Numerical homogenization of multiscale problems

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New trends in asymptotic methods for multiscale PDEs

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2019-10-23

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We consider applications such as composite materials and flow in a porous medium that require numerical solution of partial differential equations with rough data (module of elasticity, conductivity, or permeability).

Major challenge: Features on multiple non-separated scales.
Outline

1. Elliptic problem: Homogenization and FEM
2. Introduction to LOD
3. High contrast data
4. Applications
5. Conclusions
Elliptic model problem

The Poisson equation

\[-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial \Omega\]

with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$. 
Homogenization of an elliptic model problem

The Poisson equation

\[-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial \Omega\]

with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$.

Homogenization theory Let $A = A(x/\epsilon)$ be $\epsilon$-periodic, and consider

\[-\nabla \cdot (A(x/\epsilon) \nabla u_\epsilon(x, x/\epsilon)) = f(x).\]

It can be shown that as $\epsilon \to 0$, $u_\epsilon \to v$ solves

\[-\nabla \cdot (A^* \nabla v(x)) = f(x),\]

- In 1D, $A^* = \frac{1}{\langle 1/A \rangle}$, i.e. the harmonic average.
- Otherwise $A^*$ is computed by solving a cell problem.
Homogenization of an elliptic model problem

The Poisson equation

\[-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial \Omega\]

with data \(0 < \alpha \leq A \leq \beta < \infty\) and \(f \in L^2(\Omega)\).

Homogenization theory

- Considers \(\epsilon \to 0\), i.e. large scale separation.
- Periodic data leads to a references cell problem.
- Single scale rather than a continuum of scales in data.
- Numerical approaches inspired by homogenization theory (e.g. HMM and MsFEM).

Multiscale finite element method, (Hou & Wu), 1996.
Heterogeneous multiscale method, (Engquist & E), 2003.
The Poisson equation (weak form): \( u \in V = H^1_0(\Omega) \) such that

\[
a(u, v) := \int_{\Omega} (A \nabla u) \cdot \nabla v \, dx = \int_{\Omega} f \cdot v \, dx \quad \text{for all } v \in V
\]

with data \( 0 < \alpha \leq A \leq \beta < \infty \) and \( f \in L^2(\Omega) \).
The Poisson equation (FE approximation): \( u_h \in V_h \subset V \) such that

\[
a(u_h, v) := \int_{\Omega} (A \nabla u_h) \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \text{for all } v \in V_h
\]

with data \( 0 < \alpha \leq A \leq \beta < \infty \) and \( f \in L^2(\Omega) \).

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Numerical error (piecewise linear continuous FE approximation)

- For solution \( u \in H^2(\Omega) \) we have

\[
\|u - u_h\| := \|A^{1/2} \nabla (u - u_h)\|_{L^2(\Omega)} \leq C \beta^{1/2} h \|D^2 u\|_{L^2(\Omega)} \sim C(\alpha, \beta, A') h.
\]

- The mesh size \( h \) has to resolve the variations in \( A \), e.g. \( h < \epsilon \) if \( A \) is \( \epsilon \)-periodic.
The Poisson equation (FE approximation): $u_h \in V_h \subset V$ such that

$$a(u_h, v) := \int_\Omega (A \nabla u_h) \cdot \nabla v \, dx = \int_\Omega f v \, dx \text{ for all } v \in V_h$$

with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$.

Example (periodic coefficient): $A(x) = (2 - \cos(2\pi x/\epsilon))^{-1}$, $\epsilon = 0.02$, $f = 1$
Objectives:

- Find a subspace of $V_{H}^{ms} \subset V_{h}$ for which the Galerkin approximation fulfills

  $$\|u_{h} - u_{H}^{ms}\| \leq C(\alpha, \beta)H \approx C(\alpha, \beta, A')h,$$

  but with $\dim(V_{H}^{ms}) \ll \dim(V_{h}).$

- Show that a basis for $V_{H}^{ms}$ can be constructed by local parallel computations.

- Demonstrate efficiency for applications where $V_{H}^{ms}$ is reused (eigenvalue, time dependent, semi-linear, systems).

