Progress in the SSF project ”CutFEM, Geometry, and Optimal design”

Multiscale eigenvalue problems and complex geometry

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Research group and Outline

My research group consists of:

Daniel Elfverson, Uppsala, Multiscale and UQ, 2011-2015
Fredrik Hellman, Uppsala, Multiscale and UQ, 2012-2017
Anna Persson, Göteborg, Time dep. ms problems, 2013-2018
Gustav Kettil, Göteborg, Simulation of paper, 2014-2016
Tony Stillfjord, Göteborg, Time dep. PDE and splitting, 2015-2017

Support from SSF project 2016-2018.

- Multiscale techniques for solving eigenvalue problems (Axel)
- Multiscale problems with complex geometry (Daniel)

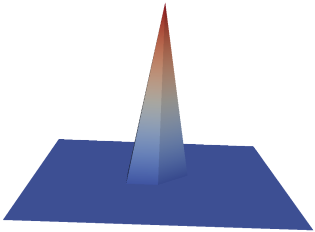
Localized Orthogonal Decomposition

Let A be a diffusion coefficient and consider the Poisson equation,

$$a(u,v) := (A\nabla u, \nabla v) = (f,v), \quad \forall v \in V := H_0^1(\Omega).$$

Decomposition

$$V = V_H \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \{v \in V \mid \mathfrak{I}_{\mathcal{T}} v = 0\} \quad (V_H, V^{\mathrm{f}}) = 0.$$



- $\mathfrak{I}_{\mathcal{T}}$ is an interpolant
- V_H is P1-FEM
- variations in A are not resolved
- $\|\nabla(u - u_H)\| \leq C(A, A', f)H.$

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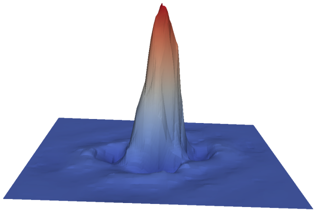
$$a(u, v) := (A\nabla u, \nabla v) = (f, v), \quad \forall v \in V := H_0^1(\Omega).$$

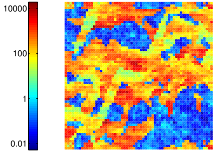
Decomposition

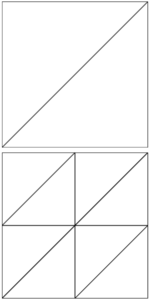
$$V = V_H^{\mathrm{ms}} \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \{v \in V \mid \mathfrak{I}_{\mathcal{T}} v = 0\} \quad a(V_H, V^{\mathrm{f}}) = 0.$$

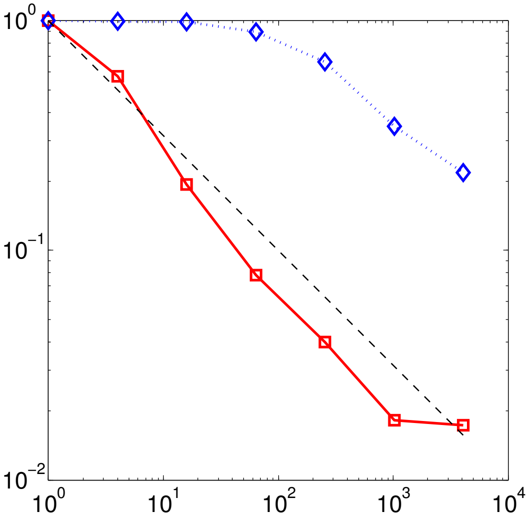


- $\dim(V_H^{\mathrm{ms}}) = \dim(V_H)$
- exponential decay
- independent subgrid problems
- $\|\nabla(u - u_H^{\mathrm{ms}})\| \leq C(A, f)H.$

Numer. exp. (Poisson, $f \in L^2$ (H), $f \in H^1_0$ (H^2)!)



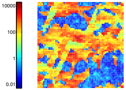


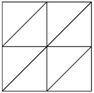


$H = 2^{-1}, 2^{-2}, \dots, 2^{-7}$
$h = 2^{-9}$, $k = \log(1/H)$

$\|\nabla(u_h - u^{\mathrm{ms},h}_{H,k})\|$ vs. #dof

Numer. exp. (linear eigenvalue problem, H^4)





ℓ	$\lambda_h^{(\ell)}$	$e^{(\ell)}(1/2\sqrt{2})$	$e^{(\ell)}(1/4\sqrt{2})$	$e^{(\ell)}(1/8\sqrt{2})$	$e^{(\ell)}(1/16\sqrt{2})$
1	21.4144522	5.472755371	0.237181706	0.010328293	0.000781683
2	40.9134676	-	0.649080539	0.032761482	0.002447049
3	44.1561133	-	1.687388874	0.097540102	0.004131422
4	60.8278691	-	1.648439518	0.028076168	0.002079812
5	65.6962136	-	2.071005692	0.247424446	0.006569640
6	70.1273082	-	4.265936007	0.232458016	0.016551520
7	82.2960238	-	3.632888104	0.355050163	0.013987920
8	92.8677605	-	6.850048057	0.377881216	0.049841235
9	99.6061234	-	10.305084010	0.469770376	0.026027378
10	109.1543283	-	-	0.476741452	0.005606426
11	129.3741945	-	-	0.505888044	0.062382302
12	138.2164330	-	-	0.554736550	0.039487317
13	141.5464639	-	-	0.540480876	0.043935515
14	145.7469718	-	-	0.765411709	0.034249528
15	152.6283573	-	-	0.712383825	0.024716759
16	155.2965039	-	-	0.761104705	0.026228034
17	158.2610708	-	-	0.749058367	0.091826207
18	164.1452194	-	-	0.840736127	0.118353184
19	171.1756923	-	-	0.946719951	0.111314058
20	179.3917590	-	-	0.928617606	0.119627862

