

Adaptive Variational Multiscale Methods

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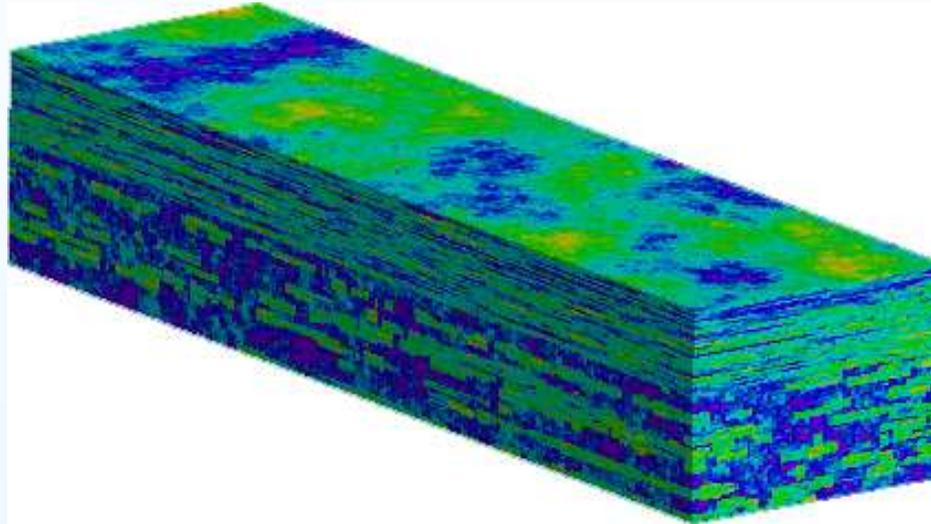
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Outline

- Multiscale application
- Model problem: Poisson equation
- Multiscale method
- Implementation
- A posteriori error estimation and adaptivity
- Extension to mixed formulation
- Numerical examples
- (Extension to convection-dominated problems)
- Conclusions

An application

The figure illustrates data taken from a model oil reservoir.



The size of the reservoir is about $368m \times 671m \times 52m$. The problem features many different scales. We see the x -component of the permeability a .

An application

We seek the water concentration c that solves the system of a flow and a transport equation,

$$(*) \quad \begin{cases} \frac{1}{a} \boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma, \end{cases}$$

$$\begin{cases} \dot{c} + \nabla \cdot (\boldsymbol{\sigma} k(c)) - \epsilon \Delta c = g & \text{in } \Omega \times (0, T], \\ \boldsymbol{n} \cdot \nabla c = 0 & \text{on } \Gamma, \\ c = c_0 & \text{for } t = 0. \end{cases}$$

This is a simple model, e.g. there is only one way coupling, in general a will depend on c .

The model problem

Model problem: The Poisson equations with coefficient $a > 0$,

$$\begin{aligned} -\nabla \cdot a \nabla u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Weak form: Find $u \in V = H_0^1(\Omega)$ such that,

$$a(u, v) = l(v) \quad \text{for all } v \in V,$$

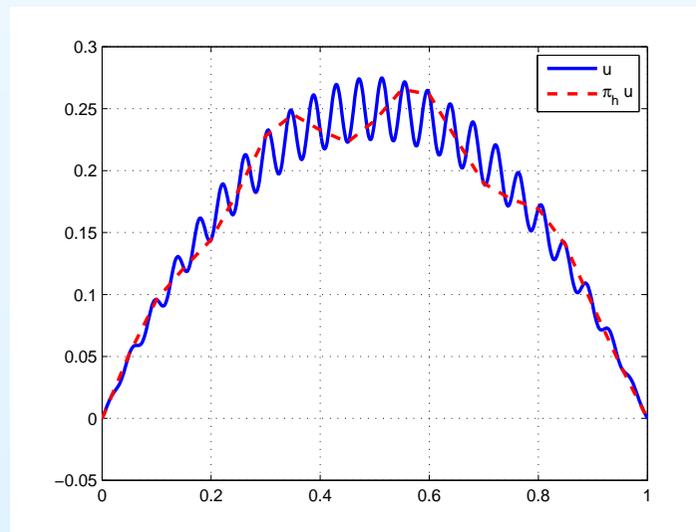
where $a(v, w) = \int_{\Omega} a \nabla v \cdot \nabla w \, dx$, $l(v) = \int_{\Omega} f v \, dx$, $f \in L^2(\Omega)$ and Ω is a domain in \mathbf{R}^d , $d = 1, 2, 3$. We will also use the notation (v, w) for dual pairing between v and w , in most cases $(v, w) = \int_{\Omega} v w \, dx$.

Why multiscale method?

- If we assume a to oscillate at a characteristic scale ϵ we have (Hou),

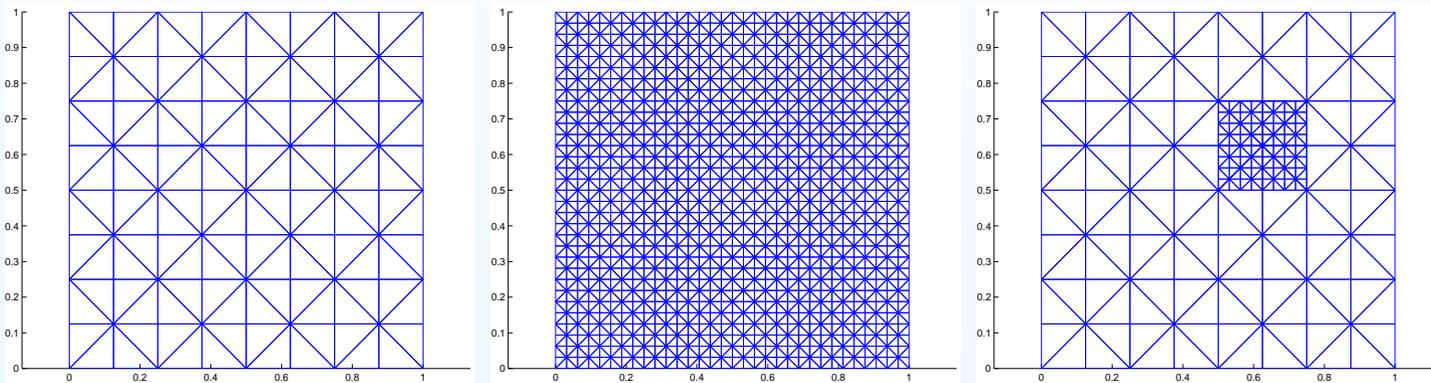
$$\|\nabla u - \nabla U\| \leq C \frac{H}{\epsilon} \|f\|.$$

- $H > \epsilon$ will give unreliable results even with exact quadrature.
- $H < \epsilon$ will be too computationally expensive to solve on a single mesh.



Conclusion of the simple estimate

We need to solve PDE:s on a scale that captures the oscillations but we can not afford to do it on the entire domain.



Coarse $H > \epsilon$ and fine $h < \epsilon$ mesh. This will not be done by meshrefinement but by solving local problems decoupled from each other and from the coarse mesh.

Various multiscale methods

- Upscaling techniques: Durlofsky et al, Nielsen et al. Here an effective permeability \bar{a} is computed by local solves and then a coarse scale equation is solved.
- Multiscale finite element method: Hou et al., Efendiev-Ginting, Aarnes-Lie. Here local solves are used to modify coarse basis functions.
- Multiscale finite volume method: Jenny et al., as above.
- Variational multiscale method: Hughes et al., Arbogast, Larson-Målqvist. Here the weak form is modified with new stabilizing terms. It can sometimes also be viewed as modification of basis functions.

