Localization of multiscale problems

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Applied Mathematics Seminar

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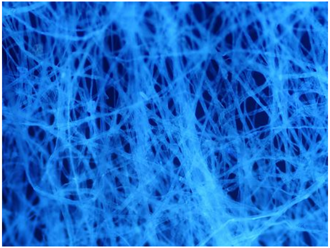
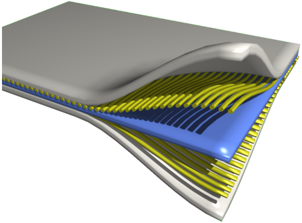
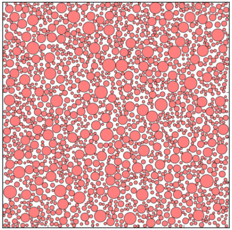
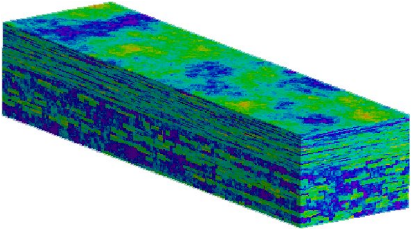
Multiscale materials


Multiscale problems

We consider applications such as





▹ composite materials ▹ flow in a porous medium

that require numerical solution of partial differential equations with rough data (module of elasticity, conductivity, or permeability).

Major challenge: Features on multiple non-separated scales.

Outline

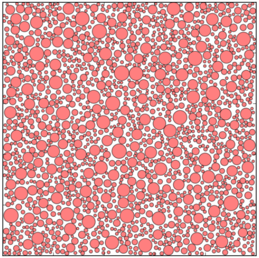
1. **Elliptic model problem**
2. Introduction to LOD
3. High contrast data
4. Applications
5. Conclusions

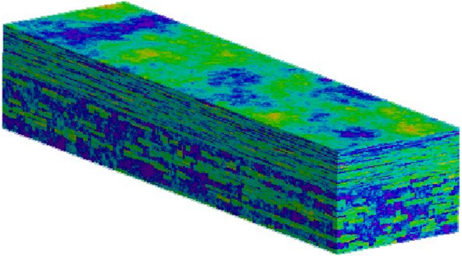
Elliptic model problem

The Poisson equation

$$-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$.





Finite element method

The Poisson equation (weak form): $u \in V = H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega (A\nabla u) \cdot \nabla v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x \quad \text{for all } v \in V$$

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The Poisson equation (FE approximation): $u_h \in V_h \subset V$ such that

$$a(u_h, v) := \int_\Omega (A \nabla u_h) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \text{for all } v \in V_h$$

with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$.

Numerical error (piecewise linear continuous FE approximation)

- For solution $u \in H^2(\Omega)$ we have

$$|||u - u_h||| := \|A^{1/2} \nabla (u - u_h)\|_{L^2(\Omega)} \leq C\beta^{1/2} h \|D^2 u\|_{L^2(\Omega)} \sim C(\alpha, \beta, A')h.$$

- The mesh size h has to resolve the variations in A, e.g. $h < \epsilon$ if A is ϵ-periodic.

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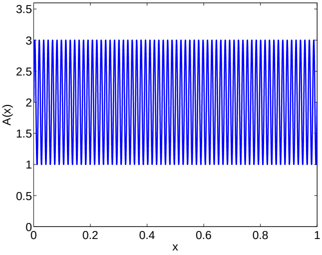
with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$.

Example (periodic coefficient): $A(x) = 2 + \sin(2\pi x/\varepsilon)$, $\varepsilon = 2^{-6}$, $f = 1$



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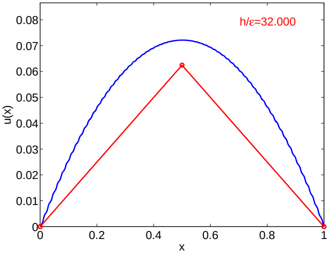
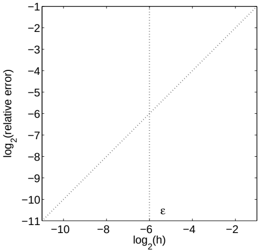


solution and P1-FEM-approximation



$\log_2(H^1(\Omega) - \text{error})$ vs. $\log_2(h)$

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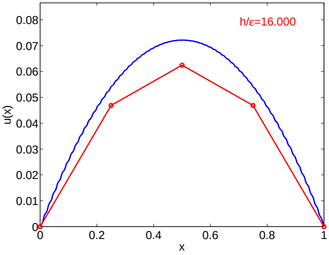

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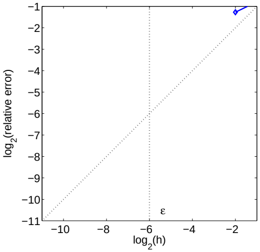

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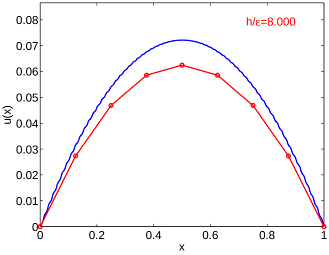

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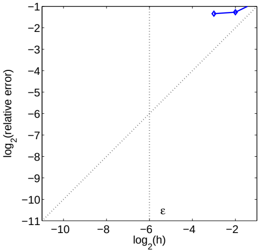

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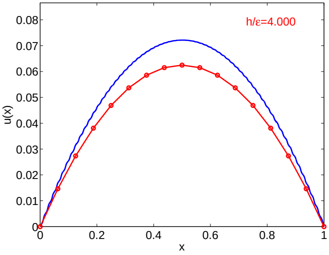

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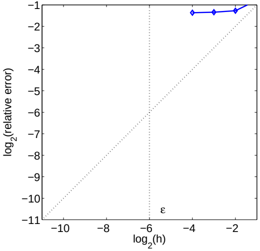