Variational multiscale method, (Hughes), 1995
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Multiscale decomposition

- (coarse) FE mesh $\mathcal{T}$ with parameter $H > h$
- P1-FE space $V_H := \{ v \in V \mid \forall T \in \mathcal{T}, v|_T \in P_1(T) \}$
- $\mathcal{I}_T : V \rightarrow V_H$ some interpolation operator

Decomposition

$$V = V_H \oplus V^f$$ with $V^f := \text{kernel } \mathcal{I}_T = \{ v \in V \mid \mathcal{I}_T v = 0 \}$

Example:

rough coefficient
Multiscale decomposition

- (coarse) FE mesh $\mathcal{T}$ with parameter $H > h$
- P1-FE space $V_H := \{ v \in V \mid \forall T \in \mathcal{T}, v|_T \in P_1(T) \}$
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Decomposition

$$V = V_H \oplus V^f \quad \text{with} \quad V^f := \text{kernel} \mathcal{I}_T = \{ v \in V \mid \mathcal{I}_T v = 0 \}$$

Example:

$u \in V = u_H \oplus u^f \in V_H \oplus V^f$
Orthogonalization

- For each \( v \in V \) define finescale projection \( Qv \in V^f \) by

\[
a(Qv, w) = a(v, w) \quad \text{for all } w \in V^f
\]

**a-Orthogonal Decomposition**

\( V = V_{ms}^H \oplus V^f \) with \( V_{ms}^H := (V_H - QV_H) \)

---

Example:

\[
\begin{align*}
    u \in V & \quad \rightarrow \quad u_{ms}^H \in V_{ms}^H \\
    & \quad + \quad u^f \in V^f
\end{align*}
\]
Given the space $V_{H}^{\text{ms}}$ we construct a Galerkin approximation:

**Ideal method**

Find $u_{H}^{\text{ms}} \in V_{H}^{\text{ms}}$ such that

$$a(u_{H}^{\text{ms}}, v) = (f, v), \ \forall v \in V_{H}^{\text{ms}}.$$  

We have that $u - u_{H}^{\text{ms}} = u_{f} \in V^{f}$ since $u_{H}^{\text{ms}}$ is the $a$-orthogonal projection of $u$ onto $V_{H}^{\text{ms}}$. Therefore

$$\|u_{f}\|^2 = a(u, u_{f}) = (f, u_{f}) = (f, u_{f} - \mathcal{T}_{\mathcal{T}} u_{f}) \leq \frac{C_{\mathcal{T}}}{\alpha^{1/2}} \|Hf\|_{L^2(\Omega)} \|u_{f}\|.$$  

For $V_{H}^{\text{ms}}$ to be useful we need a discrete local basis.
Modified nodal basis

- $\mathcal{N}$ denotes set of interior vertices of $\mathcal{T}$
- $\phi_x \in V_H$ denotes classical nodal basis function ($x \in \mathcal{N}$)
- $Q\phi_x \in V^f$ denotes the finescale correction of $\phi_x$ ($x \in \mathcal{N}$)

**Ideal multiscale FE space**

$$V_{ms}^H = \text{span} \{ \phi_x - Q\phi_x \mid x \in \mathcal{N} \}$$

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**Example**

\[
\phi_x - Q\phi_x \in V_{ms}^H = \phi_x \in V_H - Q\phi_x \in V^f
\]
Modified nodal basis

- $\mathcal{N}$ denotes set of interior vertices of $\mathcal{T}$
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Ideal multiscale FE space

$$V_{ms}^H = \text{span} \{ \phi_x - Q\phi_x \mid x \in \mathcal{N} \}$$

Example

$$\phi_x - Q\phi_x \in V_{ms}^H = \phi_x \in V_H - Q\phi_x \in V^f$$
Define nodal patches of \( \ell \)-th order \( \omega_{T,\ell} \) about \( T \in \mathcal{T} \)

Correctors \( Q_T^\ell \phi_x \in V^f(\omega_{T,\ell}) := \{ v \in V^f \mid v|_{\Omega \setminus \omega_{T,\ell}} = 0 \} \) solve

\[
a(Q_T^\ell \phi_x, w) = \int_T A \nabla \phi_x \cdot \nabla w \, dx \quad \text{for all } w \in V^f(\omega_{T,\ell})
\]

Localized multiscale FE spaces

\[
V_{H,\ell}^{ms} = \text{span}\{ \phi_x - \sum_{T \in \mathcal{T}} Q_T^\ell \phi_x \mid x \in \mathcal{N} \}
\]
Fine scale discretization