The variational multiscale method

Find $u_c \in V_c$ and $u_f \in V_f$, $V_c \oplus V_f = V$ such that,

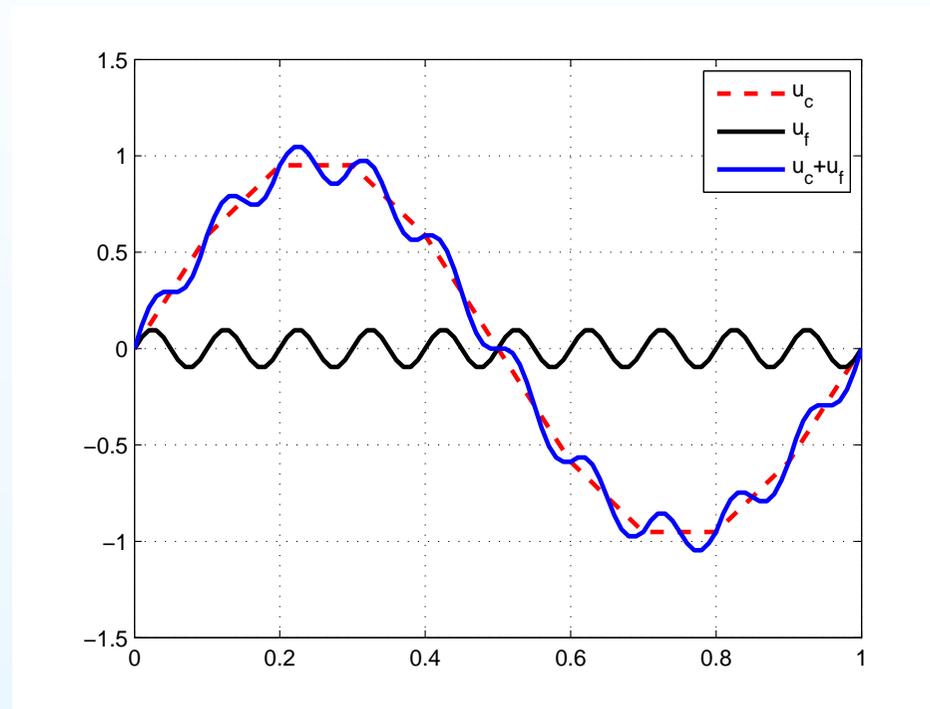
$$a(u_c + u_f, v_c + v_f) = l(v_c + v_f),$$

for all $v_c \in V_c$ and $v_f \in V_f$.

$$\begin{aligned} a(u_c, v_c) + a(u_f, v_c) &= l(v_c) \quad \text{for all } v_c \in V_c, \\ a(u_f, v_f) &= (R(u_c), v_f) \quad \text{for all } v_f \in V_f. \end{aligned}$$

where we introduce the residual distribution $R : V \rightarrow V'$,
 $(R(v), w) = l(w) - a(v, w)$, for all $v, w \in V$. Here V_c is a coarse finite element space of piecewise linear basis functions and V_f is therefore zero in all coarse nodes.

The variational multiscale method



Example on what u_c , u_f , and $u_c + u_f$ may look like.

Approximation (our version)

We derive the method in two steps. Remember the fine scale equations, $a(u_f, v_f) = (R(u_c), v_f)$.

- We decouple the fine scale equations by introducing a partition of unity $\sum_{i \in \mathcal{N}} \varphi_i = 1$ (typically consists of coarse basis functions),

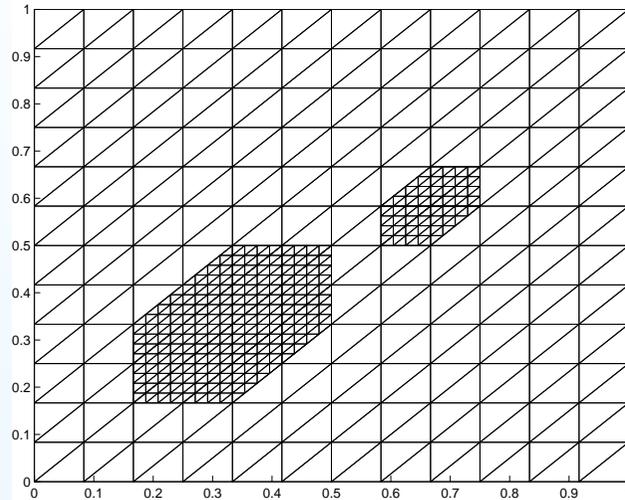
$$a(u_{f,i}, v_f) = (R(u_c), \varphi_i v_f) \quad \text{for all } v_f \in V_f.$$

- For each $i \in \mathcal{N}$ we discretize V_f and solve the resulting problem on a patch ω_i rather than Ω ,

$$a(U_{f,i}, v_f) = (R(U_c), \varphi_i v_f) \quad \text{for all } v_f \in V_f^h(\omega_i).$$

We use homogeneous Dirichlet bc.

The patch ω_i



To the right we see a mesh star to the left what we call a two layer mesh star. The coarse mesh size is denoted H and the fine mesh size is denoted h .

This leads to an overlapping method.

Our method

The resulting method reads: find $U_c \in V_c$ and $U_f = \sum_{i \in \mathcal{N}} U_{f,i}$ where $U_{f,i} \in V_f^h(\omega_i)$ such that

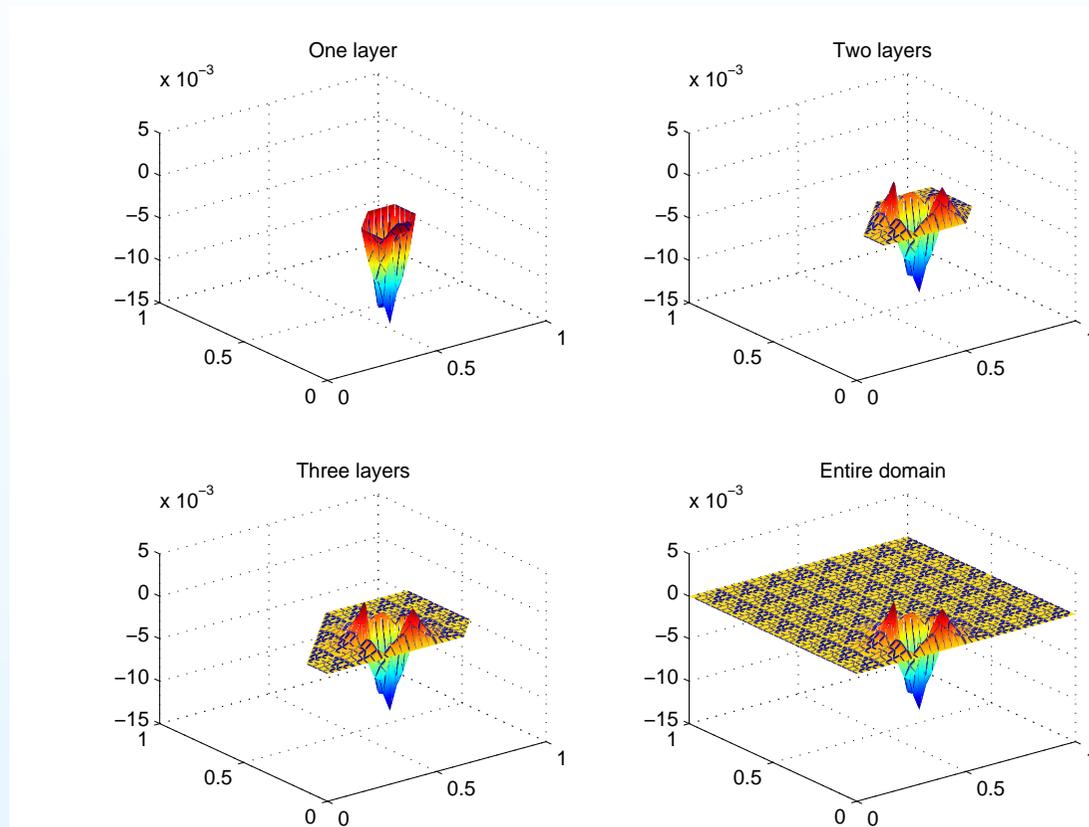
$$\begin{aligned} a(U_c, v_c) + a(U_f, v_c) &= l(v_c), \\ a(U_{f,i}, v_f) &= (R(U_c), \varphi_i v_f), \end{aligned}$$

for all $v_c \in V_c$, $v_f \in V_f^h(\omega_i)$, and $i \in \mathcal{N}$.

The patch is chosen such that $\text{supp}(\varphi_i) \subset \omega_i \subset \Omega$.

An iterative approach where we start with a given U_c is possible but we will instead consider a direct solution approach.

The local solution $U_{f,i}$



The solution improves as the patch size increases.

Motivation for the method

Why do we expect the method to work?

- The right hand side of the fine scale equations has support on a coarse mesh star, " $\varphi_i R(U_c)$ ".
- The fine scale solution $U_{f,i} \in V_f^h(\omega_i)$ which is a slice space. This means that $U_{f,i} = 0$ in all coarse nodes.