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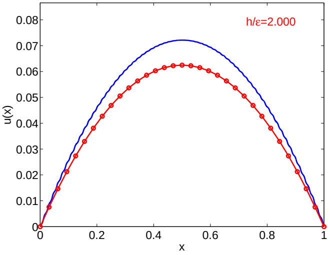

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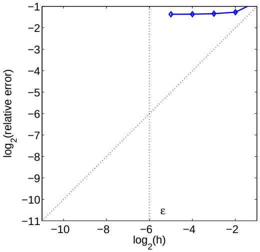

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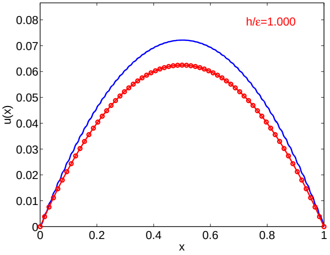

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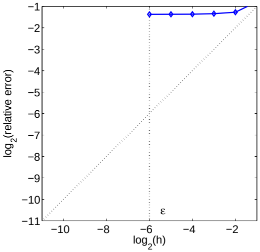

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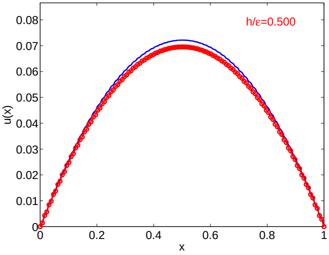

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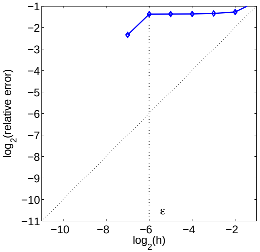

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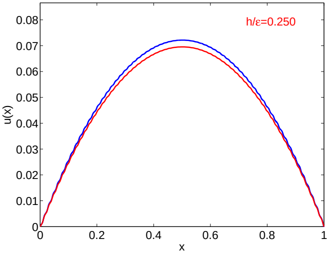

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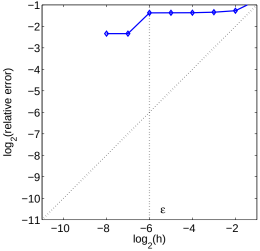

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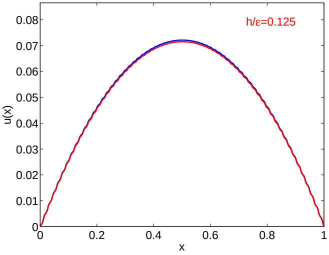

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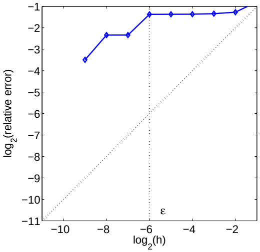

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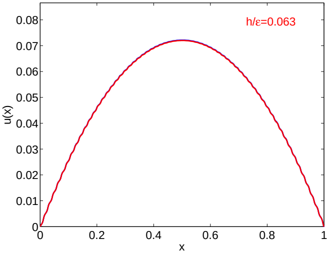

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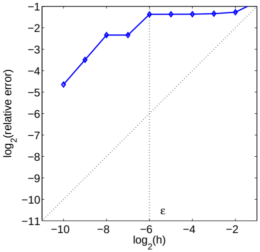

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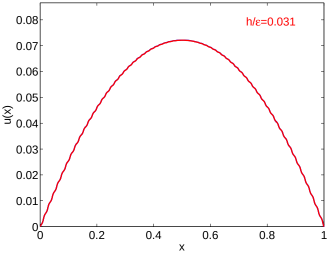

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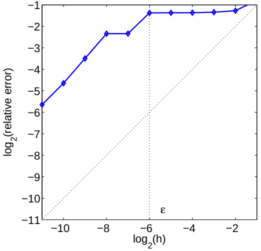

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Comments

- No convergence until fine scale variations are resolved.
- FEM uses the average of the diffusion ($= 2$) on coarse meshes while Homogeization theory suggests harmonic average ($= 1.5$).
- The error is clearly large in H^1 but as seen also in L^2.

New numerical approach

Objectives:

- Find a subspace of $V_H^{\mathrm{ms}} \subset V_h$ for which the Galerkin approximation fulfills

$$|||u_h - u_H^{\mathrm{ms}}||| \leq C(\alpha,\beta)H \approx C(\alpha,\beta,A')h,$$

  but with $\dim(V_H^{\mathrm{ms}}) \ll \dim(V_h)$.
- Show that a basis for V_H^{ms} can be constructed by local parallel computations.
- Demonstrate efficiency for applications where V_H^{ms} is reused (eigenvalue, time dependent, semi-linear, systems).

Variational multiscale method, (Hughes), 1995
Local problems on stars, (Pedro, Nochetto, Kunibert), 2003

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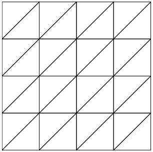
Multiscale decomposition

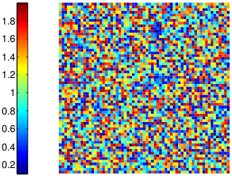
- (coarse) FE mesh $\mathcal{T}$ with parameter $H > h$
- P1-FE space $V_H := \{v \in V \mid \forall T \in \mathcal{T}, v|_T \in P_1(T)\}$
- $\mathfrak{I}_\mathcal{T} : V \to V_H$ some interpolation operator



Decomposition

$$V = V_H \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \text{kernel}\, \mathfrak{I}_\mathcal{T} = \{v \in V \mid \mathfrak{I}_\mathcal{T} v = 0\}$$

Example:



rough coefficient

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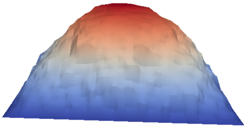

$u \in V$

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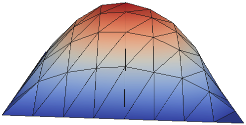
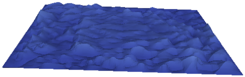


