Iterative solution of spatial network models by subspace decomposition

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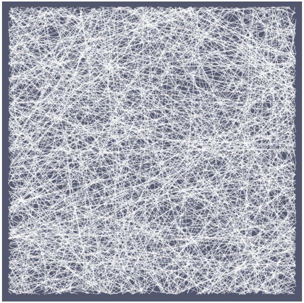
Motivation: simulation of paper/paperboard



- A simplified spatial network model: $Ku = f$, where u is displacement and f is applied force
- K is ill-conditioned (geometry and material data variation), direct solver used by Fraunhofer
- The goal is to develop and analyze an iterative solver

Outline

Graph Laplacian and norms

- Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a graph of nodes and edges, $x \in \Omega \subset \mathbb{R}^d$
- Let $\hat{V} : \mathcal{N} \to \mathbb{R}$ be scalar functions on $\mathcal{N}$. For $v, w \in \hat{V}$

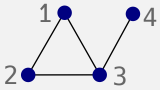
$$(v, w) = \sum_x v(x)w(x)$$

$$(L^g v, v) = \sum_{(x,y)\in\mathcal{E}} (v(x) - v(y))^2$$

$$(Lv, v) = \sum_{(x,y)\in\mathcal{E}} \frac{(v(x) - v(y))^2}{|x - y|}$$

$$|v|_L = (Lv, v)^{1/2}$$

Example:



$$L^g = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

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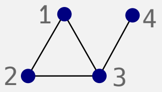

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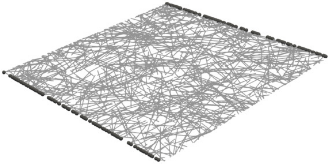
- Let M be diagonal with $M_{xx} = \frac{1}{2} \sum_{y\sim x} |x - y|$, $|v|_M = (Mv, v)^{1/2}$
- $|v|_L$ corresponds to the H^1 semi-norm in P1-FEM and M to the lumped mass matrix (L^2-norm)

Model problem

Find $u \in V := \{v \in \hat{V} : v(x) = 0 \text{ for } x \in \Gamma_D\}$:

$$(Ku, v) = (f, v), \quad v \in V.$$



Assume: $(K\cdot,\cdot)$ is scalar product on V and

$$\alpha |v|_L^2 \leq (Kv, v) \leq \beta |v|_L^2, \quad \forall v \in V.$$

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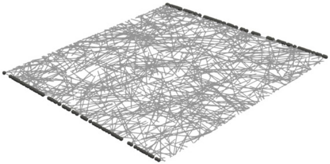

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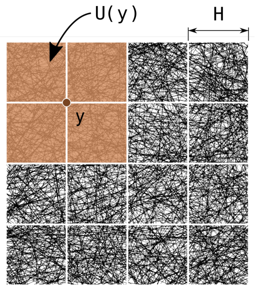
Example:

- $$(Kv, v) = \sum_{(x,y)\in\mathcal{E}} \gamma_{xy} \frac{(v(x) - v(y))^2}{|x - y|}, \quad \alpha \leq \gamma_{xy} \leq \beta$$
- 1D diffusion model on network with continuity and Kirchhoff flux constraint in junctions with P1-FEM

Outline

Geometric coarsening



- $\mathcal{T}_H$ is a mesh of squares
- $\hat{V}_H$ is Q1-FEM with basis $\{\varphi_y\}_y$
- $V_H \subset \hat{V}_H$ satisfy the boundary conditions
- Clément type interpolation operator

$$\mathcal{I}_H v = \sum_{\text{free DoFs } y} \bar{v}_{U(y)} \varphi_y \in V_H$$

Lemma (Stability and approximability of $\mathcal{I}_H$)

For all $v \in V$ and for $H > R_0$,

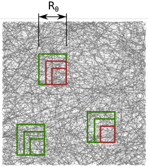
$$H^{-1}|v - \mathcal{I}_H v|_M + |\mathcal{I}_H v|_L \leq C|v|_L,$$

where $C = C_d \mu \sqrt{\sigma}$.

