A posteriori error estimates for mixed finite element approximations of elliptic and parabolic problems

Mats G. Larson **Axel Målqvist**

Division of Scientific Computing

Uppsala University

Sweden

Outline and Papers

Outline

- Background
- Previous work
- A posteriori error bound for an elliptic model problem
- A posteriori error bound for a parabolic model problem
- Conclusions and future work

Papers

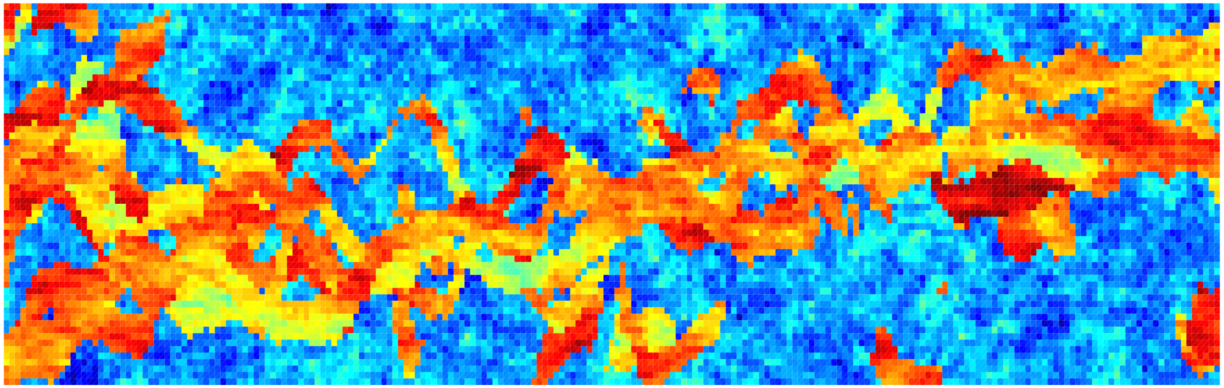
- M.G. Larson and A. Målqvist, *A posteriori error estimates for mixed finite element approximations of elliptic problems,* Numer. Math. 108 (2008) 487–500
- M.G. Larson and A. Målqvist, *A posteriori error estimates for mixed finite element approximations of parabolic problems,* Numer. Math. 118 (2011) 33–48

Motivation

Elliptic and parabolic problems on mixed form:

$$\begin{cases} \dot{u} - \nabla \cdot \sigma = f, \\ a^{-1}\sigma - \nabla u = 0, \end{cases}$$

Arising e.g. in porous media flow problems, nearly elliptic pressure coupled to nearly hyperbolic saturation equation.



Challenges

Comments:

- mass conservation is important
- a good approximation of the flux σ (rather than pressure) is needed, since it is passed to the saturation equation
- Reliable solutions are difficult to compute because of the multiscale coefficients
- Data is typically very localized in space

Conclusions:

- Mixed finite elements seems appropriate, e.g. RTN
- Adaptivity will be important because of the size of the problems and the localized features
- A posteriori bound for the error in the flux (L^2 norm) is therefore needed

Previous work

Elliptic:

- Braess-Verfürth, Alonso, Carstensen 96-97, two spatial dimensions and bound in mesh dependent norms and $H(\text{div}, \Omega)$ norm of the flux
- Lovadina and Stenberg 2006, three dimensions and bound in L^2-norm of the flux
- Vohralik 2007, with local efficiency

Parabolic:

- Nochetto, Makridakis, Lakkis, Karakatsani, Akrivis, ... 2003–, elliptic reconstruction
- Casćon et. al. 2006, bound for the divergence of the flux in a weak norm is derived
- Ern-Vohralik 2010, framework for a posteriori estimation including mixed methods

Model problem: elliptic

Given a computational domain $\Omega \in \mathbf{R}^d$ and functions $a > 0$ and f (with zero mean), we seek σ and u such that,

$$\begin{cases} \frac{1}{a}\sigma - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \sigma = f & \text{in } \Omega, \\ \nu \cdot \sigma = 0 & \text{on } \Gamma, \end{cases}$$

Weak form: find
$\sigma \in H_0(\text{div}, \Omega) = \{w \in H(\text{div}, \Omega) : \nu \cdot w = 0 \text{ on } \Gamma\}$ and $u \in L^2(\Omega)$ with zero mean such that,

$$\begin{cases} (\frac{1}{a}\sigma, v) + (u, \nabla \cdot v) = 0 & \text{for all } v \in H_0(\text{div}, \Omega), \\ (-\nabla \cdot \sigma, w) = (f, w) & \text{for all } w \in L^2(\Omega), \end{cases}$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ scalar product.