$u_H \in V_H$

+



$u^{\mathrm{f}} \in V^{\mathrm{f}}$

Orthogonalization

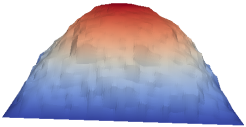
- For each $v \in V$ define finescale projection $Qv \in V^{\mathrm{f}}$ by

$$a(Qv, w) = a(v, w) \quad \text{for all } w \in V^{\mathrm{f}}$$

a-Orthogonal Decomposition

$$V = V_H^{\mathrm{ms}} \oplus V^{\mathrm{f}} \quad \text{with } V_H^{\mathrm{ms}} := (V_H - QV_H)$$

Example:



$u \in V$

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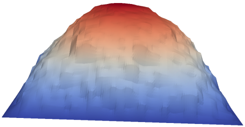


$u_H^{\mathrm{ms}} \in V_H^{\mathrm{ms}}$

$+$

$u^{\mathrm{f}} \in V^{\mathrm{f}}$

Ideal multiscale representation

Given the space V_H^{ms} we construct a Galerkin approximation:

Ideal method

Find $u_H^{\mathrm{ms}} \in V_H^{\mathrm{ms}}$ such that

$$a(u_H^{\mathrm{ms}}, v) = (f, v), \ \forall v \in V_H^{\mathrm{ms}}.$$

We have that $u - u_H^{\mathrm{ms}} = u_{\mathrm{f}} \in V^{\mathrm{f}}$ since u_H^{ms} is the a-orthogonal projection of u onto V_H^{ms}. Therefore

$$|||u_{\mathrm{f}}|||^2 = a(u, u_{\mathrm{f}}) = (f, u_{\mathrm{f}}) = (f, u_{\mathrm{f}} - \mathfrak{I}_{\mathcal{T}} u_{\mathrm{f}}) \leq \frac{C_{\mathfrak{I}_{\mathcal{T}}}}{\alpha^{1/2}} \|Hf\|_{L^2(\Omega)} |||u_{\mathrm{f}}|||.$$

For V_H^{ms} to be useful we need a discrete local basis.

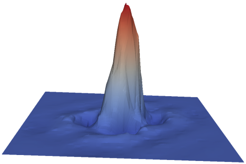
Modified nodal basis

- $\mathcal{N}$ denotes set of interior vertices of $\mathcal{T}$
- $\phi_x \in V_H$ denotes classical nodal basis function ($x \in \mathcal{N}$)
- $Q\phi_x \in V^{\mathrm{f}}$ denotes the finescale correction of ϕ_x ($x \in \mathcal{N}$)

Ideal multiscale FE space

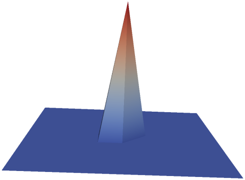
$$V_H^{\mathrm{ms}} = \mathrm{span}\,\{\phi_x - Q\phi_x \mid x \in \mathcal{N}\}$$

Example



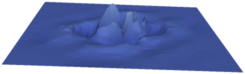
$\phi_x - Q\phi_x \in V_H^{\mathrm{ms}}$

$=$



$\phi_x \in V_H$

$-$



$Q\phi_x \in V^{\mathrm{f}}$

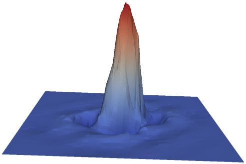
Modified nodal basis

- $\mathcal{N}$ denotes set of interior vertices of $\mathcal{T}$
- $\phi_x \in V_H$ denotes classical nodal basis function ($x \in \mathcal{N}$)
- $Q\phi_x \in V^{\mathrm{f}}$ denotes the finescale correction of ϕ_x ($x \in \mathcal{N}$)

Ideal multiscale FE space

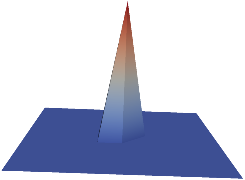
$$V_H^{\mathrm{ms}} = \operatorname{span}\{\phi_x - Q\phi_x \mid x \in \mathcal{N}\}$$

Example



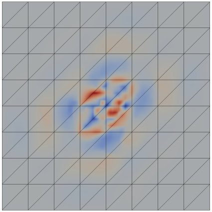
$\phi_x - Q\phi_x \in V_H^{\mathrm{ms}}$

$=$



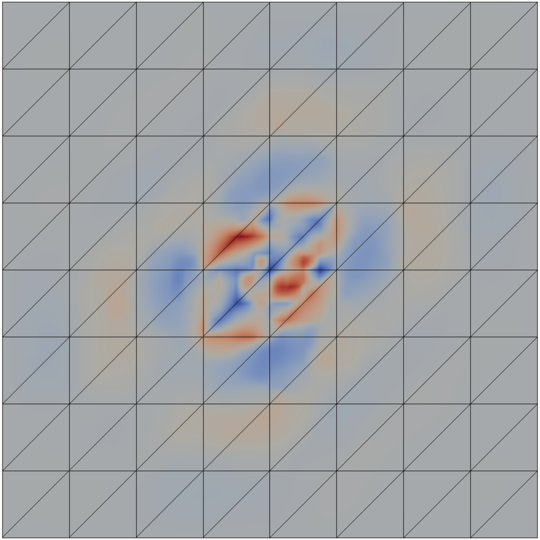
$\phi_x \in V_H$

$-$



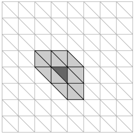
$Q\phi_x \in V^{\mathrm{f}}$

Modified nodal basis

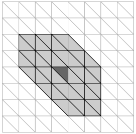


Localization

- Define nodal patches of ℓ-th order $\omega_{T,\ell}$ about $T \in \mathcal{T}$



$\omega_{T,1}$



$\omega_{T,2}$

- Correctors $Q_\ell^T \phi_x \in V^{\mathrm{f}}(\omega_{T,\ell}) := \{v \in V^{\mathrm{f}} \mid v|_{\Omega \setminus \omega_{T,\ell}} = 0\}$ solve

