Numerical methods in poroelasticity

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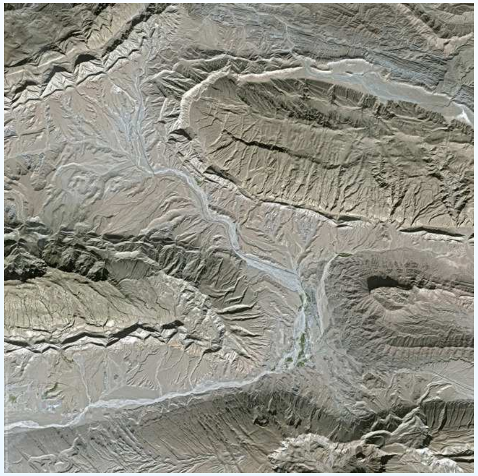
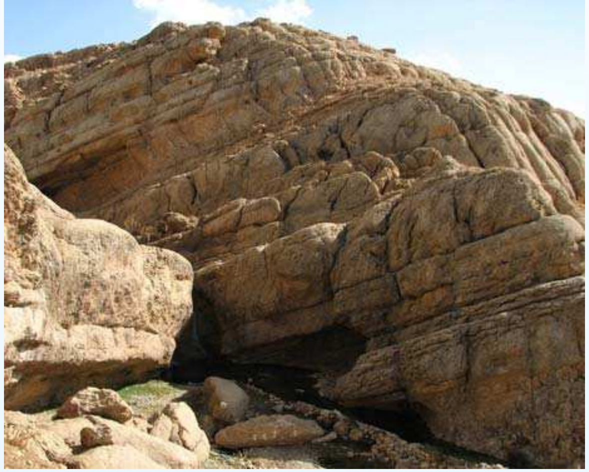
Numerical simulation in the earths subsurface

Understanding physical processes in the earths subsurface is very challenging and very important research field.


- Challenges: Rock formations often consist of porous material with multiscale features. Several physical processes are active at the same time.
- Apps: ground water flow, CO_2 storage, and oil recovery.

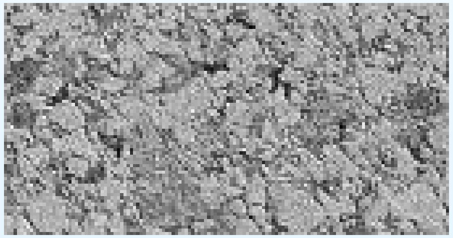
Numerical simulation is an increasingly powerful tool.

Poroelasticity

Linear elasticity for a porous medium with a fluid phase.

$$\begin{cases} S\dot{p} - \nabla \cdot \mathbf{\Lambda} \nabla p + \alpha \nabla \cdot \dot{\mathbf{u}} = q, \\ \nabla(\lambda + \mu)\nabla \cdot \mathbf{u} + \nabla \cdot \mu \nabla \mathbf{u} - \alpha \nabla p = \mathbf{r}, \end{cases}$$

where p is fluid pressure, $\mathbf{u}$ porous media displacement, and $\mathbf{\Lambda}$ is flow mobility, directly proportional to the permeability.

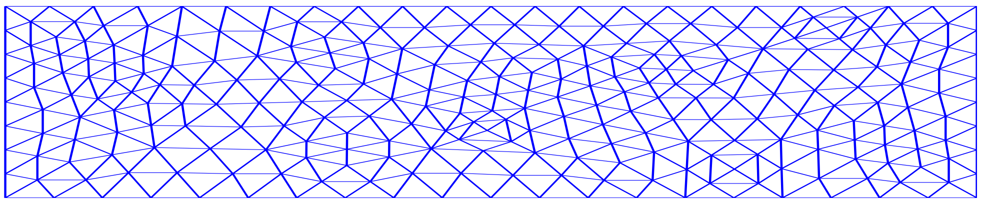


M. A. Biot, *General theory of three-dimensional consolidation,* J. Appl. Phys. 12, (1941) 155–164.

Numerical simulation

Given an iterate $(\hat{p}, \hat{\mathbf{u}})$, backward Euler in time gives:

$$\begin{cases} Sp - \triangle t \nabla \cdot \mathbf{\Lambda} \nabla p + \nabla \cdot \mathbf{u} = q \triangle t + S\hat{p} + \nabla \cdot \hat{\mathbf{u}}, \\ \nabla (\lambda + \mu) \nabla \cdot \mathbf{u} + \nabla \cdot \mu \nabla \mathbf{u} - \alpha \nabla p = \mathbf{r} \end{cases}$$



FEM is used to discretize in space using n nodes. In each time step we need to solve a linear $n \times n$ system: $Ax = b$.

Y. Yokoo, K. Yamagata, and H. Nagaoka, *Finite element method applied to Biot's consolidation theory,* Soils and Foundations, 11, (1971), 29–46.

Direct computation requires $\mathcal{O}(n^2)$ operations with $n > 10^6$.

Iterative methods and preconditioning

Richardson iteration: $x_{k+1} = x_k + \omega(b - Ax_k)$ gives

$$|x - x_k| \leq \left(\frac{\kappa(A) - 1}{\kappa(A) + 1}\right)^k |x - x_0|, \quad \text{for optimal } \omega.$$

If the number of iterations k to reach a tolerance is independent of n we would get $\mathcal{O}(n)$. However $\kappa(A) = \mathcal{O}(n^{2/d})$.