- **Finescale mesh**
  \[ \mathcal{T} \]  
  mesh refinement  
  \[ \mathcal{T}_h \text{ with } h \leq H \]

- **Reference FE space**
  \[ V_h := \{ v \in V \mid \forall T \in \mathcal{T}(\Omega), v|_T \in P_1(T) \} \]

- **Reference FE solution** \( u_h \in V_h \) solves
  \[ a(u_h, v) = (f, v) \quad \text{for all } v \in V_h \]

- **Fully discrete correctors** \( Q^{T,\ell}_h \phi_x \in V^f_h(\omega_T, \ell) := V^f(\omega_T, \ell) \cap V_h : \)
  \[ a(Q^{T,\ell}_h \phi_x, w) = (A \nabla \phi_x, \nabla w)^T \quad \text{for all } w \in V^f_h(\omega_T, \ell) \]
Localized Orthogonal Decomposition (LOD)

Fully discrete multiscale FE spaces

\[ V_{H,\ell}^{ms,h} = \text{span}\{\phi_x - \sum_{T \in T} Q_{\ell,h}^T \phi_x \mid x \in \mathcal{N}\} \]

Fully discrete multiscale approximation \( u_{H,\ell}^{ms,h} \in V_{H,\ell}^{ms,h} \)

\[ a(u_{H,\ell}^{ms,h}, v) = (f, v) \quad \text{for all } v \in V_{H,\ell}^{ms,h} \]

Remarks:

- \( \dim V_{H,\ell}^{ms,h} = |\mathcal{N}| = \dim V_H \)
- The basis functions have local support, with overlap depending on \( \ell \), and are independent.
Lemma (Truncation error)

$$\| | Q_h v_H - Q_{\ell,h} v_H | | \leq C_1 \gamma^\ell | | Q_h v_H | | , \quad \forall v_h \in V_H$$

$C_1 < \infty$ and $\gamma < 1$ depends on $\beta/\alpha$ but not $A'$. By choosing $\ell = C_2 \log(H^{-1})$ with appropriate $C_2$ we guarantee that the truncation leads to a higher order perturbation:

Theorem (A priori error bound)

$$\| | u_h - u_{H,\ell}^{ms} | | \leq C(\alpha, \beta) H,$$

Numerical experiment: Poisson’s equation

\[ H = 2^{-1}, 2^{-2}, \ldots, 2^{-7} \]
\[ h = 2^{-9}, \ell = \log(1/H) \]

\[ \|u_h - u_{ms,h}^{H,\ell}\| \text{ vs. } \#\text{dof} \]
Numerical experiment: Poisson’s equation

\[ H = 2^{-1}, 2^{-2}, \ldots, 2^{-7} \]
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\[ \|u_h - u_{ms, h}^{H, \ell}\| \text{ vs. } \#\text{dof} \]
Trilinear shape functions on cubes.


Storage of all basis function is not needed. The full solution can be recomputed (at a lower cost) once $\tilde{T} u_{H,\ell}^{ms,h}$ is computed.

Corrector function $Q^T \phi_x$, implementation by Fredrik Hellman.
Numerical experiment: Poisson’s equation 3D

\[
H = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}
\]
\[
h = 2^{-6}, \ell = \log(1/H)
\]

\[
\|u_h - u_{H,\ell}^{\text{ms},h}\| \text{ vs. } \#\text{dof}
\]
Numerical experiment: Poisson’s equation 3D

\[ H = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5} \]
\[ h = 2^{-6}, \ell = \log(1/H) \]

\[ \|u_h - u_{H,\ell}^{ms,h}\| \text{ vs. } \#\text{dof} \]
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High contrast data

Poisson equation:

\[-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial \Omega.\]

\[A = 1 \text{ in } \Omega_1 \text{ (black), } A = \alpha \text{ in } \Omega_\alpha, \alpha \ll 1, \text{ and } f = \chi_{[1/4,3/4]^2}.\]

- High contrast data with channels leads to non-local behaviour.
- The decay rate of the basis functions determines the accuracy of LOD.
- The choice of interpolant \( \mathcal{Z}_T v = \sum_{x \in \mathcal{N}} \bar{v}_{\omega_x} \phi_x \) affects the decay.
Numerical example: High contrast