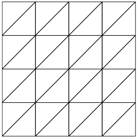
$$a(Q_\ell^T \phi_x, w) = \int_T A \nabla \phi_x \cdot \nabla w \, dx \quad \text{for all } w \in V^{\mathrm{f}}(\omega_{T,\ell})$$

Localized multiscale FE spaces

$$V_{H,\ell}^{\mathrm{ms}} = \mathrm{span}\{\phi_x - \sum_{T \in \mathcal{T}} Q_\ell^T \phi_x \mid x \in \mathcal{N}\}$$

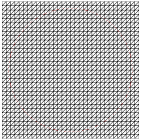
Fine scale discretization

- Finescale mesh



mesh refinement

$\mathcal{T} \quad \rightsquigarrow \quad \mathcal{T}_h$ with $h \leq H$



- Reference FE space

$$V_h := \{v \in V \mid \forall T \in \mathcal{T}(\Omega), v|_T \in P_1(T)\}$$

- Reference FE solution $u_h \in V_h$ solves

$$a(u_h, v) = (f, v) \quad \text{for all } v \in V_h$$

- Fully discrete correctors $Q^T_{\ell,h}\phi_x \in V^f_h(\omega_{T,\ell}) := V^f(\omega_{T,\ell}) \cap V_h$:

$$a(Q^T_{\ell,h}\phi_x, w) = (A\nabla\phi_x, \nabla w)_T \;\; \text{for all } w \in V^f_h(\omega_{T,\ell})$$

Localized Orthogonal Decomposition (LOD)

Fully discrete multiscale FE spaces

$$V_{H,\ell}^{\mathrm{ms},h} = \mathrm{span}\{\phi_x - \sum_{T\in\mathcal{T}} Q_{\ell,h}^{T}\phi_x \mid x \in \mathcal{N}\}$$

Fully discrete multiscale approximation $u_{H,\ell}^{\mathrm{ms},h} \in V_{H,\ell}^{\mathrm{ms},h}$

$$a(u_{H,\ell}^{\mathrm{ms},h}, v) = (f, v) \quad \text{for all } v \in V_{H,\ell}^{\mathrm{ms},h}$$

Remarks:

- $\dim V_{H,\ell}^{\mathrm{ms},h} = |\mathcal{N}| = \dim V_H$
- The basis functions have local support, with overlap depending on ℓ, and are independent.

A priori error analysis

Lemma (Truncation error)

$$|||Q_h v_H - Q_{\ell,h} v_H||| \leq C_1 \gamma^{\ell} |||Q_h v_H|||, \quad \forall v_h \in V_H$$

$C_1 < \infty$ and $\gamma < 1$ depends on β/α but not A'.

By choosing $\ell = C_2 \log(H^{-1})$ with appropriate C_2 we guarantee that the truncation leads to a higher order perturbation:

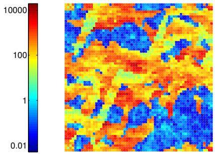
Theorem (A priori error bound)

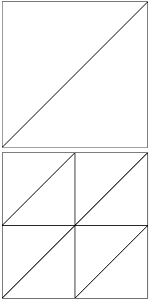
$$|||u_h - u_{H,\ell}^{\mathrm{ms},h}||| \leq C(\alpha,\beta)H,$$

with C independent of A'.

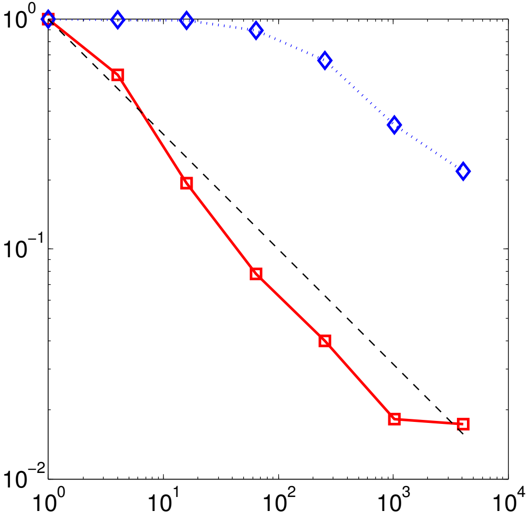
M. & Peterseim, Localization of elliptic multiscale problems, 2014.

Numerical experiment: Poisson's equation



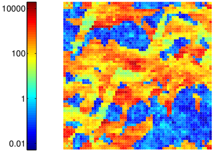


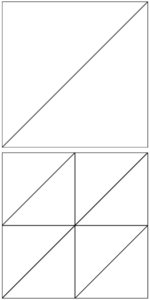
$H = 2^{-1}, 2^{-2}, \ldots, 2^{-7}$
$h = 2^{-9}$, $\ell = \log(1/H)$