This makes $U_{f,i}$ decay rapidly, which makes it possible to get a good approximation using small patches.

Implementation

We have: find $U_{f,i} \in V_f^h(\omega_i)$ such that

$$a(U_{f,i}, v_f) = (f, v_f \varphi_i) - a(U_c, v_f \varphi_i)$$

for all $v_f \in V_f^h(\omega_i)$. Instead we solve: find $\chi_i^k, \eta_i \in V_f^h(\omega_i)$ such that

$$\begin{cases} a(\chi_i^k, v_f) = -a(\varphi_k, v_f \varphi_i) \\ a(\eta_i, v_f) = (f, v_f \varphi_i). \end{cases}$$

for all $v_f \in V_f^h(\omega_i)$ and $\text{supp}(\varphi_k) \cap \text{supp}(\varphi_i) \neq \emptyset$. In the first equation we replace f by 0 and $U_c = \sum U_c^k \varphi_k$ by φ_k (coarse basis function) and in the second equation we keep f and replace U_c by 0.

Implementation

This means that: $\sum_{k \in \mathcal{N}} U_c^k \chi_i^k + \eta_i$ solves:

$$a\left(\sum_{k \in \mathcal{N}} U_c^k \chi_i^k + \eta_i, v_f\right) = (f, v_f \varphi_i) - a(U_c, v_f \varphi_i),$$

so $U_{f,i} = \sum_{k \in \mathcal{N}} U_c^k \chi_i^k + \eta_i$ and

$$U_f = \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{N}} U_c^k \chi_i^k + \eta_i = \sum_{k \in \mathcal{N}} U_c^k \chi^k + \eta,$$

where $\chi^k = \sum_{i \in \mathcal{N}} \chi_i^k$ and $\eta = \sum_{i \in \mathcal{N}} \eta_i$. All this works because the equation is linear.

Implementation

We include this in the coarse scale equations: Find

$U_c = \sum_{i \in \mathcal{N}} U_c^i \varphi_i$ such that,

$$\begin{aligned}(f, \varphi_j) &= a(U_c, \varphi_j) + a(U_f, \varphi_j) \\ &= a\left(\sum_{i \in \mathcal{N}} U_c^i \varphi_i, \varphi_j\right) + a\left(\sum_{i \in \mathcal{N}} U_c^i \chi^i + \eta, \varphi_j\right),\end{aligned}$$

for all $j \in \mathcal{N}$ or

$$\sum_{i \in \mathcal{N}} U_c^i a(\varphi_i + \chi^i, \varphi_j) = (f, \varphi_j) - a(\eta, \varphi_j),$$

which gives a modified system. The degrees of freedom is the same but more non-zero elements compared to Galerkin on coarse mesh.

Implementation

This can now be written on matrix form as,

$$(A + T)x = b - d$$

where,

$$\begin{cases} A_{mn} = a(\varphi_m, \varphi_n), \\ T_{mn} = a(\chi^m, \varphi_n), \\ b_n = (f, \varphi_n), \\ d_n = a(\eta, \varphi_n). \end{cases}$$

or alternatively a symmetric formulation,

$$\begin{cases} A_{mn} + T_{mn} = a(\varphi_m + \chi^m, \varphi_n + \chi^n), \\ b_n = (f, \varphi_n + \chi^n), \\ d_n = a(\eta, \varphi_n + \chi^n). \end{cases}$$

Implementation

Implementing the method comes down to calculating T and d on patches ω_i , $T = \sum_{i \in \mathcal{N}} T^i$ and $d = \sum_{i \in \mathcal{N}} d^i$, where

$$T_{mn}^i = a(\chi_i^m, \varphi_n)$$

and

$$d_n^i = a(\eta_i, \varphi_n).$$

1. Compute A and b .
2. Compute T^i and d^i on the patches ω_i .
3. Solve $(A + T)x = b - d$.

Given the vector x , $\{\chi^m\}_{m=1}^N$, and η we can reconstruct the entire solution U locally or globally.

How do we choose patchsize and h ?

Our aim is to create a method that tunes critical parameters by itself.

- A posteriori error estimation bounds the error from above in terms of known quantities.
- Based on this we formulate an adaptive algorithm.
- The algorithm tunes the critical parameters, such as coarse mesh size, fine mesh size, and patch sizes, automatically.

Energy norm estimate: $\|e\|_a^2 = a(e, e)$

We introduce the coarse and fine scale error $e_c = u_c - U_c$, $e_{f,i} = u_{f,i} - U_{f,i}$, and $e = e_c + \sum_{i \in \mathcal{N}} e_{f,i}$. We have the following orthogonality properties,

$$a(e_c, v_c) + a(e_f, v_c) = 0, \quad \text{for all } v_c \in V_c$$

and

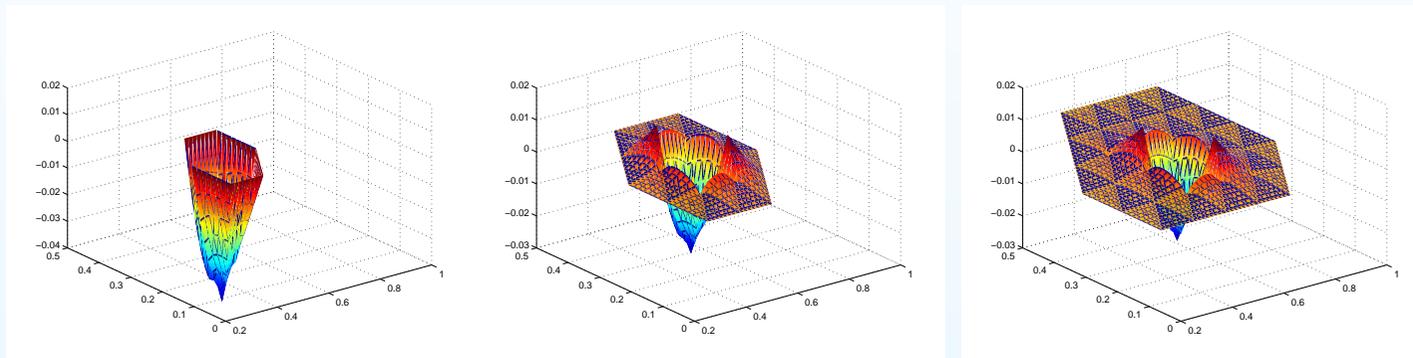
$$a(e_{f,i}, v_f) + a(e_c, \varphi_i v_f) = 0, \quad v_f \in V_f^h(\omega_i).$$

and estimate,

$$\begin{aligned} \|e\|_a^2 &\leq \sum_{i \in \mathcal{C}} C_a \|H\mathcal{R}(U_c)\|_{\omega_i}^2 \\ &\quad + \sum_{i \in \mathcal{F}} C_a \left(\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2 + \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2 \right) \end{aligned}$$

Energy norm estimate

The boundary part $\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2$, where $\Sigma(U_{f,i})$ is an approximation of $a\partial_n U_{f,i}$, decays rapidly as the patch increases.



Here the patches are one, two, and three layer stars. The term $\mathcal{R}(U_c)$ is a bound of the coarse scale residual (" $f - \nabla \cdot a\nabla U_c$ ") and $\mathcal{R}_i(U_{f,i})$ is a bound of the fine scale residual.

Adaptive algorithm based on error estimate

$$\begin{aligned} \|e\|_a^2 &\leq \sum_{i \in \mathcal{C}} C_a \|HR(U_c)\|_{\omega_i}^2 \\ &\quad + \sum_{i \in \mathcal{F}} C_a \left(\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2 + \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2 \right) \end{aligned}$$

1. Start with given H , r , and L where $h = H/2^r$.
2. Calculate U using AVMS.
3. Calculate $E_H^i = \|HR(U_c)\|_{\omega_i}^2$, $E_r^i = \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2$, and $E_L^i = \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2$.
4. Stop if E_H^i , E_h^i , and E_L^i are small enough else if E_H^i is big start solving local problems there with $r = L = 1$, if $E_h^i > E_L^i$ let $r_{\text{new}} := r + 1$ and if $E_L^i > E_h^i$ let $L_{\text{new}} = L + 1$ end return to 2.