High contrast data

Three examples: $H = 2^{-4}$, $h = 2^{-10}$,

We let $\alpha = 10^{-1}, \ldots, 10^{-6}$ and plot $\| u_h - u_{ms,h}^{H,k} \|$ vs. $k$, with $\mathcal{S}_T^S$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\end{figure}
Nodal variables:
Let $x \in \mathcal{N}$ be nodes of $\mathcal{T}$ and $\sigma_x \subset \Omega$ associated domains. We define a $L^2(\sigma_x)$-dual basis $\psi_x \in V_H$ fulfilling,

$$
\int_{\sigma_x} \psi_x \phi_y = \delta_{xy}.
$$

Let the nodal variable $N_x(v) = \int_{\sigma_x} \psi_x v$ and,

$$
\mathcal{S}_{\mathcal{T}} v = \sum_{x \in \mathcal{N}} N_x(v) \phi_x.
$$

- $\sigma_x$ does not need to be full elements $T$ or vertex patches $U_1(x)$.
- The stability of $|N_x(v)| \leq \|\psi\|_{L^2(\sigma_x)} \|v\|_{L^2(\sigma_x)}$ depends on the size and shape of $\sigma_x$ and its distance to $x$. 
Selection of $\sigma_x$:
By letting $\sigma_x \subset \Omega_1$ (frequently enough) we guarantee decay, i.e. nodes in high conductivity channels are needed.

Let $U_1(x)$ be the vertex patch at node $x$.
- **Type I node**: for $x \in \Omega_1$ let $\sigma_x \subset U_1(x) \cap \Omega_1$, connected.
- **Type II node**: for $x \in \Omega_\alpha$ let $\sigma_x = U_\delta(x), \ 0 < \delta \leq 1$. 
The following weighted Poincaré inequality holds:

\[ \| A^{1/2} v_f \|_{L^2(T)} \leq CH \| A^{1/2} \nabla v_f \|_{L^2(U_1(T))}, \quad \forall v_f \in V^f = \ker(\mathcal{F}_T). \]

This is used to prove contrast independent decay.

**Theorem**

With \( \delta < 1/2 \) we have,

\[ \| A^{1/2} \nabla Q^T v_H \|_{\Omega \setminus U_k(T)} \leq C \gamma^k \| A^{1/2} \nabla Q^T v_H \|_{L^2(\Omega)}, \]

where \( C \) and \( \gamma \) are independent of \( \beta/\alpha \).

Hellman & M. Contrast independent localization of multiscale problems, MMS 2017
Numerical example: High contrast

High contrast data Three examples: $H = 2^{-4}$, $h = 2^{-10}$,

We let $\alpha = 10^{-1}, \ldots, 10^{-6}$ and plot $\| u_h - u_{H,k}^{\text{ms},h} \|$ vs. $k$ with $\mathcal{F}_T$.
Numerical example: High contrast

High contrast data Three examples: $H = 2^{-4}$, $h = 2^{-10}$,

We let $\alpha = 10^{-1}, \ldots, 10^{-6}$ and plot $\|u_h - u_{H,k}^{\text{ms}}\|$ vs. $k$ with $\mathcal{F}^\sigma$.
Thin structures so far need to be resolved.

As $\epsilon \to 0$ we have convergence (with rate) to an interface problem.

\[
-\nabla \cdot A \nabla u = f, \quad \text{in } \Omega \\
u = 0, \quad \text{on } \partial \Omega \\
[u] = 0, \quad \text{on } \Gamma \\
-\nabla_{\Gamma} \cdot A_{\Gamma} \nabla_{\Gamma} u = f_{\Gamma} - [n \cdot A \nabla u], \quad \text{on } \Gamma.
\]
Weak form and LOD

On weak form we have: find \( u \in V = H^1_0(\Omega) \cap H^1(\Gamma) \) such that

\[
a(u, v) := \int_{\Omega} A \nabla u \cdot \nabla v \, dx + \int_{\Gamma} A_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v \, ds = \int_{\Omega} f v \, dx + \int_{\Gamma} f_{\Gamma} v \, ds,
\]

for all \( v \in V = H^1_0(\Omega) \cap H^1(\Gamma) \). We note that \( a(\cdot, \cdot) \) is a scalar product on \( V \).