Table : Errors $e^{(\ell)}(H) =: \frac{\lambda_H^{(\ell)} - \lambda_h^{(\ell)}}{\lambda_h^{(\ell)}}$ and $h = 2^{-7}\sqrt{2}$.

The quadratic eigenvalue problem

Consider discretized structure with damping,

$$Kx + \lambda Cx + \lambda^2 Mx = 0,$$

where $K_{ij} = (A\nabla\phi_j, \nabla\phi_i)$ is stiffness, $C_{ij} = c(\phi_j, \phi_i)$ is damping, $M_{ij} = (\phi_j, \phi_i)$ is mass matrix, and $V_h = \text{span}(\{\phi_i\})$.

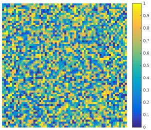
Linearization: $y = \lambda x$

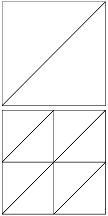
$$Ax := \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} -C & -M \\ M & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} := \lambda Bx,$$

Note that $T = A^{-1}B$ has eigenvalues λ^{-1}. T is not symmetric (even if C is).

The analysis is very different compared to the symmetric case.

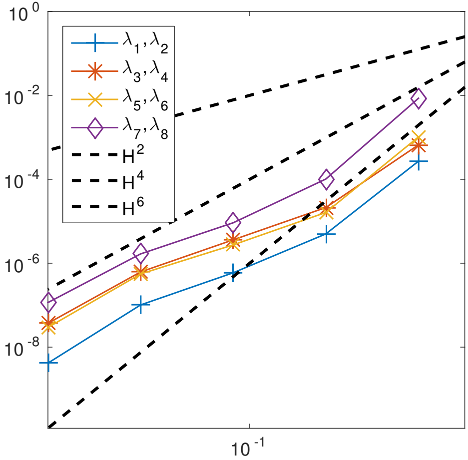
Numerical experiment





$H = 2^{-2}, 2^{-3}, \ldots, 2^{-6}$

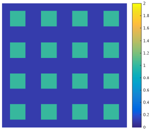
$h = 2^{-8}$, $k = 3 \log 1/H$

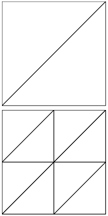


$|\lambda_h - \lambda_H^{ms}|/|\lambda_h|$ vs. #dof

A (pic), $c(u, v) = \int_\Omega (1 + sin(10x))u\, v\, dx$

Numerical experiment





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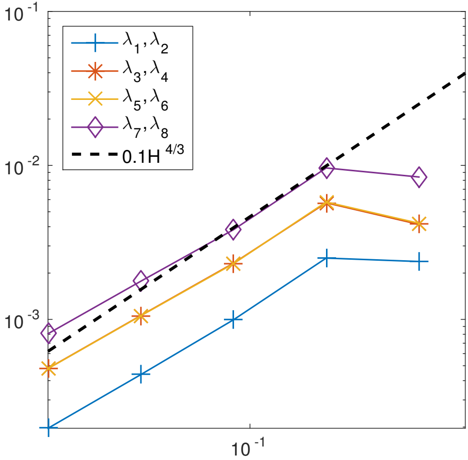
$$h = 2^{-8},\ k = 3 \log 1/H$$



$|\lambda_h - \lambda_H^{ms}|/|\lambda_h|$ vs. #dof

$$A \text{ (pic)},\ c(u,v) = \int_\Omega \frac{A+0.5}{100} \nabla u \cdot \nabla v \, dx$$

Relevant papers and future direction

Work package D: Multiscale Methods for Solution and Optimisation of Nonlinear Eigenvalue Problems.

1. Linear eigenvalue, M. & Peterseim, Numer. Math. 2015.
2. LOD for complex geom., Elfverson, Larson, & M., sub MMS
3. Quadratic eigenvalue, M. & Peterseim, preprint

- Optimization of functions of eigenvalues $f(\lambda_1, \lambda 2, \dots)$, for the Poisson type eigenvalue problem. Note that

$$\frac{d\lambda_i(s)}{ds} = \int_\Omega \frac{dA(s)}{ds} \nabla u_i(s) \cdot \nabla u_i(s), \mathrm{dx}$$

Toader, Barbarosie, *Optimization of eigenvalues and eigenmodes by using the adjoint method*, preprint 2015