Instead consider $P^{-1}Ax = P^{-1}b$, where

1. $\kappa(P^{-1}A) \ll \kappa(A)$ improving the convergence,
2. $P^{-1}v$ is easy to compute.

One can even get optimal convergence rate $\mathcal{O}(n)$ using multigrid based preconditioners for many applications.

M. Griebel, D. Oeltz, and M. A. Schweitzer, *An Algebraic Multigrid Method for Linear Elasticity,* SIAM J. Sci. Comp., 25, (2003), 385–407.

Parallel implementation

Modern computer architecture calls for careful implementation of numerical algorithms, taking advantage of the massively parallel computers available.

Communication between processors is a bottle neck. Can we still get the optimal scaling $\mathcal{O}(n)$?

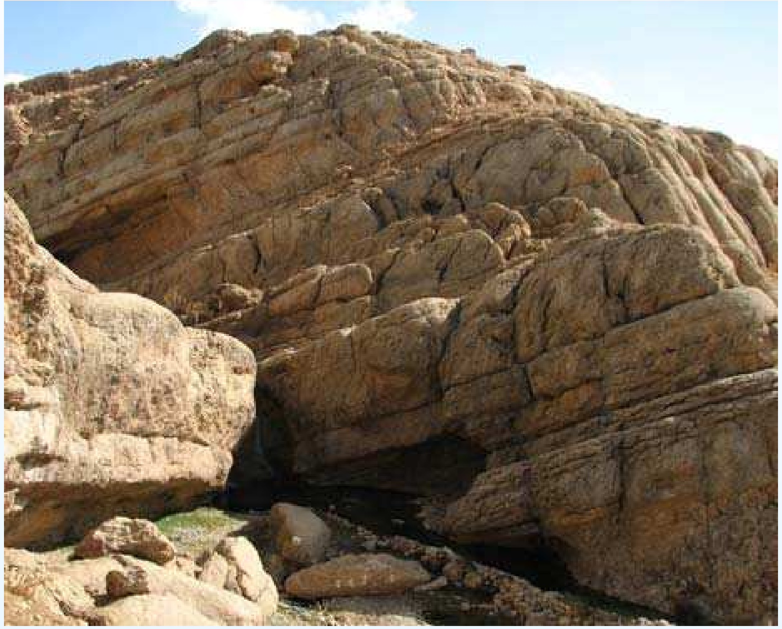
- scalar elliptic and parabolic problems
- system of linear elasticity

M. F. Adams, H. H. Bayraktar, T. M. Keaveny, and P. Papadopoulos, *Ultrascalable implicit finite element analysis in solid mechanics with over a half billion degrees of freedom,* Proc. ACM/IEEE Conference on Supercomputing, IEEE Computer Society, 2004.

W. Joubert and J. Cullum, *Scalable algebraic multigrid on 3500 processors,* Electronic Transactions on Numerical Analysis, 23, (2006), 105–128.

Does it work for the poroelastic problem?

Poroelasticity



Two major complications:

- the problem consists of two coupled equations and we need to precondition the whole system.
- multiscale features in the permeability, in particular regions with low permeability ϵ are present $\Rightarrow \kappa(A) \sim \epsilon^{-1}$.

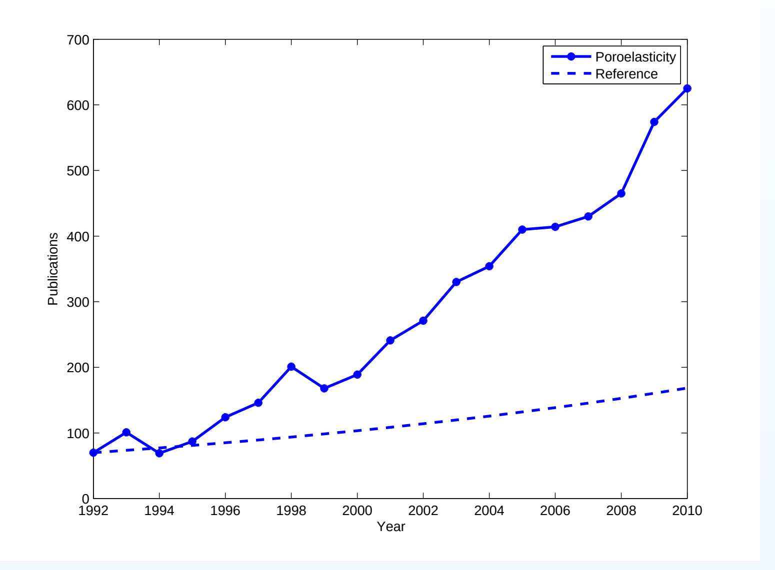
Contributions in this thesis

C. J. B. Haga and coauthors,

- develop algebraic multigrid based pre-conditioners for the system of poroelasticity
- analyze how low permeable regions affect convergence of algebraic multigrid
- perform large scale computations with over 500 processors showing promising results in terms of scalability
- perform careful numerical investigation using several different formulations of the problem, iterative solvers, finite elements, and considering both 2D and 3D geometries

The result is a very useful guide on how to solve these challenging problems numerically.

General comments



- It is a vibrant research field
- Spans over a wide range of disciplines including geoscience, mathematical modeling, mathematical analysis, numerical analysis, and parallel implementation
- Poroelastic computations is an important tool for understanding geological processes
- Important engineering applications relevant for Norway