Summary of the main idea

- We want to avoid to refine a given coarse mesh
- Still the problem makes it necessary to use better resolution
- We have a method for solving decoupled fine scale problems that are used to modify the coarse scale equation
- We have an error estimate that tells us where and how accurate these fine scale problems need to be solved
- We have an adaptive algorithm that makes these decisions automatically.

We now study other equations and some applications.

Extension to a mixed setting with application

We have also extended this theory to the mixed formulation of the Poisson equation (which is used in oil reservoir simulation),

$$\begin{cases} \frac{1}{a}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma. \end{cases}$$

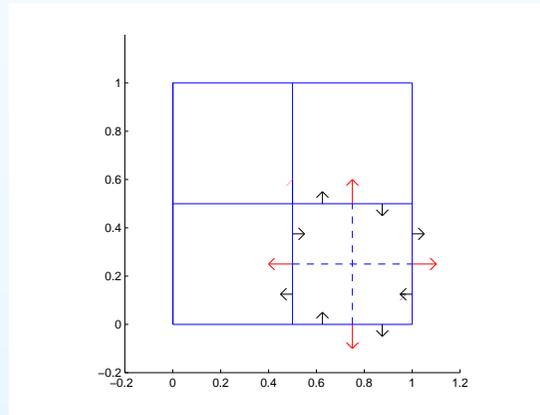
Find $\boldsymbol{\sigma}_c \in \mathbf{V}_c$, $\boldsymbol{\sigma}_f \in \mathbf{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_c, \mathbf{v}_c) + (\frac{1}{a}\boldsymbol{\sigma}_f, \mathbf{v}_c) + (u_c, \nabla \cdot \mathbf{v}_c) + (u_f, \nabla \cdot \mathbf{v}_c) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) - (\nabla \cdot \boldsymbol{\sigma}_f, w_c) = (f, w_c) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \mathbf{v}_f) + (u_f, \nabla \cdot \mathbf{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \mathbf{v}_f) - (u_c, \nabla \cdot \mathbf{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla \cdot \boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\mathbf{v}_c \in \mathbf{V}_c$, $\mathbf{v}_f \in \mathbf{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Basis for V_c , V_f , W_c , and W_f .

- For V_c and W_c : Lowest order Raviart-Thomas elements on rectangles together with piecewise constants.
- For V_f and W_f : Hierarchical extension.



We note, $(w_c, \nabla \cdot \mathbf{v}_f) = \sum_K w_c^K \int_{\partial K} \mathbf{n} \cdot \mathbf{v}_f dx = 0$ where w_c^K is the constant at coarse element K , $(w_f, \nabla \cdot \mathbf{v}_c) = \sum_K \nabla \cdot \mathbf{v}_c^K \int_K w_f dx = 0$.

Variational multiscale method

Find $\sigma_c \in V_c$, $\sigma_f \in V_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\left\{ \begin{array}{l} (\frac{1}{a}\sigma_c, \mathbf{v}_c) + (\frac{1}{a}\sigma_f, \mathbf{v}_c) + (u_c, \nabla \cdot \mathbf{v}_c) + (u_f, \nabla \cdot \mathbf{v}_c) = 0 \\ \quad -(\nabla \cdot \sigma_c, w_c) - (\nabla \cdot \sigma_f, w_c) = (f, w_c) \\ (\frac{1}{a}\sigma_f, \mathbf{v}_f) + (u_f, \nabla \cdot \mathbf{v}_f) = -(\frac{1}{a}\sigma_c, \mathbf{v}_f) - (u_c, \nabla \cdot \mathbf{v}_f) \\ \quad -(\nabla \cdot \sigma_f, w_f) = (f, w_f) + (\nabla \cdot \sigma_c, w_f) \end{array} \right.$$

for all $\mathbf{v}_c \in V_c$, $\mathbf{v}_f \in V_f$, $w_c \in W_c$, and $w_f \in W_f$.

If we let φ_i be a coarse Raviart-Thomas base function on rectangles (bricks),

$\varphi_i = \begin{bmatrix} \varphi_i^x \\ \varphi_i^y \end{bmatrix}$, $\varphi_i = \begin{bmatrix} \varphi_i^x & 0 \\ 0 & \varphi_i^y \end{bmatrix}$, will be a partition

of unity, $\sum_{i \in \mathcal{N}} \varphi_i = 1$. We let $\psi_i = 1/(2d)$ on the support of φ_i .

Partition of unity and discrete formulation

Find $\boldsymbol{\sigma}_c, u_c, \boldsymbol{\sigma}_f = \sum_{i \in \mathcal{N}} \boldsymbol{\sigma}_{f,i}$, and $u_f = \sum_{i \in \mathcal{N}} u_{f,i}$ such that,

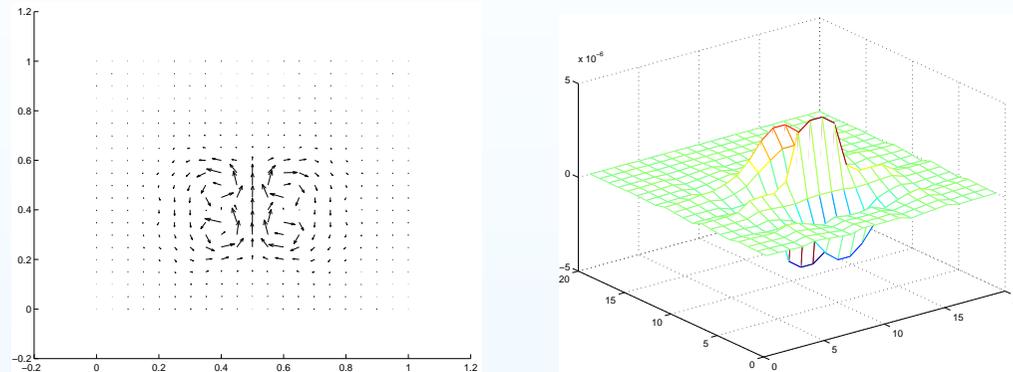
$$\left\{ \begin{array}{ll} (\frac{1}{a} \boldsymbol{\sigma}_c, \mathbf{v}_c) + (\frac{1}{a} \boldsymbol{\sigma}_f, \mathbf{v}_c) + (u_c, \nabla \cdot \mathbf{v}_c) = 0, & \text{for all } \mathbf{v}_c \in \mathbf{V}_c \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) = (f, w_c), & \text{for all } w_c \in W_c \\ (\frac{1}{a} \boldsymbol{\sigma}_{f,i}, \mathbf{v}_f) + (u_{f,i}, \nabla \cdot \mathbf{v}_f) = -(\frac{1}{a} \boldsymbol{\sigma}_c, \varphi_i \mathbf{v}_f), & \text{for all } \mathbf{v}_f \in \mathbf{V}_f \\ -(\nabla \cdot \boldsymbol{\sigma}_{f,i}, w_f) = (f, \psi_i w_f), & \text{for all } w_f \in W_f. \end{array} \right.$$

Find $\boldsymbol{\Sigma}_c \in \mathbf{V}_H, \boldsymbol{\Sigma}_{f,i} \in \mathbf{V}_h(\omega_i), U_c \in W_H$, and $U_{f,i} \in W_h(\omega_i)$

$$\left\{ \begin{array}{l} (\frac{1}{a} \boldsymbol{\Sigma}_c, \mathbf{v}_c) + (\frac{1}{a} \boldsymbol{\Sigma}_f, \mathbf{v}_c) + (U_c, \nabla \cdot \mathbf{v}_c) = 0, \\ -(\nabla \cdot \boldsymbol{\Sigma}_c, w_c) = (f, w_c), \\ (\frac{1}{a} \boldsymbol{\Sigma}_{f,i}, \mathbf{v}_f) + (U_{f,i}, \nabla \cdot \mathbf{v}_f) = -(\frac{1}{a} \boldsymbol{\Sigma}_c \varphi_i, \mathbf{v}_f), \\ -(\nabla \cdot \boldsymbol{\Sigma}_{f,i}, w_f) = (f, w_f \psi_i), \end{array} \right.$$

for all $\mathbf{v}_c \in \mathbf{V}_H, \mathbf{v}_f \in \mathbf{V}_h(\omega_i), w_c \in W_H$, and $w_f \in W_h(\omega_i)$.

