{\beta}, \boldsymbol{\phi}_i + \boldsymbol{\xi}_i),$$
$$\mathbf{b}_k = (f, \varphi_k),$$

We need to compute approximations to $\boldsymbol{\xi}_i$ and $\boldsymbol{\beta}_i$ numerically in smaller spaces then $\boldsymbol{V}_f$.

Local Solutions

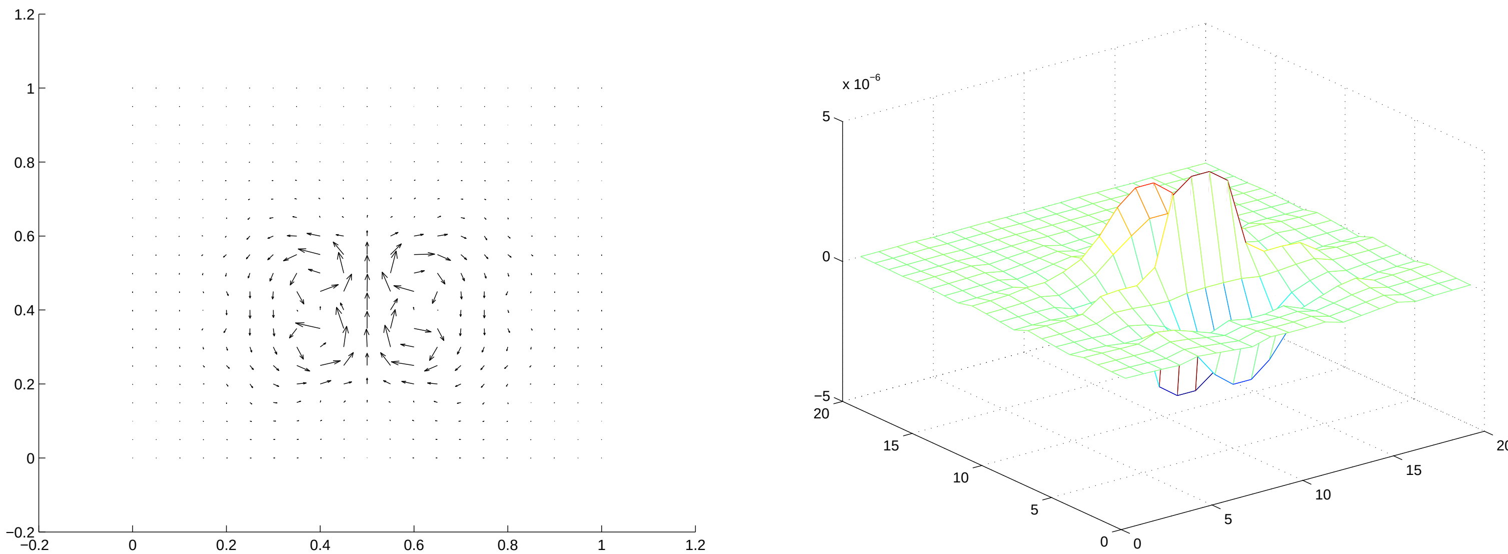
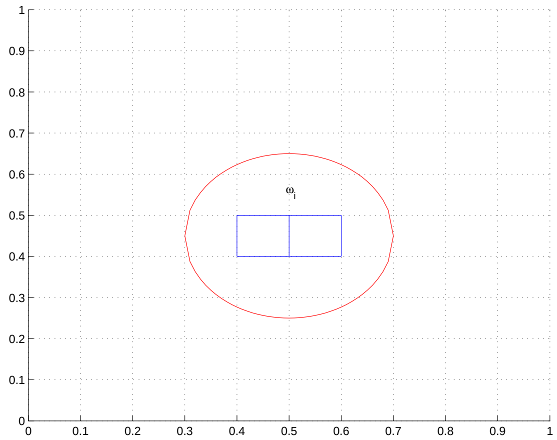


Figure 2: The local solutions $\boldsymbol{\sigma}_{f,i} = \sigma_c^i \boldsymbol{\xi}_i + \boldsymbol{\beta}_i$ and $u_{f,i} = \sigma_c^i \zeta_i + \rho_i$. Remember that rhs is zero outside the support of ϕ_i.

Introducing the Patch

The equations are solved in a slice space where solutions decay rapidly, $\int_E \boldsymbol{n} \cdot \boldsymbol{\sigma}_{f,i}\, dx = 0$ and $\int_K u_{f,i}\, dx = 0$.