Finite element approximation

We let $\mathcal{K}$ be a discretization of Ω with $h_K = \text{diam}(K)$ for all $K \in \mathcal{K}$ and use k order Raviart-Thomas elements together with the space of piecewise polynomials of degree k,

$$\mathcal{RT}_k \subset H_0(\text{div}, \Omega),$$
$$\mathcal{P}_k \subset L^2(\Omega).$$

The finite element method reads: find $\Sigma \in \mathcal{RT}_k$ and $U \in \mathcal{P}_k$ such that,

$$\begin{cases} \left(\frac{1}{a}\Sigma, v\right) + (U, \nabla \cdot v) = 0 & \text{for all } v \in \mathcal{RT}_k, \\ (-\nabla \cdot \Sigma, w) = (f, w) & \text{for all } w \in \mathcal{P}_k. \end{cases}$$

The goal is to bound the error in the flux $\|a^{-1/2}(\sigma - \Sigma)\|_{L^2(\Omega)}$ in terms of computable quantities.

Two technical results

In the proof two technical results are used:

Lemma: It holds,

$$\|\nu \cdot w\|^2_{H^{-1/2}(\partial K)} \lesssim \|w\|^2_{L^2(K)} + h_K^2 \|\nabla \cdot w\|^2_{L^2(K)},$$

for all $w \in H(\text{div}, K)$ with constant independent of h_K. Prove the result on a reference element followed by scaling argument.

Lemma: It holds,

$$\inf_{v \in H^1(\Omega)} \sum_{K \in \mathcal{K}} \|Q - v\|^2_{H^{1/2}(\partial K)} \lesssim \sum_{K \in \mathcal{K}} h_K^{-1} \| [Q] \|^2_{L^2(\partial K)},$$

for all Q such that $Q|_K$ is polynomial. Here the jump denoted by $[\cdot]$ is the difference in function value over a face in the mesh.

A posteriori error analysis

For any $Q \in \bigoplus_{K\in\mathcal{K}} \mathcal{P}_l(K)$, with $l \geq 0$ it holds,

$$\|a^{-1/2}(\sigma - \Sigma)\|^2_{L^2(\Omega)}$$

$$\lesssim \sum_{K\in\mathcal{K}} h_K^2 \|f + \nabla\cdot\Sigma\|^2_{L^2(K)} + \|\Sigma - a\nabla Q\|^2_{L^2(K)} + h_K^{-1} \|\,[Q]\,\|^2_{L^2(\partial K)}$$

- Choosing $Q = U$ gives a suboptimal bound (e.g. $k = 0$ gives $\nabla U|_K = 0$).
- A continuous Q would eliminate the third term but may not give an optimal bound.
- A discontinuous Q can be chosen to eliminate the second term (under appropriate assumptions on a) but may not necessarily lead to an optimal bound.

Postprocessing

There have been several works on post-processing of the pressure variable in order to get optimal bounds.

In Lovadina and Stenberg's paper *Energy norm a posteriori error estimates for mixed finite element methods* from 2006 the following construction is proven to give optimal bounds:

Definition: Let $U^* \in \bigoplus_{K\in\mathcal{K}} \mathcal{P}_{k+1}(K)$ such that the projection on each element K, $P_k U^*|_K$ is equal to $U|_K$ and,

$$(\nabla U^*, \nabla v)_K = (a^{-1}\Sigma, \nabla v)_K \quad \text{for all } v \in (I - P_k)\mathcal{P}_{k+1}(K).$$

$$\|a^{-1/2}(\sigma - \Sigma)\|^2_{L^2(\Omega)}$$
$$\lesssim \sum_{K\in\mathcal{K}} h_K^2 \|f + \nabla\cdot\Sigma\|^2_{L^2(K)} + \|\Sigma - a\nabla U^*\|^2_{L^2(K)} + h_K^{-1}\|\,[U^*]\,\|^2_{L^2(\partial K)}$$