Localized Orthogonal Decomposition

Given an interpolant \( \mathcal{I}_T : V \to V_H \) we can formulate the LOD method: find \( u_{ms}^H \in V_{ms}^H \) such that

\[
a(u_{ms}^H, v) = \int_{\Omega} f v \, dx + \int_{\Gamma} f_{\Gamma} v \, ds, \quad \text{for all } v \in V_{ms}^H.
\]
Edge based Scott-Zhang interpolation

Integration on edges/faces. We have the interpolation bound:

\[ \| \mathbf{v} - \mathcal{S}_T \mathbf{v} \|_{L^2(\Omega)} + \| \mathbf{v} - \mathcal{S}_T \mathbf{v} \|_{L^2(\Gamma)} \leq CH \left( \| \mathbf{v} \|_{H^1(\Omega)}^2 + \| \mathbf{v} \|_{H^1(\Gamma)}^2 \right)^{1/2}. \]

Decay follows if the nodal variables are restricted to the interfaces.

Element based vs. edge based in log scale with interface is at \( x = 0.5 \).

Numerical example

Let $A_\Gamma = 2$ and $A$ be random between 0.1 and 0.9 with forcing in the bulk and on the interfaces.

Coarses mesh has 237 dof and finest mesh has 219345 dof, $k = \log(H^{-1})$.

Error in energy norm vs number of uniform refinements.
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Prototypical self-adjoint eigenvalue problem

\[- \nabla \cdot A \nabla u = \lambda u \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial \Omega\]

with data $0 < \alpha \leq A \leq \beta < \infty$
Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem (variational form): find $u \in V := H^1_0(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$a(u, v) := \int_{\Omega} (A \nabla u) \cdot \nabla v \, dx = \lambda \int_{\Omega} u \cdot v \, dx \quad \text{for all } v \in V$$

with data $0 < \alpha \leq A \leq \beta < \infty$
Prototypical self-adjoint eigenvalue problem (FE approximation): $u_h \in V_h \subset V$ and $\lambda_h \in \mathbb{R}$ such that

$$a(u_h, v) := \int_\Omega (A \nabla u_h) \cdot \nabla v \, dx = \lambda_h \int_\Omega u_h \cdot v \, dx \quad \text{for all } v \in V_h$$

with data $0 < \alpha \leq A \leq \beta < \infty$. 

![Image of a heat map with color scale from 0.01 to 10000]
Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem (FE approximation):

\[ u_h \in V_h \subset V \text{ and } \lambda_h \in \mathbb{R} \text{ such that} \]

\[ a(u_h, v) := \int_{\Omega} (A \nabla u_h) \cdot \nabla v \, dx = \lambda_h \int_{\Omega} u_h \cdot v \, dx \text{ for all } v \in V_h \]

with data \( 0 < \alpha \leq A \leq \beta < \infty \)

Numerical error (piecewise linear continuous FE approximation)

- For an eigenpair \((u^{(k)}, \lambda^{(k)})\) with \(u^{(k)} \subset H^2(\Omega)\) it holds

\[ \lambda^{(k)} \leq \lambda_h^{(k)} \leq \lambda^{(k)} + C(\alpha, \beta, A', k) h^2, \]

\[ \|\| u^{(k)} - u_h^{(k)} \|\| := \|A^{1/2} \nabla (u^{(k)} - u_h^{(k)})\|_{L^2(\Omega)} \leq C(\alpha, \beta, A', k) h. \]

- The mesh size \( h \) has to resolve the variations in \( A \), e.g. \( h < \epsilon \) if \( A \) is periodic.
**LOD approximation**

**LOD:** Find \( u_{H,\ell}^{ms,h} \in V_{H,\ell}^{ms,h}, \lambda_{H,\ell}^{ms,h} \in \mathbb{R} \)

\[
a(u_{H,\ell}^{ms,h}, v) = \lambda_{H,\ell}^{ms,h} (u_{H,\ell}^{ms,h}, v) \quad \text{for all } v \in V_{H,\ell}^{ms,h}
\]

**Theorem**

\[
\lambda_{h}^{(k)} \leq \lambda_{H,\ell}^{ms,h,(k)} \leq \lambda_{h}^{(k)} + CH^4,
\]

\[
\|\| u_h^{(k)} - u_{H,\ell}^{ms,h,(k)} \|\| \leq CH^2,
\]

with \( C \) independent of \( A' \) and the regularity of the eigenfunctions and \((\lambda_h, u_h)\) is the reference solution.