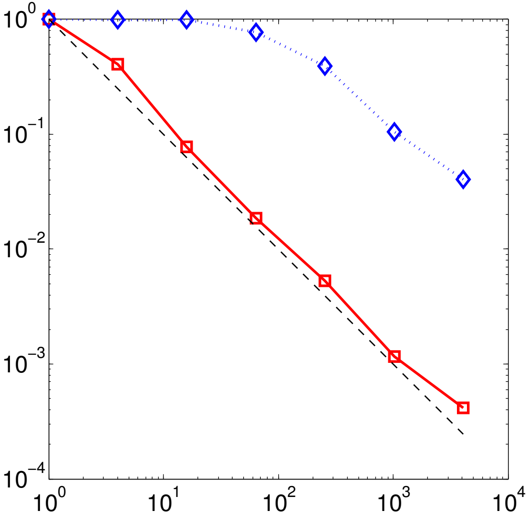


$|||u_h - u^{\mathrm{ms},h}_{H,\ell}|||$ vs. #dof

Numerical experiment: Poisson's equation



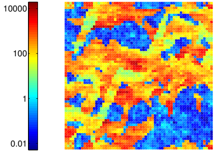


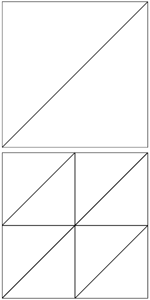


$H = 2^{-1}, 2^{-2}, \ldots, 2^{-7}$
$h = 2^{-9}$, $\ell = \log(1/H)$

$\|u_h - u^{\mathrm{ms},h}_{H,\ell}\|$ vs. #dof

Numerical experiment: Poisson's equation





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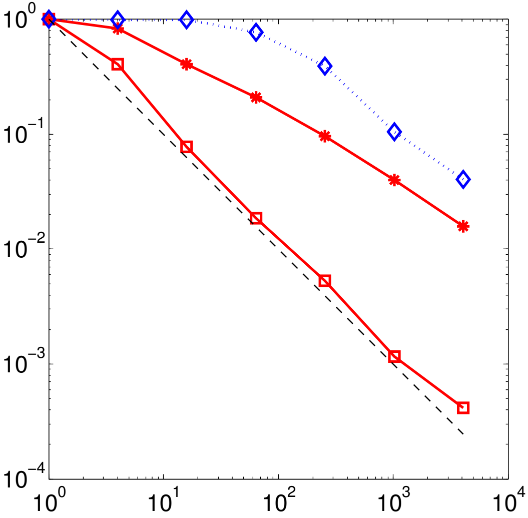
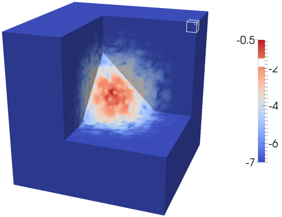


$\|u_h - \mathfrak{I}_{\mathcal{T}} u^{ms,h}_{H,\ell}\|$ vs. #dof

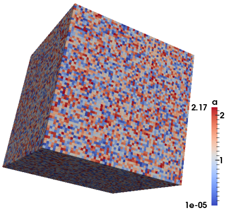
3D implementation in python

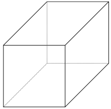
- Trilinear shape functions on cubes.
- Petrov-Galerkin formulation reduces communication, Elfverson et.al. Numer. Math. 2016.
- Storage of all basis function is not needed. The full solution can be recomputed (at a lower cost) once $\mathfrak{I}_{\mathcal{T}} u_{H,\ell}^{\mathrm{ms},h}$ is computed.

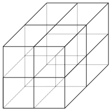


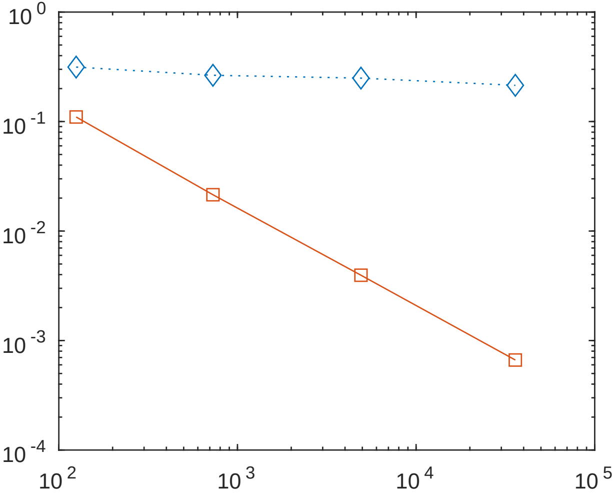
Corrector function $Q^T \phi_x$, implementation by Fredrik Hellman.

Numerical experiment: Poisson's equation 3D





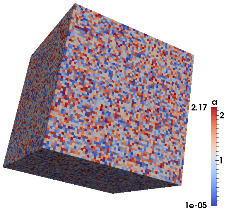


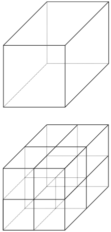


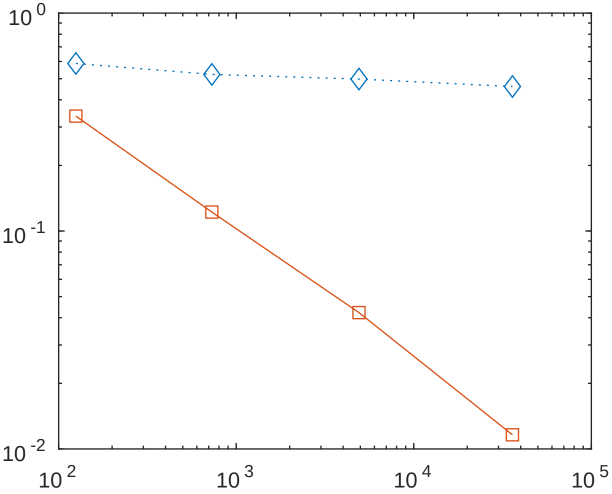
$H = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$
$h = 2^{-6}$, $\ell = \log(1/H)$

$|||u_h - u^{\mathrm{ms},h}_{H,\ell}|||$ vs. #dof

Numerical experiment: Poisson's equation 3D



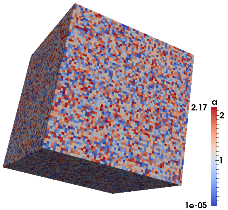


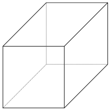


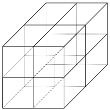
$H = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$
$h = 2^{-6}$, $\ell = \log(1/H)$

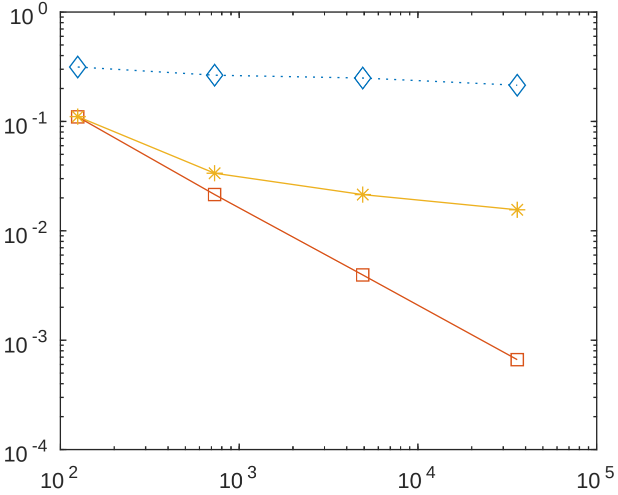
$\|u_h - u_{H,\ell}^{\mathrm{ms},h}\|$ vs. #dof