Comments

- We reproduce the Lovadina-Stenberg result as a corollary
- Valid for RTN and BDM elements
- Extended to the stabilized method of Masud-Hughes 2002
- In Lovadina-Stenberg the method (using the postprocessed U^*) is proven to be of optimal order for RTN and BDM elements

Model problem: parabolic

Given a computational domain $\Omega \in \mathbf{R}^d$ and functions $a > 0$ (independent of time for simplicity), u_0, and f with zero mean, we seek σ and u such that,

$$\begin{cases} \dot{u} - \nabla \cdot \sigma = f & \text{in } \Omega, \quad t > 0, \\ a^{-1}\sigma - \nabla u = 0 & \text{in } \Omega, \quad t > 0, \\ \nu \cdot \sigma = 0 & \text{on } \Gamma, \quad t > 0, \\ u = u_0 & \text{in } \Omega, \quad t = 0, \end{cases}$$

for $0 < t < T$. The corresponding weak form reads: for each $t > 0$, find $\sigma(t) \in H_0(\text{div}, \Omega)$, $u(t) \in L^2(\Omega)$ with mean zero such that,

$$\begin{cases} (a^{-1}\sigma, v) + (u, \nabla \cdot v) = 0 & \text{for all } v \in H_0(\text{div}, \Omega), \\ (\dot{u}, w) - (\nabla \cdot \sigma, w) = (f, w) & \text{for all } w \in L^2(\Omega), \end{cases}$$

Discretization in space and time

We let $\mathcal{K}$ be a discretization of Ω and again consider the k order Raviart-Thomas space together with piecewise polynomials of degree k,

$$\mathcal{RT}_k \subset H_0(\text{div}, \Omega),$$

$$\mathcal{P}_k \subset L^2(\Omega).$$

In time we discretize a finite interval $[0, T]$ by letting $0 = t_0 < t_1 < \cdots < t_N = T$ with corresponding time steps $\tau_n = t_n - t_{n-1}$ for $n = 1, \ldots, N$.

We let $v = \frac{t-t_{n-1}}{\tau_n} v^n + \frac{t_n - t}{\tau_n} v^{n-1}$, for $n = 1, \ldots, N$.

We use the Backward Euler method in time and let $\partial_t v^n = \frac{v^n - v^{n-1}}{\tau_n}, \quad n = 1, \ldots, N$.

Finite element approximation

With this notation the numerical method reads: find $u_h^n \in \mathcal{P}_k$ and $\sigma_h^n \in \mathcal{RT}_k$ for $n = 1, \ldots, N$, such that,

$$(a^{-1}\sigma_h^0, v) = (\nabla u_0, v), \quad \text{for all } v \in \mathcal{RT}_k,$$
$$(u_h^0, w) = (u_0, w), \quad \text{for all } w \in \mathcal{P}_k,$$

and,

$$\left(a^{-1}\sigma_h^n, v\right) + (u_h^n, \nabla \cdot v) = 0, \quad \text{for all } v \in \mathcal{RT}_k,$$
$$(\partial_t u_h^n, w) - (\nabla \cdot \sigma_h^n, w) = (f^n, w), \quad \text{for all } w \in \mathcal{P}_k.$$

Again we wish to control the flux in $L^2(\Omega)$ norm, $\int_0^T \|a^{-1/2}(\sigma - \Sigma)\|_{L^2(\Omega)}^2 \, dt$, in terms of computable quantities, using the result we have for the corresponding elliptic problem.