Numerical example: eigenvalues

\[ k \quad \lambda_{h}^{(k)} \quad e^{(k)}(1/2 \sqrt{2}) \quad e^{(k)}(1/4 \sqrt{2}) \quad e^{(k)}(1/8 \sqrt{2}) \quad e^{(k)}(1/16 \sqrt{2}) \]

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<td>0.928617606</td>
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Table: Errors \( e^{(k)}(H) =: \frac{\lambda_{ms}^{(k)} - \lambda_{h}^{(k)}}{\lambda_{h}^{(k)}} \) and \( h = 2^{-7} \sqrt{2} \).
The parabolic problem: Find $u \in V$ such that

$$(\dot{u}, v) + (A \nabla u, \nabla v) = (f(t), v), \quad \forall v \in V, \quad t > 0$$

and $u(0) = u_0 \in L^2(\Omega)$. We assume $A$ to be independent of $t$.

**FE Backward Euler:** Find $u^n_h \in V_h$ such that

$$(\bar{\partial}_t u^n_h, v) + a(u^n_h, v) = (f^n, v), \quad \forall v \in V_h,$$

and $u^0_h \in V_h$ some approximation of $u_0$.

**LOD:** Find $(u^{ms}_H)^n \in V^{ms,h}_{H,\ell}$ such that

$$(\bar{\partial}_t (u^{ms}_H)^n, v) + a((u^{ms}_H)^n, v) = (f^n, v), \quad \forall v \in V^{ms,h}_{H,\ell},$$

and $(u^{ms}_H)^0 \in V^{ms,h}_{H,\ell}$ some approximation of $u_0$. 
Theorem

\[ \|u^n_h - (u^{ms}_H)^n\|_{L^2(\Omega)} \leq C(1 + \log(t_n/\tau))H^2(t_n^{-1}\|u^0_h\|_{L^2(\Omega)} + \|f\|_{W^{1,\infty}(L^2(\Omega))}) \]

with \(C\) independent of \(A'\).

- The analysis uses classic a priori error estimation techniques and the elliptic results.
- The term \(t_n^{-1}\) appears also in \(u - u_h\) bounds if \(u_0 \in L^2(\Omega)\). The log term can be avoided if \(f(t) \in H^1_0(\Omega)\).
- The case \(f = f(u)\) can also be treated, under certain growth conditions on \(f'(u)\) and \(f''(u)\).
- The case \(A = A(t)\) or \(A = A(u)\) is not covered and would require updates of \(V^{ms,h}_{H,\ell}\).
Numerical experiment: The heat equation

\[ H = 2^{-1}, 2^{-2}, \ldots, 2^{-6} \]
\[ h = 2^{-7}, \ell = \log(1/H), u_0 = 1, \]
\[ T = 1, \tau = 0.001 \]
\[ ||u_h^n - (u_{H}^{ms})^n|| \text{ vs. } \#dof \]

More applications

Stationary/eigenvalue problems

- Semilinear, (Henning, M., Peterseim), 2014.
- Gross-Pitaevskii, (Henning, M., Peterseim), 2014.
- Helmholtz, (Gallistl & Peterseim), 2015.
- Reduced basis, (Abdulle & Henning), 2015.
- Elasticity, (Henning & Persson), 2016.
- High contrast, (Peterseim & Scheichl), 2016.
- Iterative solvers, (Kornhuber & Yserentant), 2017.
- Network models, (Kettil et. al.), 2019.

Time-dependent problems

- Two phase flow, (Hellman & M.), 2019.
Outline

1. Elliptic problem: Homogenization and FEM
2. Introduction to LOD
3. High contrast data
4. Applications
5. Conclusions
Conclusion and outlook

- By LOD we compute an effective stiffness matrix on a coarse scale. (numerical homogenization)
- Rapidly varying diffusion of low contrast is well understood.
- New ideas for high contrast problems (which is a great challenge for any method).
- Great reduction in computational cost when the basis is reused (load cases, eigenvalues, time dependent, non-linear, control).
- Future challenges: random diffusion, interfaces, multiscale in time, and fully non-linear problems.

Thank you for your attention!