The patch ω_i typically consists of coarse elements but could have any geometry.

Solving Local Neumann Problems

Find $\boldsymbol{\xi}_i^h \in \boldsymbol{V}_h(\omega_i)$ and $\zeta_i^h \in W_h(\omega_i)$ such that

$$\begin{cases} (\frac{1}{a}\boldsymbol{\xi}_i^h, \boldsymbol{v}_f) + (\zeta_i^h, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\xi}_i^h, w_f) = 0, \end{cases}$$

and: find $\boldsymbol{\beta}_i^h \in \boldsymbol{V}_h(\omega_i)$ and $\rho_i^h \in W_h(\omega_i)$ such that

$$\begin{cases} (\frac{1}{a}\boldsymbol{\beta}_i^h, \boldsymbol{v}_f) + (\rho_i^h, \nabla \cdot \boldsymbol{v}_f) = 0, \\ -(\nabla \cdot \boldsymbol{\beta}_i^h, w_f) = (f\psi_i, w_f), \end{cases}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_h(\omega_i)$, and $w_f \in W_h(\omega_i)$ and $i \in \mathcal{N}$.

Example of Local Solutions

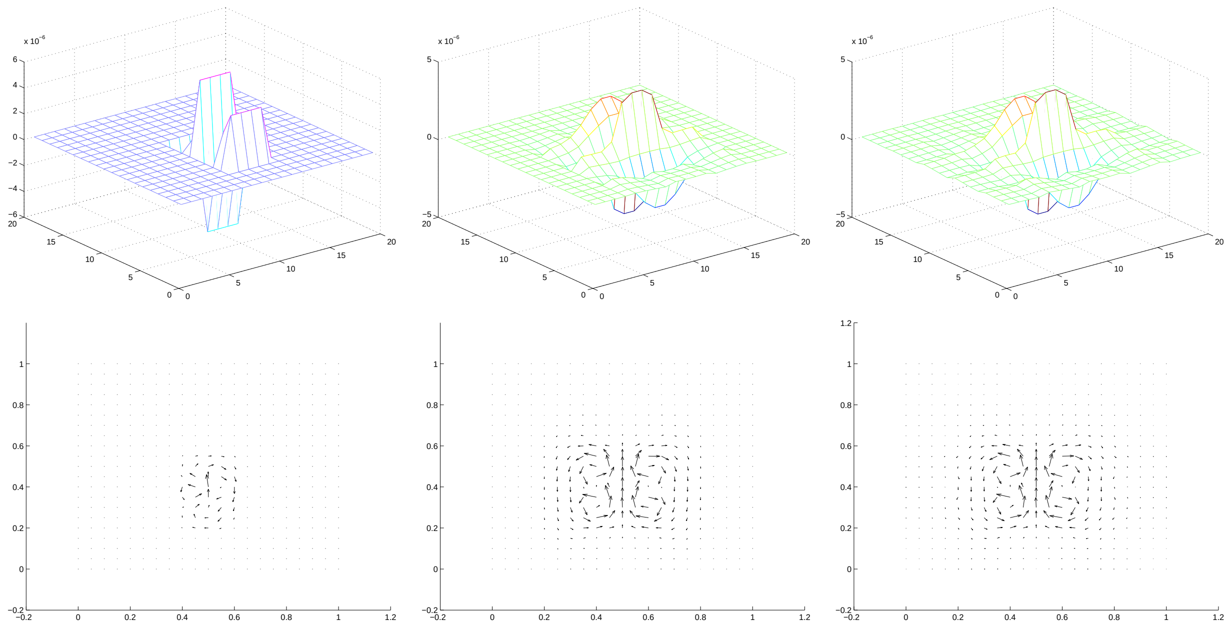
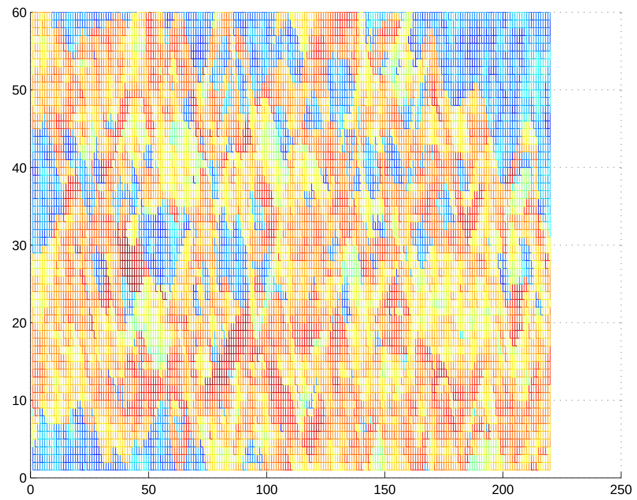


Figure 3: 1, 2, and 3 layer patches. Above $U_{f,i}$ below $\Sigma_{f,i}$.