Numerical experiment: Poisson's equation 3D









$H = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$
$h = 2^{-6}$, $\ell = \log(1/H)$

$\|u_h - \mathfrak{I}_{\mathcal{T}} u^{\mathrm{ms},h}_{H,\ell}\|$ vs. #dof

Some relevant references

- Variational Multiscale Method, (Hughes et.al.), 1995.
- Multiscale Finite Element Method, (Hou & Wu), 1996.
- Local problems on stars (AFEM), (Morin, Nochetto, Siebert), 2003.
- Heterogeneous Multiscale Method, (Engquist & E), 2003.
- Adaptive Variational Multiscale Method, (Larson & M.), 2004-2007.
- VMS, Fine scale Green's functions, (Hughes & Sangalli), 2007.
- Flux-norm approach, (Berlyand-Owhadi) 2010, (Owhadi-Zhang) 2011.
- GFEM, local eig., (Babuška & Lipton), 2011, (Efendiev et.al.), 2013.
- AL-Basis (Grasedyck, Greff, Sauter), 2012.
- Localized Orthogonal Decomposition, (M. & Peterseim), 2014.
- Baysian numerical homogenization, (Owhadi), 2014.
- Iterative numerical homogenization, (Kornhuber & Yserentant), 2015.
- LOD, High contrast, (Peterseim & Scheichl), 2016.

There are numerous other related methods not listed here.

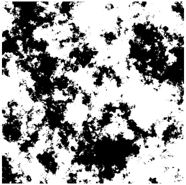
Outline

High contrast data

Poisson equation:

$$-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega.$$

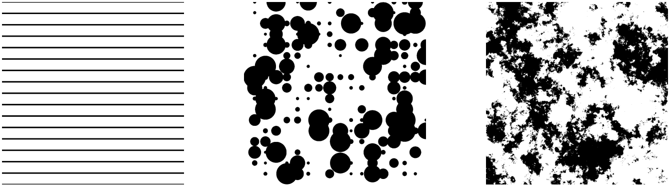
$A = 1$ in Ω_1 (black), $A = \alpha$ in Ω_α, $\alpha \ll 1$, and $f = \chi_{[1/4,3/4]^2}$.



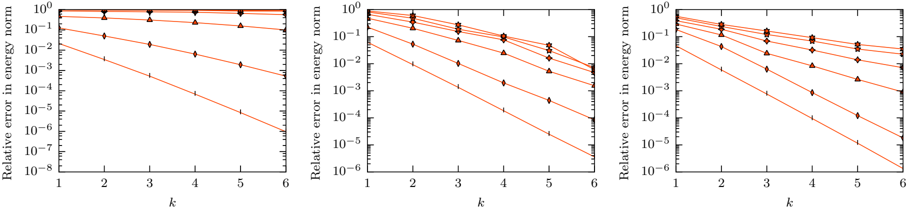
- High contrast data with channels leads to non-local behaviour.
- The decay rate of the basis functions determines the accuracy of LOD.
- The choice of interpolant $\mathfrak{I}_{\mathcal{T}} v = \sum_{x \in \mathcal{N}} \bar{v}_{\omega_x} \phi_x$ affects the decay.

Numerical example: High contrast

High contrast data Three examples: $H = 2^{-4}$, $h = 2^{-10}$,



We let $\alpha = 10^{-1}, \ldots, 10^{-6}$ and plot $|||u_h - u_{H,k}^{\mathrm{ms},h}|||$ vs. k, with $\mathfrak{I}_{\mathcal{T}}^{SZ}$,



Heuristic motivation for lack of decay

Fine scale equation: Correctors $Q^T v_H \in V^{\mathrm{f}} = \ker(\mathfrak{I}_{\mathcal{T}})$ solve

$$a(Q^T v_H, w) = \int_T A \nabla v_H \cdot \nabla w \, dx \quad \text{for all } w \in V^{\mathrm{f}}$$

Decay because localized rhs and $\mathfrak{I}_{\mathcal{T}}(Q^T v_H) = 0 \rightarrow Q^T v_H(x) \approx 0$.

If we define $g := Q^T v_H|_{\partial T}$ we note that $Q^T v_H$ minimizes

$$\frac{1}{2} \| A^{1/2} \nabla Q^T v_H \|^2_{L^2(\Omega \setminus T)} = \min_{v_f \in V^{\mathrm{f}} : v|_{\partial T = g}} \frac{1}{2} \| A^{1/2} \nabla v_f \|^2_{L^2(\Omega \setminus T)}.$$

- High derivatives in Ω_1 are penalized.
- With $\mathfrak{I}_{\mathcal{T}} v = \sum_{x \in \mathcal{N}} \bar{v}_{\omega_x} \phi_x$ and ω_x containing both Ω_1 and Ω_α, $\mathfrak{I}_{\mathcal{T}}(Q^T v_H) = 0$ still allows large values (and small derivatives) in Ω_1 and high derivatives in Ω_α.
- To make $Q^T v_H$ decay in Ω_1 we need $\omega_x \subset \Omega_1$.