Network homogeneity and connectivity

Let $B_H(x)$ be a box at x of side length $2H$, with $H \geq R_0$.

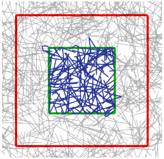
1. Limited mass variation



$$1 \leq \frac{\max_x |1|^2_{M,B_H(x)}}{\min_x |1|^2_{M,B_H(x)}} \leq \sigma(R_0)$$

2. The Poincaré-type inequality

$$|v - \bar{v}|_{M,B_H} \leq \mu(R_0) H |v|_{L,B_{H+R_0}}, \quad \forall v \in \hat{V}$$

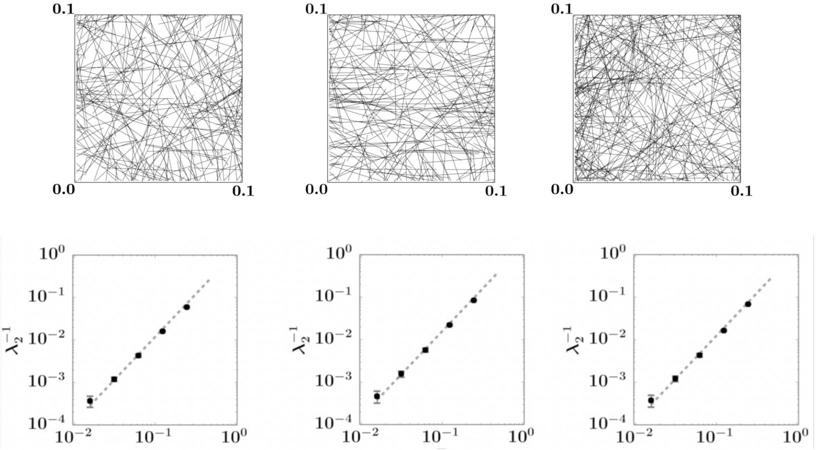


Algebraic connectivity:[a]
$L'\phi = \lambda M'\phi$, $\lambda_1 = 0$, $\lambda_2 = (\mu H)^{-2}$.

[a] Cheeger 1970, Fiedler 1973, Chung, Spectral graph theory, AMS, 1997

Example: Connectivity $\lambda_2^{-1/2} \approx \mu H$

Finite length fibers $r = 0.05$ and $|1|_M^2 = 1000$, $\Omega = [0,1]^2$

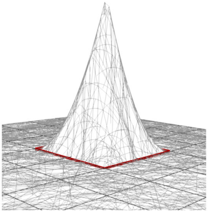


H varies from 2^{-2} to 2^{-6}. Here $R_0 \sim 2^{-6}$.

Subspace decomposition preconditioner[1]

Let $V_0 = V_H$ and

$$V_j = V(U(y_j)), \quad j = 1, \ldots, m.$$



Define projections $P_j : V \to V_j$ by

$$(KP_j v, v_j) = (Kv, v_j), \quad \forall v_j \in V_j \subset V.$$

We add the projections to form

$$P = P_0 + P_1 + \cdots + P_m.$$

- $BKu = Bf$, with preconditioner $P = BK$
- Preconditioned conjugate gradient method.
- Semi-iterative: direct method on decoupled problems

[1] Kornhuber & Yserentant, MMS, 2016

Convergence analysis

Lemma (Spectral bound of P)

For $H > 2R_0$ it holds

$$C_1^{-1}|v|_K^2 \leq (KPv, v) \leq C_2|v|_K^2, \quad \forall v \in V,$$

where $C_1 = C_d\beta\alpha^{-1}\sigma\mu^2$ and $C_2 = C_d$.

Interpolation bound is a crucial component of the proof.