Local solutions



The local solutions $\Sigma_{f,i}$ and $U_{f,i}$ for $a = 1$. We introduce patches since,

- The right hand side has support on $supp(\varphi_i) = supp(\psi_i)$.
- The equations are solved in a slice space where solutions decay rapidly. $\int_E \mathbf{n} \cdot \Sigma_{f,i} dx = 0$ and $\int_K U_{f,i} dx = 0$.

The smallest patch now consist of two coarse elements since the RT basis functions have support on two elements.

Energy norm estimate $\|\mathbf{v}\|_a^2 = \left(\frac{1}{a}\mathbf{v}, \mathbf{v}\right)$

Next we present an estimate of the error.

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_a^2 &\leq \sum_i C_a \left\| \frac{1}{a} (\boldsymbol{\Sigma}_c \varphi_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U_{f,i}^* \right\|_{\omega_i}^2 \\ &\quad + \sum_i C_a \left\| h (f \psi_i + \nabla \cdot (\boldsymbol{\Sigma}_c \varphi_i + \boldsymbol{\Sigma}_{f,i})) \right\|_{\omega_i}^2 \\ &\quad + \sum_i C_a \left\| \frac{1}{2\sqrt{h}} U_{f,i}^* \right\|_{\partial\omega_i \setminus \Gamma}^2\end{aligned}$$

U^* is a post processed version of U modified by information from the flux according to paper by Stenberg et. al.

Adaptive Strategy

- Calculate Σ .
- Calculate the error indicators on each patch,

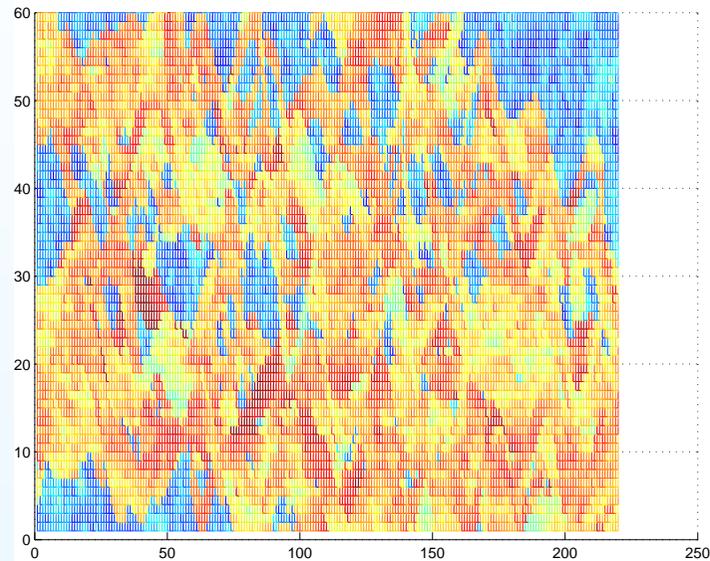
$$X_i(h) = \left\| \frac{1}{a} (\Sigma_c \varphi_i + \Sigma_{f,i}) - \nabla U_{f,i}^* \right\|_{\omega_i}^2$$

$$Y_i(h) = \left\| h(f\psi_i + \nabla \cdot (\Sigma_c \varphi_i + \Sigma_{f,i})) \right\|_{\omega_i}^2$$

$$Z_i(L) = \left\| \frac{1}{2\sqrt{h}} U_{f,i}^* \right\|_{\partial\omega_i \setminus \Gamma}^2$$

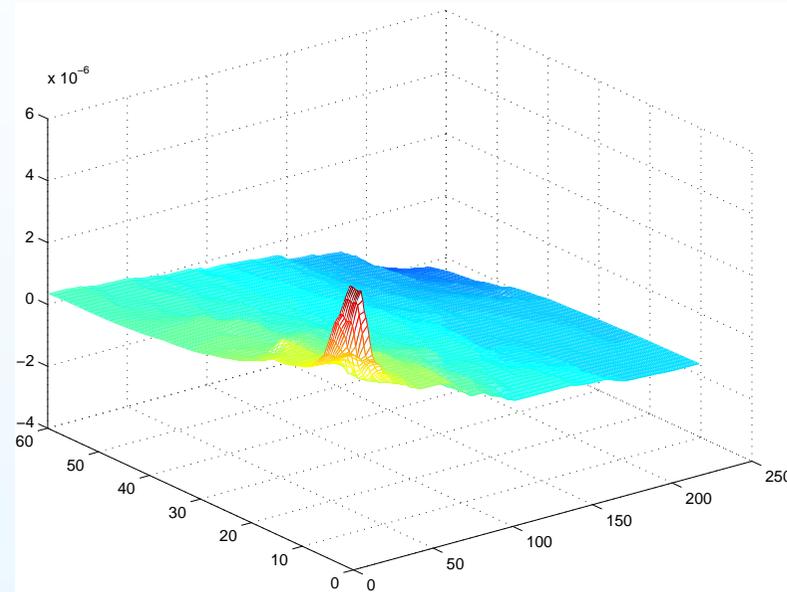
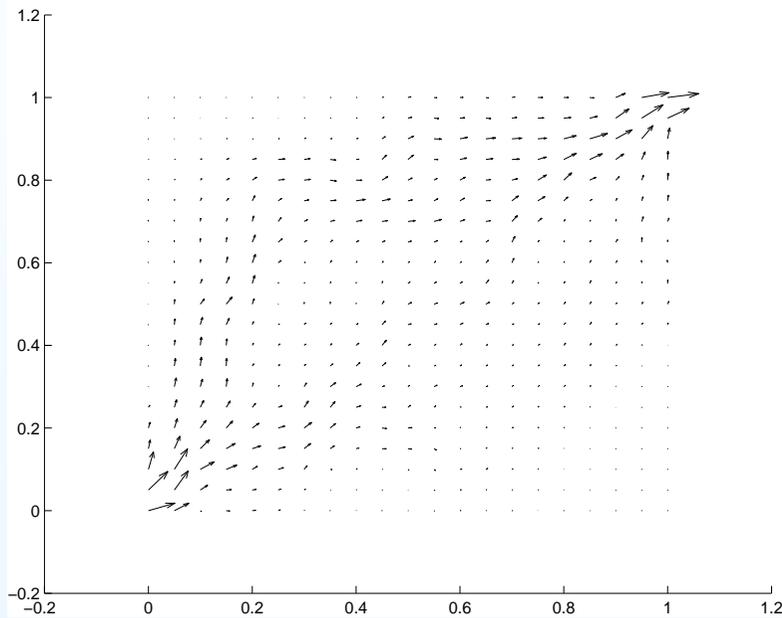
- If indicators $X_i(h)$ or $Y_i(h)$ are big on a patch we decrease h .
- If indicator $Z_i(L)$ is big we increase the size of the patch.
- Go back to the first step or stop if the solution is good enough.

Numerical examples



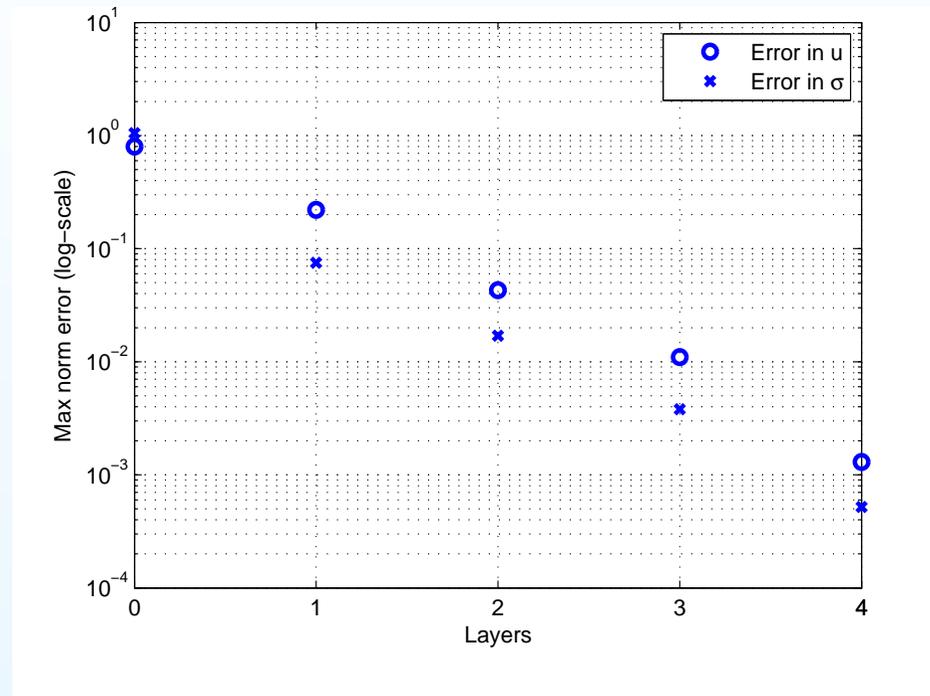
2D slice of the x -component of the permeability α (in log scale)
taken from the tenth SPE comparative solution project.