Scott-Zhang type interpolation

Nodal variables:

Let $x \in \mathcal{N}$ be nodes of $\mathcal{T}$ and $\sigma_x \subset \Omega$ associated domains. We define a $L^2(\sigma_x)$-dual basis $\psi_x \in V_H$ fulfilling,

$$\int_{\sigma_x} \psi_x \phi_y = \delta_{xy}.$$

Let the nodal variable $N_x(v) = \int_{\sigma_x} \psi_x v$ and,

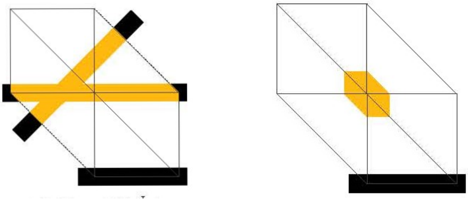
$$\mathfrak{I}_{\mathcal{T}}^{\sigma} v = \sum_{x \in \mathcal{N}} N_x(v) \phi_x.$$

- σ_x does not need to be full elements T or vertex patches $U_1(x)$.
- The stability of $|N_x(v)| \leq ||\psi||_{L^2(\sigma_x)} ||v||_{L^2(\sigma_x)}$ depends on the size and shape of σ_x and its distance to x.

Geometry dependent interpolation

Selection of σ_x:

By letting $\sigma_x \subset \Omega_1$ (frequently enough) we guarantee decay, i.e. nodes in high conductivity channels are needed.



Let $U_1(x)$ be the vertex patch at node x.

- Type I node: for $x \in \Omega_1$ let $\sigma_x \subset U_1(x) \cap \Omega_1$, connected, and chosen so $\inf_{q \in \mathbb{R}} \|v - q\|_{L^2(\sigma_x)} \leq CH\|\nabla v\|_{L^2(\sigma_x)}$ holds (Poincaré).
- Type II node: for $x \in \Omega_\alpha$ let $\sigma_x = U_\delta(x), \quad 0 < \delta \leq 1,$

Weighted Poincaré inequality and decay

The following weighted Poincaré inequality holds:

$$\|A^{1/2}v_f\|_{L^2(T)} \leq CH\|A^{1/2}\nabla v_f\|_{L^2(U_1(T))}, \quad \forall v_f \in V^{\mathrm{f}} = \ker(\mathfrak{I}_{\mathcal{T}}^{\sigma}).$$

This is used to prove contrast independent decay.

Theorem

With $\delta < 1/2$ we have,

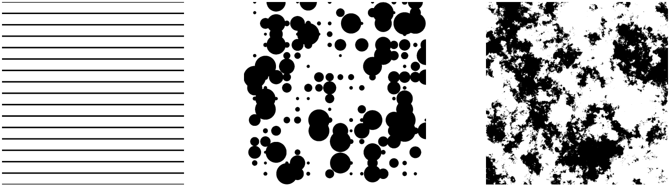
$$\|A^{1/2}\nabla Q^T v_H\|_{\Omega\setminus U_k(T)} \leq C\gamma^k\|A^{1/2}\nabla Q^T v_H\|_{L^2(\Omega)},$$

where C and γ are independent of β/α.

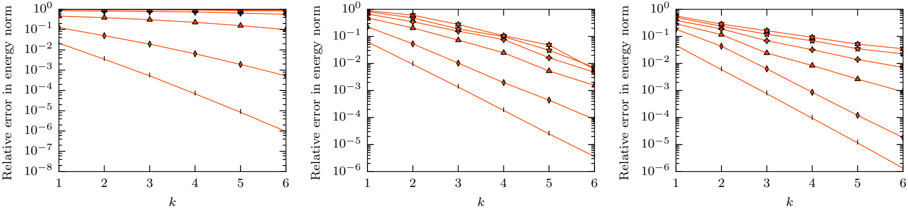
Hellman & M. Contrast independent localization of multiscale problems, arXiv

Numerical example: High contrast

High contrast data Three examples: $H = 2^{-4}$, $h = 2^{-10}$,



We let $\alpha = 10^{-1}, \ldots, 10^{-6}$ and plot $|||u_h - u^{\mathrm{ms},h}_{H,k}|||$ vs. k with $\mathfrak{I}^{SZ}_{\mathcal{T}}$,



Numerical example: High contrast

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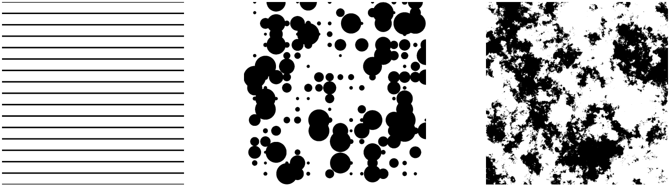
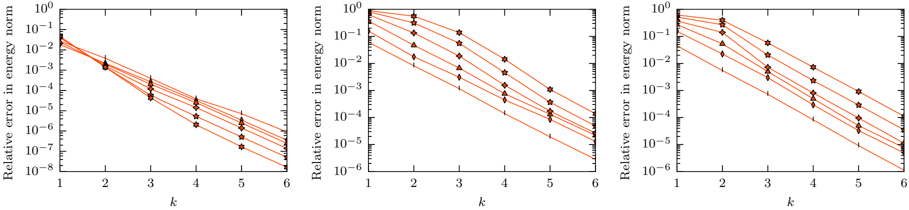


We let $\alpha = 10^{-1}, \ldots, 10^{-6}$ and plot $|||u_h - u_{H,k}^{\mathrm{ms},h}|||$ vs. k with $\mathfrak{I}_{\mathcal{T}}^{\sigma}$,