Theorem (Convergence of PCG)

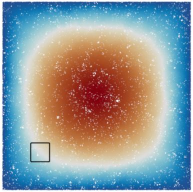
With $\kappa = C_1C_2$ and $H > 2R_0$ it holds

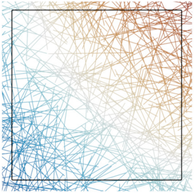
$$|u - u^{(\ell)}|_K \leq 2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^\ell |u - u^{(0)}|_K.$$

Outline

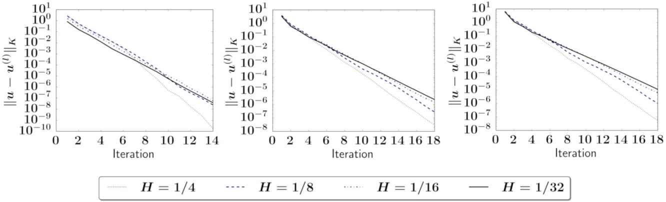
Example: Convergence graph Laplacian

$Ku = M1$, $(Kv, v) = \sum_{x \sim y} \gamma_{xy} \frac{((v(x)-v(y))^2}{|x-y|}$, $u|_{\partial\Omega} = 0$, $|1|_M^2 = 1000$.

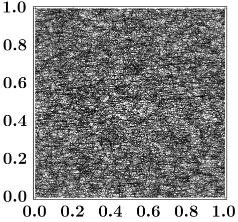


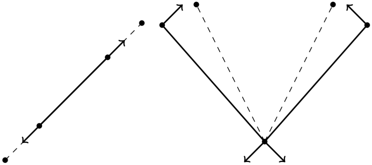


Grid $\gamma = 1$ (left), rand $\gamma = 1$ (center), rand $\gamma \in U([0.1, 1])$ (right)



Example: A fibre network model[2]





- $2 \cdot 10^4$ fibres, biased angle (x-axis), length 0.05, $3 \cdot 10^5$ nodes.
- Two forces in the model: edge extension and angular deviation.
- Find displacement u: $Ku = f$ (tensile, distributed load)
- DD with $H = 1/4, 1/8, 1/16, 1/32$.

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[2] Görtz et. al. *Network models for predicting structural properties of paper,* Nordic Pulp and Paper 2022

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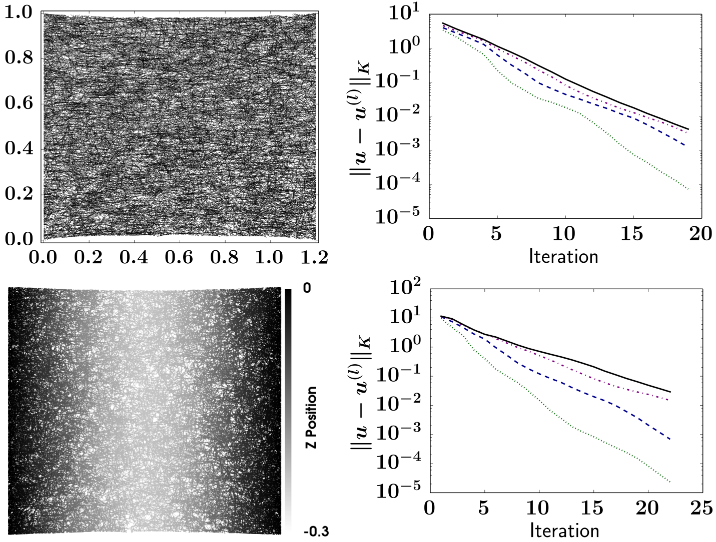
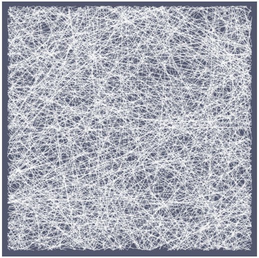


Outline

Conclusions



- Efficient preconditioner when network resembles a homogeneous material on coarse scales $H > 2R_0$
- Direct solver on fine scales (localized, in parallell)
- Poincaré type inequality plays a crucial role in the analysis
- Algebraic coarsening algorithms are useful (cardboard)

Görtz-Hellman-M., *Iterative solution of spatial network models by subspace decomposition,* Math. Comp. 2023 (online)