Reference solutions



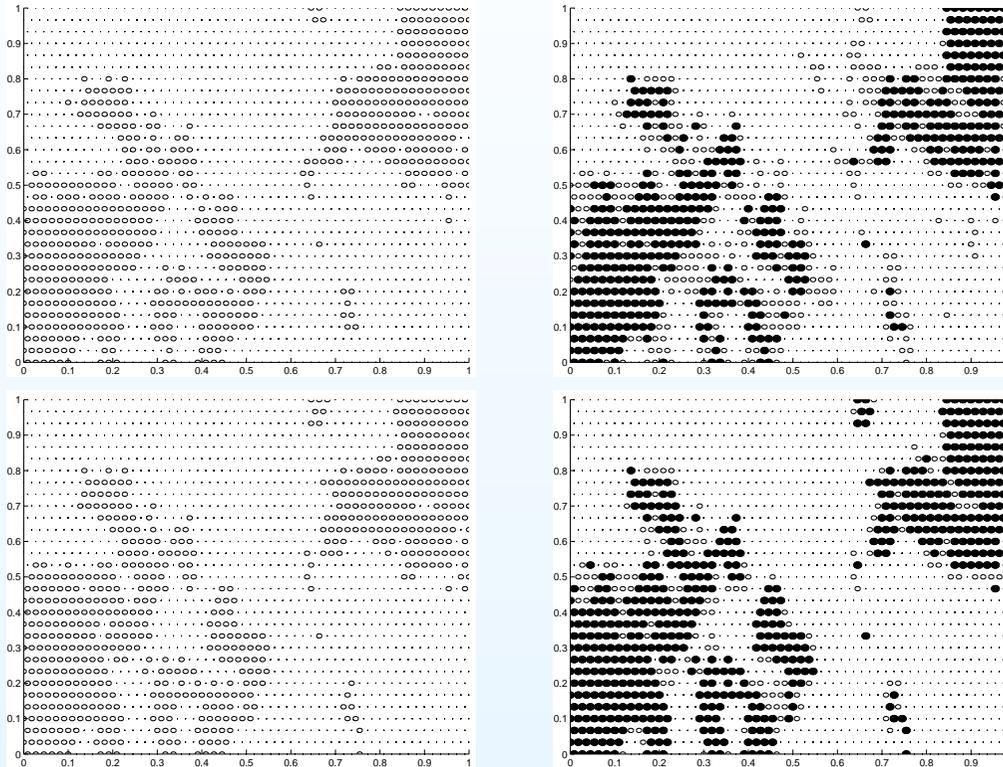
Above we see the reference solution, (left) flux $-\Sigma$ and (right) pressure u .

Convergence



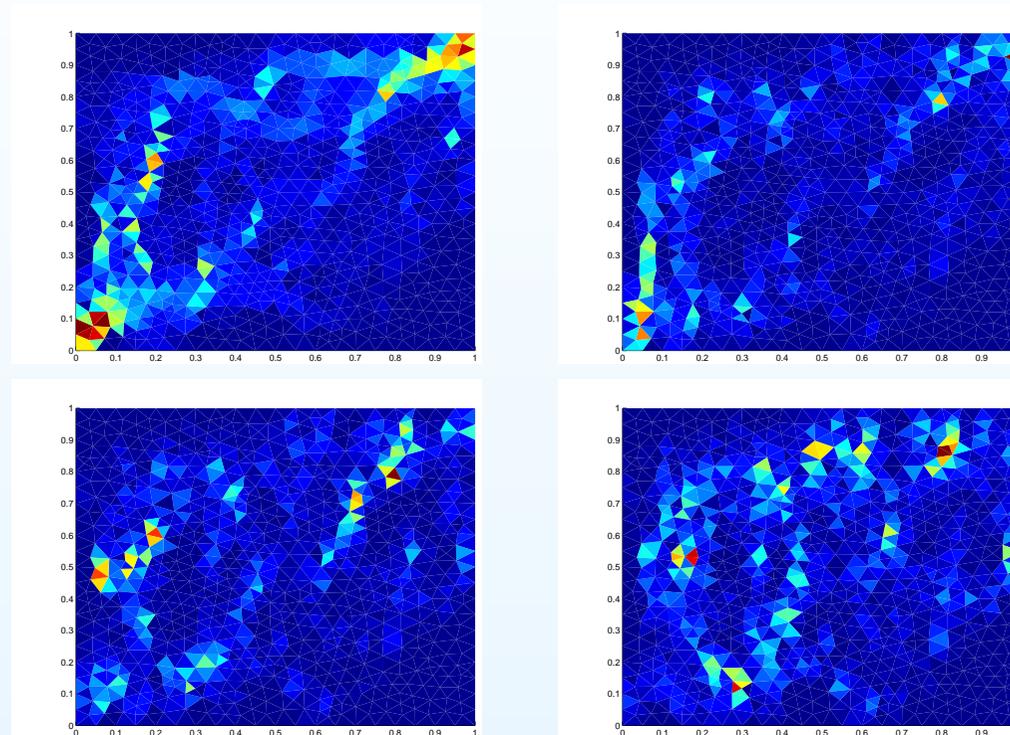
Max norm error (compared to reference solution) in log scale versus number of layers. The coarse mesh has 55×15 elements and the reference mesh has 220×60 elements.

Example using the adaptive algorithm



35% of the patches increased in each iteration and 35% of the fine scale meshes refined in each iteration.

Relative error in energy norm



Relative error in energy norm: 106%, 16%, 10%, and 8%.

Extension to convection dominated problem

Model problem: Convection-Diffusion problem with multiscale features in $b, \epsilon > 0$,

$$\begin{aligned} -\epsilon \Delta u + \nabla \cdot (bu) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

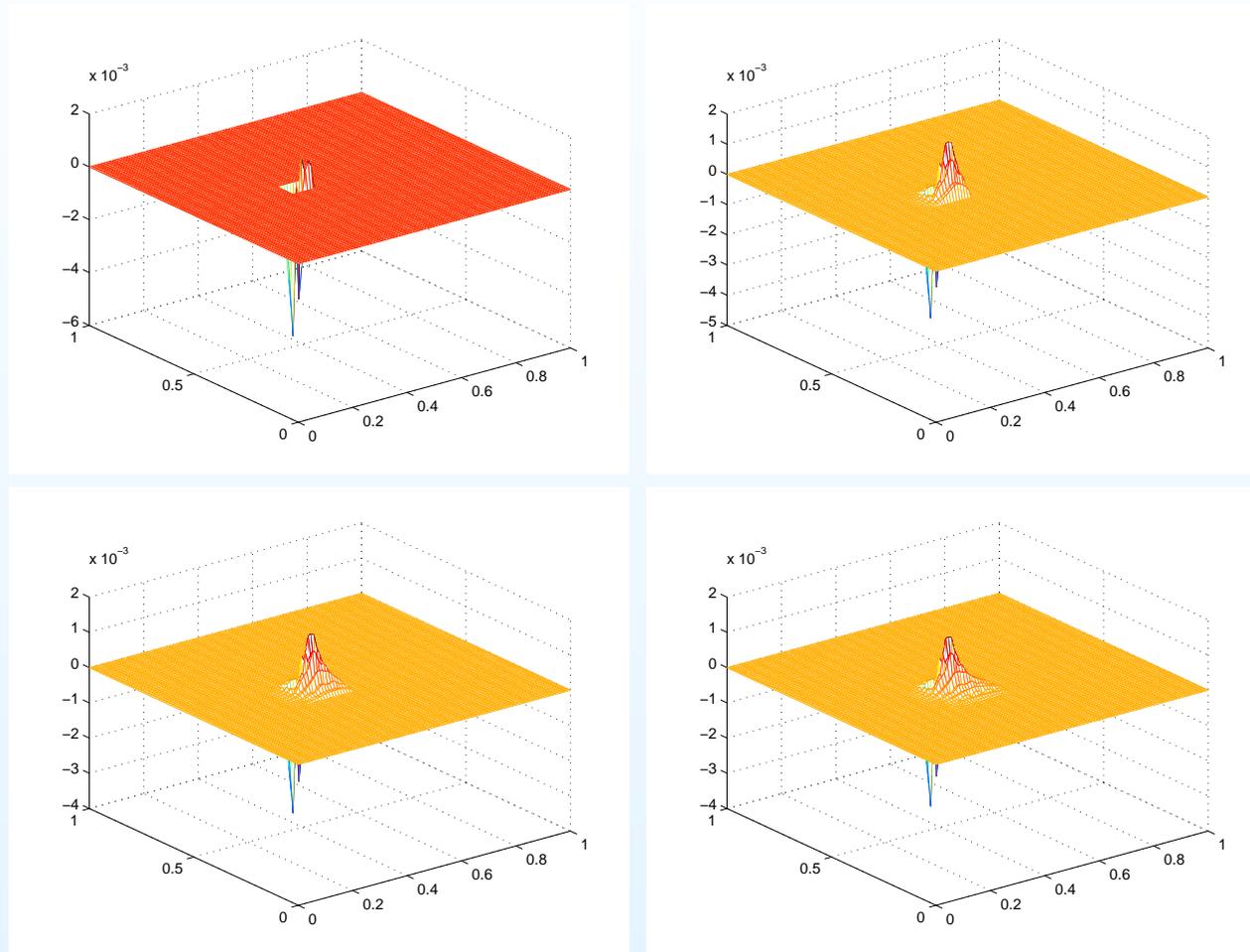
Weak form: Find $u \in V = H_0^1(\Omega)$ such that,