Elliptic reconstruction

Let ω^n, η^n be the elliptic reconstruction of σ_h^n, u_h^n defined by: find ω^n, η^n with $\int_\Omega \eta^n \, dx = 0$ such that

$$\begin{cases} -\nabla \cdot \omega^n = -\nabla \cdot \sigma_h^n & \text{in } \Omega, \\ a^{-1}\omega^n - \nabla \eta^n = 0 & \text{in } \Omega, \\ \nu \cdot \omega^n = 0 & \text{on } \Gamma. \end{cases}$$

The problem is well posed in $H_0(\text{div}, \Omega) \times L^2(\Omega)$ since $\int_\Omega \nabla \cdot \sigma_h^n \, dx = \int_\Gamma \nu \cdot \sigma_h^n \, ds = 0$.

The elliptic reconstruction is defined in such a way that $\{\sigma_h^n, u_h^n\}$ are precisely the finite element approximation of $\{\omega^n, \eta^n\}$, using $\mathcal{RT}_k$ and $\mathcal{P}_k$ elements.

Bound for the elliptic reconstruction

Lemma: Let $\{\omega^n, \eta^n\}$ be the elliptic reconstruction of $\{\sigma_h^n, u_h^n\}$. Then

$$\|a^{-1/2}(\omega^n - \sigma_h^n)\|_{L^2(\Omega)}^2 \lesssim \sum_{K \in \mathcal{K}_h} \rho_K^2(\sigma_h^n, u_h^{n,*}),$$

$$\|\partial_t^m(\eta^n - u_h^{n,*})\|_{L^2(\Omega)}^2 \lesssim \sum_{K \in \mathcal{K}_h} h_K^{2\alpha} \rho_K^2(\partial_t^m \sigma_h^n, \partial_t^m u_h^{n,*}), \quad m = 0, 1,$$

for $n = 1, \ldots, N$. Where

$$\rho_K^2(v, w) = \|a^{-1}v - \nabla w\|_{L^2(K)}^2 + h_K^{-1}\|[w]\|_{L^2(\partial K)}^2,$$

and $1/2 < \alpha \leq 1$ is the regularity parameter of the corresponding elliptic problem ($\alpha = 1$ on convex domain with a, f smooth enough).

Error analysis

We split the error using the elliptic reconstruction:

$$e = u - u_h^* = (u - \eta) + (\eta - u_h^*) = e_1 + e_2,$$
$$\varepsilon = (\sigma - \omega) + (\omega - \sigma_h) = \varepsilon_1 + \varepsilon_2.$$

Second part:

$$\|a^{-1/2}\varepsilon_2\|_{L^2(\Omega)} \leq \|a^{-1/2}(\omega^n - \sigma_h^n)\|_{L^2(\Omega)} + \|a^{-1/2}(\omega^{n-1} - \sigma_h^{n-1})\|_{L^2(\Omega)}$$
$$\lesssim \left(\sum_{K\in\mathcal{K}_h} \rho_K^2(\sigma_h^n, u_h^{n,*})\right)^{1/2} + \left(\sum_{K\in\mathcal{K}_h} \rho_K^2(\sigma_h^{n-1}, u_h^{n-1,*})\right)^{1/2},$$

for $t \in (t_{n-1}, t_n]$ where $n = 1, \ldots, N$. Thus

$$\int_0^T \|a^{-1/2}\varepsilon_2\|_{L^2(\Omega)}^2 \, dt \lesssim \sum_{n=1}^N \sum_{K\in\mathcal{K}_h} \tau_n \rho_K^2(\sigma_h^n, u_h^{n,*}).$$

Error analysis

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$$\varepsilon = (\sigma - \omega) + (\omega - \sigma_h) = \varepsilon_1 + \varepsilon_2.$$

First part:

$$\begin{aligned}(\dot{e}_1, v) - (\nabla \cdot \varepsilon_1, v) &= (f, v) - (\dot{\eta}, v) + (\nabla \cdot \omega, v) \\ &= (f^n, v) - (\dot{u}_h^*, v) + (\nabla \cdot \sigma_h^n, v) \\ &\quad + (f - f^n, v) - (\dot{\eta} - \dot{u}_h^*, v) + (\nabla \cdot (\omega - \sigma_h^n), v),\end{aligned}$$

If we let $v = e_1$ we get an energy norm bound for the error in the first component.