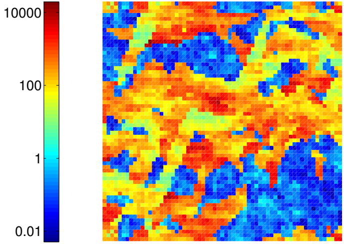
Outline

Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem

$$-\nabla \cdot A\nabla u = \lambda u \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

with data $0 < \alpha \leq A \leq \beta < \infty$

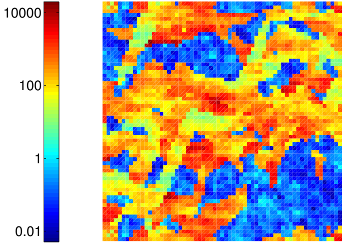


Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem (variational form): find $u \in V := H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$a(u,v) := \int_\Omega (A\nabla u)\cdot\nabla v\,dx = \lambda \int_\Omega u\cdot v\,dx \quad \text{for all } v \in V$$

with data $0 < \alpha \leq A \leq \beta < \infty$

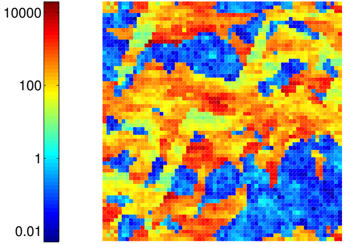


Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem (FE approximation):
$u_h \in V_h \subset V$ and $\lambda_h \in \mathbb{R}$ such that

$$a(u_h, v) := \int_\Omega (A \nabla u_h) \cdot \nabla v \, dx = \lambda_h \int_\Omega u_h \cdot v \, dx \quad \text{for all } v \in V_h$$

with data $0 < \alpha \leq A \leq \beta < \infty$



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Numerical error (piecewise linear continuous FE approximation)

- For an eigenpair $(u^{(k)}, \lambda^{(k)})$ with $u^{(k)} \subset H^2(\Omega)$ it holds

$$\lambda^{(k)} \leq \lambda_h^{(k)} \leq \lambda^{(k)} + C(\alpha, \beta, A', k) h^2,$$

$$|||u^{(k)} - u_h^{(k)}||| := \|A^{1/2} \nabla (u^{(k)} - u_h^{(k)})\|_{L^2(\Omega)} \leq C(\alpha, \beta, A', k) h.$$

- The mesh size h has to resolve the variations in A, e.g. $h < \epsilon$ if A is periodic.

LOD approximation

LOD: Find $u^{\mathrm{ms},h}_{H,\ell} \in V^{\mathrm{ms},h}_{H,\ell}$, $\lambda^{\mathrm{ms},h}_{H,\ell} \in \mathbb{R}$

$$a(u^{\mathrm{ms},h}_{H,\ell}, v) = \lambda^{\mathrm{ms},h}_{H,\ell}(u^{\mathrm{ms},h}_{H,\ell}, v) \quad \text{for all } v \in V^{\mathrm{ms},h}_{H,\ell}$$

Theorem

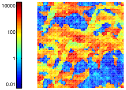
$$\lambda_h^{(k)} \leq \lambda_{H,\ell}^{ms,h,(k)} \leq \lambda_h^{(k)} + CH^4,$$

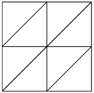
$$|||u_h^{(k)} - u_{H,\ell}^{ms,h,(k)}||| \leq CH^2,$$

with C independent of A' and the regularity of the eigenfunctions and (λ_h, u_h) is the reference solution.

M. & Peterseim, Computation of eigenvalues by nume. upscaling, 2015.

Numerical example: eigenvalues





k	$\lambda_h^{(k)}$	$e^{(k)}(1/2\sqrt{2})$	$e^{(k)}(1/4\sqrt{2})$	$e^{(k)}(1/8\sqrt{2})$	$e^{(k)}(1/16\sqrt{2})$
1	21.4144522	5.472755371	0.237181706	0.010328293	0.000781683
2	40.9134676	-	0.649080539	0.032761482	0.002447049
3	44.1561133	-	1.687388874	0.097540102	0.004131422
4	60.8278691	-	1.648439518	0.028076168	0.002079812
5	65.6962136	-	2.071005692	0.247424446	0.006569640
6	70.1273082	-	4.265936007	0.232458016	0.016551520
7	82.2960238	-	3.632888104	0.355050163	0.013987920
8	92.8677605	-	6.850048057	0.377881216	0.049841235
9	99.6061234	-	10.305084010	0.469770376	0.026027378
10	109.1543283	-	-	0.476741452	0.005606426
11	129.3741945	-	-	0.505888044	0.062382302
12	138.2164330	-	-	0.554736550	0.039487317
13	141.5464639	-	-	0.540480876	0.043935515
14	145.7469718	-	-	0.765411709	0.034249528
15	152.6283573	-	-	0.712383825	0.024716759
16	155.2965039	-	-	0.761104705	0.026228034
17	158.2610708	-	-	0.749058367	0.091826207
18	164.1452194	-	-	0.840736127	0.118353184
19	171.1756923	-	-	0.946719951	0.111314058
20	179.3917590	-	-	0.928617606	0.119627862

Table: Errors $e^{(k)}(H) =: \frac{\lambda_H^{\mathrm{ms},(k)} - \lambda_h^{(k)}}{\lambda_h^{(k)}}$ and $h = 2^{-7}\sqrt{2}$.

More applications

Stationary/eigenvalue problems

- Gross-Pitaevskii, (Henning, M., Peterseim), 2014.
- Helmholtz, (Gallistl & Peterseim), 2015.
- Reduced basis, (Abdulle & Henning), 2015.
- Quadratic eigenvalue problems, (M. & Peterseim), 2016.
- Elasticity, (Henning & Persson), 2016.
- High contrast, (Peterseim & Scheichl), 2016.
- Helmholtz, (Ohlberger & Verfürth), 2017.
- Iterative solvers, (Kornhuber & Yserentant), 2017.

Time-dependent problems

- Thermoelasticity, (M. & Persson), 2017.
- Wave equation, (Abdulle & Henning), 2017.
- Maxwell, (Gallistl & Henning), 2017. (ongoing...)

Outline

Conclusion and outlook

- By LOD we compute an effective stiffness matrix on a coarse scale. (numerical homogenization)
- Rapidly varying diffusion of low contrast is well understood.
- Recent development in high contrast problems which is a great challenge for any method.
- Great reduction in computational cost when the basis is reused (load cases, eigenvalues, time dependent, non-linear, control).
- Future challenges: weakly random diffusion/nearly periodic (defects), efficient implementation, optimization (thin structures).

Thank you for your attention!