$$a(u, v) = l(v) \quad \text{for all } v \in H_0^1(\Omega),$$

where $a(v, w) = \int_{\Omega} \epsilon \nabla v \cdot \nabla w \, dx + \int_{\Omega} \nabla \cdot (bv) w \, dx$ and $l(v) = \int_{\Omega} f v \, dx$.

We use the same method to approximate the solution.

The local solution $U_{f,i}$



The solution improves as the patch size increases.

Duality based error analysis

Find $\phi \in V$ such that

$$(\epsilon \nabla \phi, \nabla w) - (b \cdot \nabla \phi, w) = (\psi, w), \quad \text{for all } w \in V,$$

note that we use the adjoint operator i.e.

$$a^*(\phi, w) = (\psi, w).$$

We end up with the following error representation formula,

$$\begin{aligned} (e, \psi) &= a^*(\phi, e) = a(e, \phi) = l(\phi) - a(U, \phi) \\ &= \sum_{i \in \mathcal{N}} l(\varphi_i \phi) - a(U_c, \varphi_i \phi) - a(U_{f,i}, \phi). \end{aligned}$$

Error representation formula

We continue the calculation using coarse and fine scale Galerkin Orthogonality,

$$\begin{aligned}(e, \psi) &= l(\phi - \pi_c \phi) - a(U, \phi - \pi_c \phi) \\ &= \sum_{i \in \mathcal{C}} l(\varphi_i(\phi - \pi_c \phi)) - a(U_c, \varphi_i(\phi - \pi_c \phi)) \\ &\quad + \sum_{i \in \mathcal{F}} l(\varphi_i(\phi_f - \pi_{f,i}^0 \phi_f)) - a(U_c, \varphi_i(\phi_f - \pi_{f,i}^0 \phi_f)) \\ &\quad - a(U_{f,i}, \phi_f - \pi_{f,i}^0 \phi_f),\end{aligned}$$

Where $\pi_{f,i}^0$ is the interpolant onto $V_f^h(\omega_i)$ i.e. zero on $\partial\omega_i$. Remember, any function in $V_f^h(\omega_i)$ can be subtracted.

Error representation formula

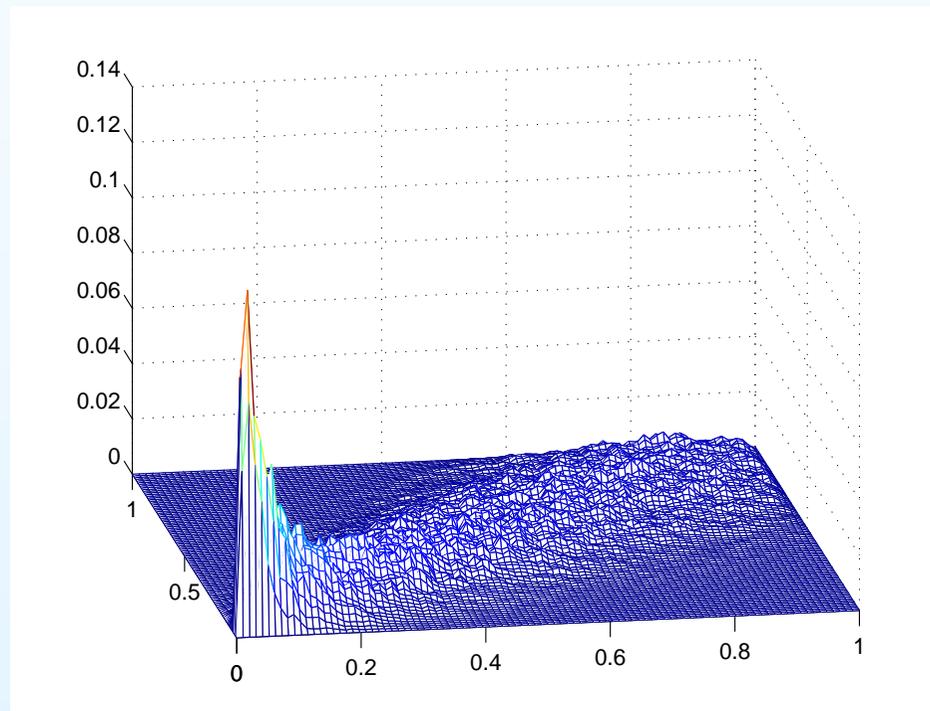
We can also introduce $\pi_{f,i}$ as the nodal interpolant on the mesh associated with $V_f^h(\omega_i)$ and express the error representation formula in terms of $\pi_{f,i}$ and $\pi_{f,i}^0 - \pi_{f,i}$.

$$\begin{aligned}(e, \psi) &= \sum_{i \in \mathcal{C}} l(\varphi_i(\phi - \pi_c \phi)) - a(U_c, \varphi_i(\phi - \pi_c \phi)) \\ &\quad + \sum_{i \in \mathcal{F}} l(\varphi_i(\phi_f - \pi_{f,i} \phi_f)) - a(U_c, \varphi_i(\phi_f - \pi_{f,i} \phi_f)) \\ &\quad \quad - a(U_{f,i}, \phi_f - \pi_{f,i} \phi_f) \\ &\quad + \sum_{i \in \mathcal{F}} l(\varphi_i(\pi_{f,i} \phi_f - \pi_{f,i}^0 \phi_f)) - a(U_c, \varphi_i(\pi_{f,i} \phi_f - \pi_{f,i}^0 \phi_f)) \\ &\quad \quad - a(U_{f,i}, \pi_{f,i} \phi_f - \pi_{f,i}^0 \phi_f).\end{aligned}$$

First depend on H second on h third on patch size.

Numerical examples

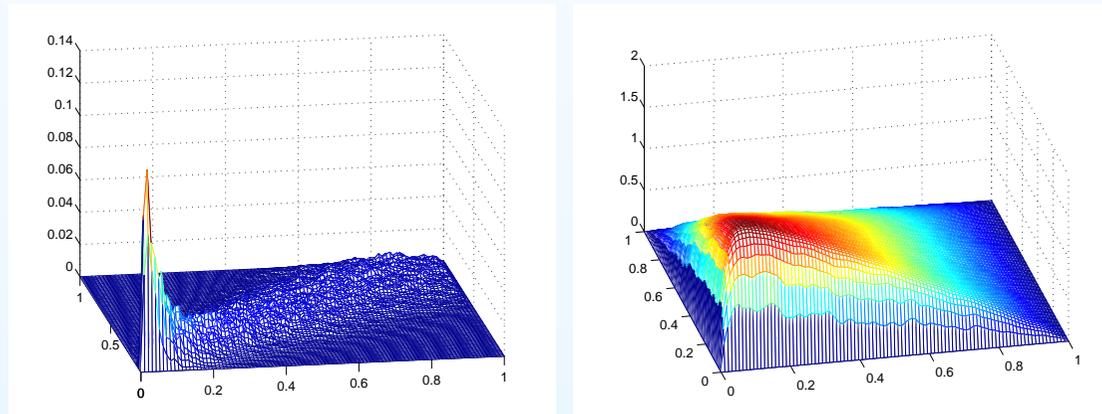
We let $\epsilon = 0.01$, $f = I_{\{x+y < 0.05\}}$, and $B = \text{rand}(96)$,
 $b = [B(i, j), B(i, j)]$ for $i/n < x < (i + 1)/n$ and
 $j/n < y < (j + 1)/n$.