Numerical Examples

In the figure we see the bottom layer of the Upper Ness formation (SPE).



We let $f = 1$ in the lower left corner and $f = -1$ in the upper right corner. $\max a / \min a = 8.8e6$.

Numerical Example

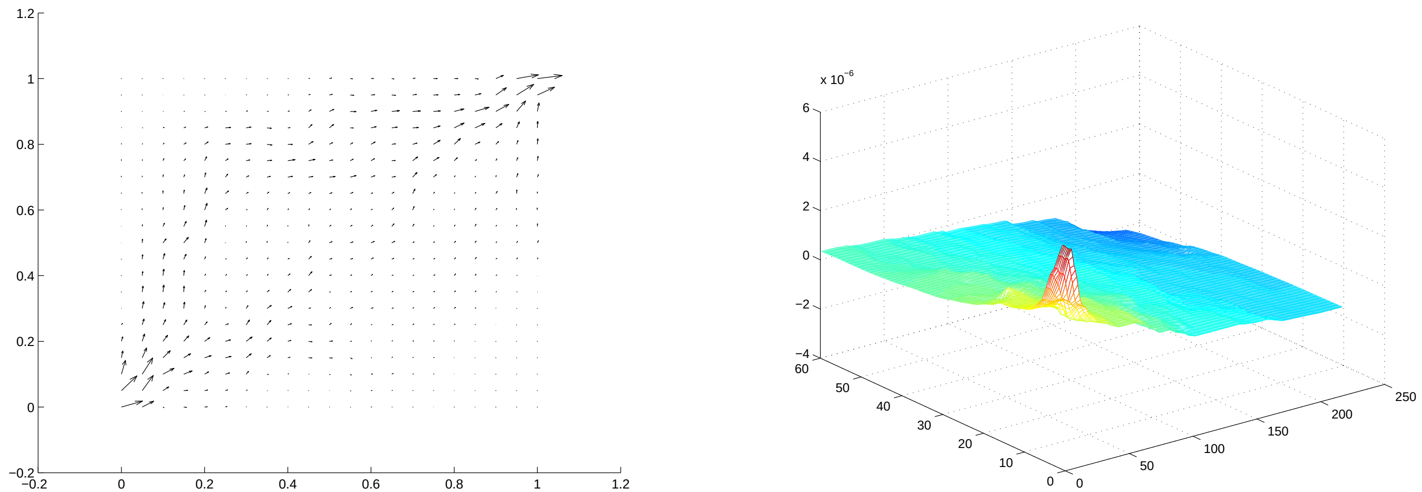


Figure 4: To the left we see the flux and to the right the pressure. We use 220×60 elements for the reference solution.

Numerical Example

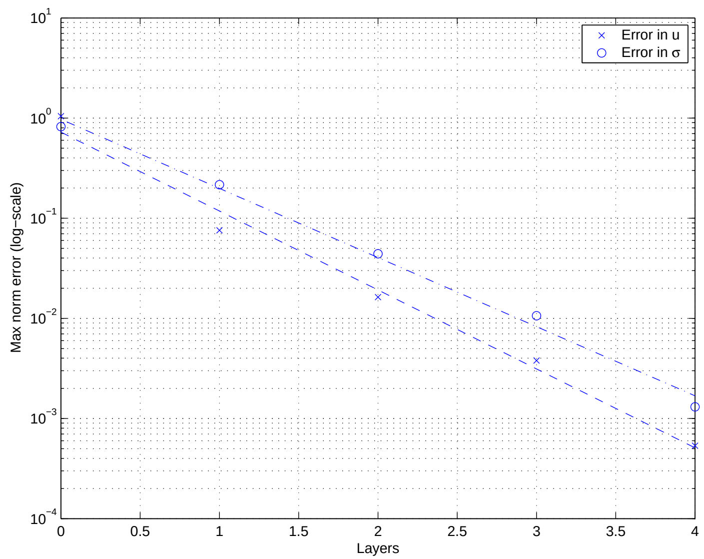


Figure 5: 55×15 coarse elements and $h = H/4$.