We will now bound the error in ε_1 using the error in the bound for the reconstruction errors $\eta - u_h^*$ and $\omega - \sigma_h$.

Error analysis cont.

For term I we get,

$$\frac{1}{2}\|e_1(T)\|^2_{L^2(\Omega)} + \int_0^T \|a^{-1/2}\varepsilon_1\|^2_{L^2(\Omega)}\,dt$$

$$= \frac{1}{2}\|u_0 - \eta_0\|^2_{L^2(\Omega)} - \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(\dot{u}_h^* - \nabla\cdot\sigma_h^n - f^n, e_1)\,dt$$

$$+ \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(f - f^n, e_1)\,dt - \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(\dot{\eta} - \dot{u}_h^*, e_1)\,dt$$

$$- \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(\omega - \sigma_h^n, \nabla e_1)\,dt$$

We get classical residual terms and elliptic reconstruction terms.

Error analysis cont.

Using the Lemma, Cauchy-Schwarz inequality, Galerkin orthogonality, and interpolation estimates, we get

$$\int_0^T \|a^{-1/2}\varepsilon_1\|^2_{L^2(\Omega)}\,dt \lesssim \|u_0 - u_h^{0,*}\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{K}_h} h_K^{2\alpha}\rho_K^2(\sigma_h^0, u_h^{0,*})$$

$$+ \sum_{n=1}^{N}\sum_{K\in\mathcal{K}_h} \tau_n h_K^2 \|\partial_t u_h^{n,*} - \nabla\cdot\sigma_h^n - f^n\|^2_{L^2(K)}$$

$$+ \sum_{n=1}^{N}\sum_{K\in\mathcal{K}_h} \tau_n h_K^{2\alpha}\rho_K^2(\partial_t\sigma_h^n, \partial_t u_h^{n,*})$$

$$+ \sum_{n=1}^{N} \tau_n^3 \|\partial_t\sigma_h^n\|^2_{L^2(\Omega)} + \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|f - f^n\|^2_{L^2(\Omega)}\,dt.$$

A posteriori error bound

The final bound reads:

$$\int_0^T \|a^{-1/2}(\sigma - \sigma_h)\|^2_{L^2(\Omega)}\,dt \lesssim \|u_0 - u_h^{0,*}\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{K}_h} h_K^{2\alpha}\rho_K^2(\sigma_h^0, u_h^{0,*})$$

$$+ \sum_{n=1}^{N}\sum_{K\in\mathcal{K}_h} \tau_n h_K^2 \|\partial_t u_h^{n,*} - \nabla\cdot\sigma_h^n - f^n\|^2_{L^2(K)}$$

$$+ \sum_{n=1}^{N}\sum_{K\in\mathcal{K}_h} \tau_n \rho_K^2(\sigma_h^n, u_h^{n,*}) + \sum_{n=1}^{N}\sum_{K\in\mathcal{K}_h} \tau_n h_K^{2\alpha}\rho_K^2\left(\partial_t\sigma_h^n, \partial_t u_h^{n,*}\right)$$

$$+ \sum_{n=1}^{N} \tau_n^3 \|\partial_t \sigma_h^n\|^2_{L^2(\Omega)} + \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|f - f^n\|^2_{L^2(\Omega)}\,dt,$$

Remarks

The error bound presented consists of seven terms.

- Two terms measuring the effect of the discrepancy between initial condition and its approximation
- the following three terms measures the error committed by the spatial discretization
- term six measures the effect of time discretization
- term seven measures the effect of approximating the right hand side f in discrete points in time.

The bound is of a similar form as the bound presented by Lakkis and Makridakis 2006, where backward Euler was used for the heat equation on standard form.

Improvements and future work

We have presented a posteriori bounds in energy norm for elliptic and parabolic model problems on mixed form.

- The coefficient a can easily be made time dependent at the cost of an additional data error in the estimate.
- We have only considered a static mesh. A natural extension would be to allow different meshes at different time intervals. Coarsening will become crucial.
- Higher order approximation in time.
- Include convection.
- Numerical tests of the adaptive algorithm.
- Multiscale basis functions and extending the analysis to the setting.