Solving the dual problem for adaptivity

Remember the dual problem: find $\phi \in V$ such that,

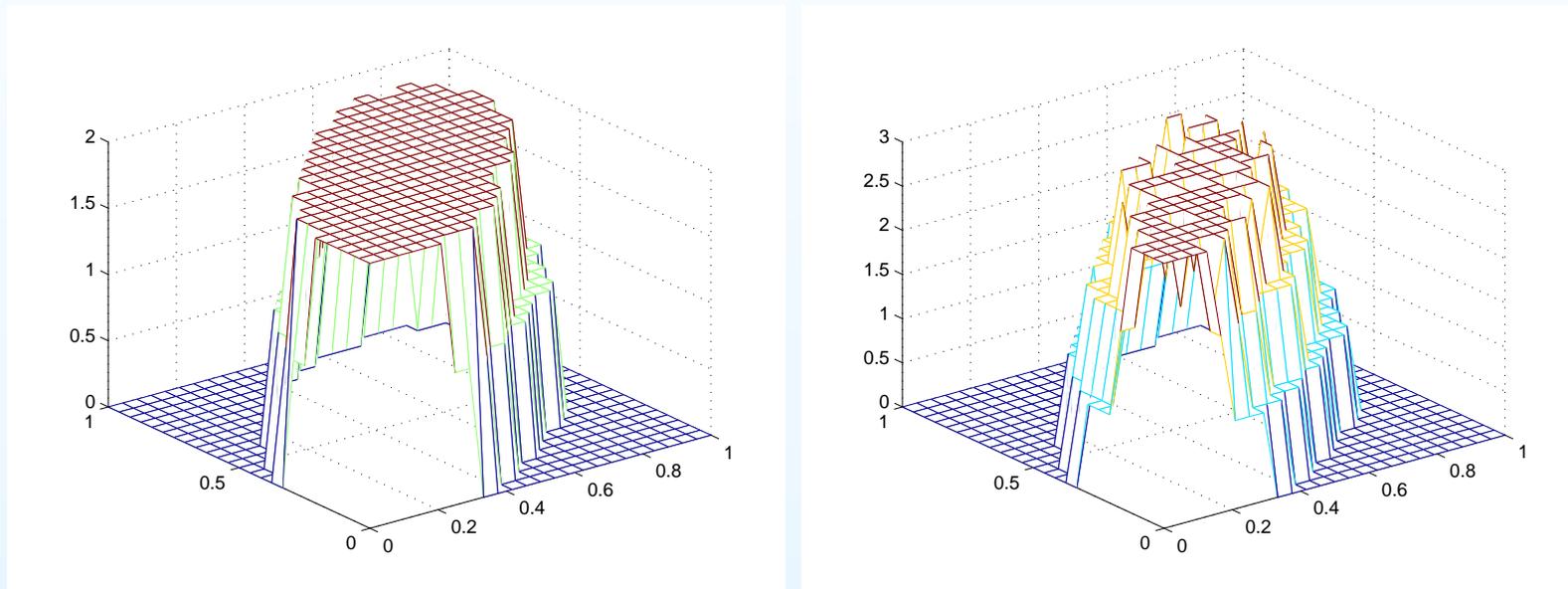
$$(\epsilon \nabla \phi, \nabla w) - (b \cdot \nabla \phi, w) = (1, w), \quad \text{for all } w \in V.$$



- Computing approximation Φ on the reference mesh or use AVMS with more refinement \rightarrow good approx. of the error.
- Computing Φ using a higher order method on coarse grid may still lead to a good indicator for adaptivity.

Numerical Examples

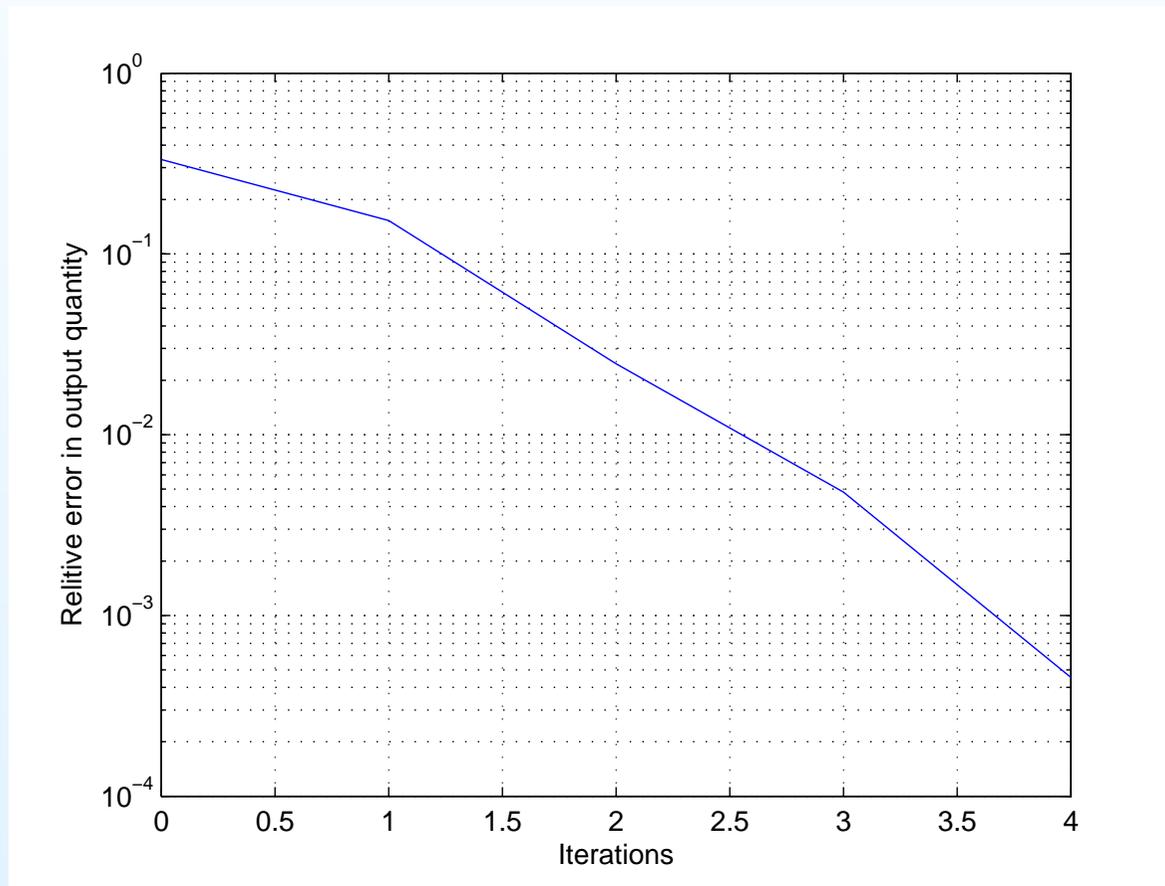
We let $\psi = 1$ and use a refinement level of 40%, $h = 1/96$,
 $H = 1/24$.



Refinements and Patchsizes chosen by the adaptive algorithm.

Numerical Examples

We plot the relative error compared to a reference solution in the quantity of interest (e, ψ) . We solve the dual and the primal using AVMS.



Future work

- Use more than two scales and consider more extreme scale separation.
- Make an evaluation of how the method performs compared to other methods.
- Prove a priori error estimates for the multiscale method.
- Numerics in 3D.
- Adaptive patch shapes for local problems in the convection dominated problem.
- Consider more challenging equations, e.g. time dependent and non-linear.
- Other interesting applications.