Do we really need to solve local problems with high resolution everywhere?

Energy Norm Estimate $\|\boldsymbol{v}\|_a^2 = (\frac{1}{a}\boldsymbol{v}, \boldsymbol{v})$

Next we present an estimate of the error.

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_a^2 \leq \sum_i C_a \|\frac{1}{a}(\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U_{f,i}^*\|_{\omega_i}^2$$

$$+ \sum_i C_a \|\frac{h}{a}(f\psi_i + \nabla \cdot (\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|_{\omega_i}^2.$$

$$+ \sum_i C_a \|\frac{1}{2\sqrt{h}} U_{f,i}^*\|_{\partial\omega_i \setminus \Gamma}^2$$

U^* is a post proc. version of U, $C_a \sim \|\sqrt{a}\|_{L^\infty(\omega_i)}$.

Adaptive Strategy

- Calculate $\mathbf{\Sigma}$.
- Calculate the error indicators on each patch,

$$X_i(h) = \|\frac{1}{a}(\Sigma_c^i \phi_i + \mathbf{\Sigma}_{f,i}) - \nabla U_{f,i}^*\|_{\omega_i}^2 \rightarrow \text{Refine } h$$

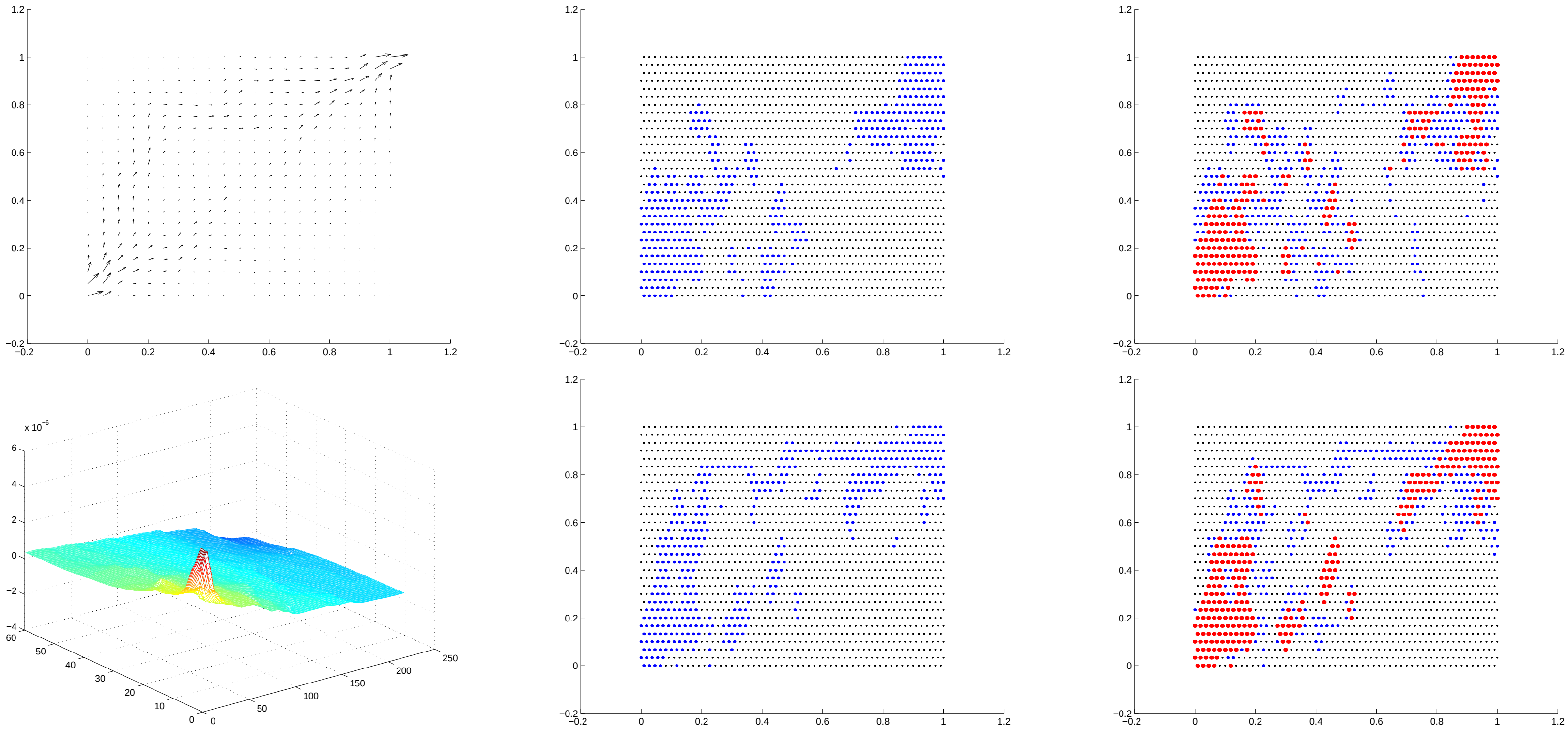
$$Y_i(h) = \|\frac{h}{a}(f\psi_i + \nabla\cdot(\Sigma_c^i \phi_i + \mathbf{\Sigma}_{f,i}))\|_{\omega_i}^2 \rightarrow \text{Refine } h$$

$$Z_i(L) = \|\frac{1}{2\sqrt{h}} U_{f,i}^*\|_{\partial\omega_i \backslash \Gamma}^2 \rightarrow \text{Increase } L$$

Back to the Numerical Example

- We start with one layer patches with one refinement in all local problems.
- We calculate the error estimators.
- Then we increase 25% of the patches with one layer and refine the mesh on 25% according to the estimators.
- We repeat this process one more time.

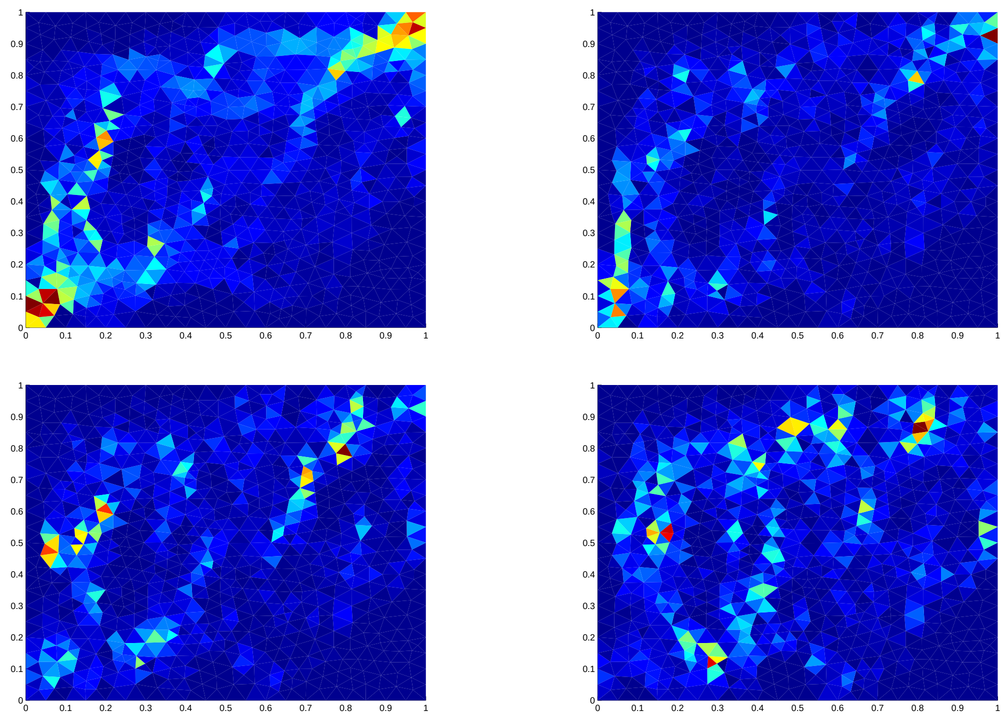
Number of Layers and Refinements



Black circle is one, blue is two and red is three layers/refinements.

Relative Error in Energy Norm

Galerkin 105.6% and one iteration 15.8%.



Two iterations 10.1% and three iterations 7.6%.

Conclusions

- The error indicators finds critical areas.
- More computational effort in these areas decreases the global error quickly.
- To get an equally good approximation without adaptivity we need to use three refinements on two layer patches.
- In the example above we still have 70% of the patches using one layer and one refinement.

Outlook

- Implementation in 3D. More extensive numerical tests.
- More scales. Should be fairly straight forward.
- Multiscale approach for the transport problem.
- Randomly perturbed coefficient a taking error in measurements into account.