Contrast independent localization of multiscale problems

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2nd Workshop on Scientific Computing in Sweden, 2018

Höör, Sweden

2018-10-09

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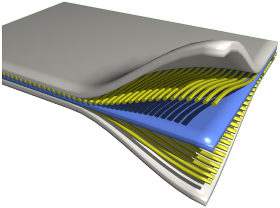
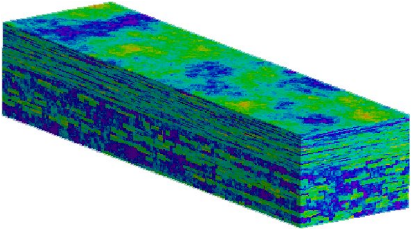
Multiscale problems

We consider applications such as





▷ composite materials ▷ flow in a porous medium

that require numerical solution of partial differential equations with rough data (module of elasticity, conductivity, or permeability).

Major challenges: High contrast and thin structures.

Outline

The finite element method

The Poisson equation: let $\alpha < A < \beta$, $f \in L^2(\Omega)$

$$-\nabla \cdot A\nabla u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega.$$

On weak form: find $u \in V := H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega (A\nabla u) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \text{for all } v \in V.$$

FE approximation: find $u_h \in V_h \subset V$ such that

$$a(u_h, v) := \int_\Omega (A\nabla u_h) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \text{for all } v \in V_h.$$

Error bound if $u \in H^2(\Omega)$:

$$|||u - u_h||| := \|A^{1/2}\nabla(u - u_h)\|_{L^2(\Omega)} \sim C(A')h.$$

Multiscale methods

Objectives:

- Find a subspace of $V_H^{\mathrm{ms}} \subset V_h$ for which $u_H^{\mathrm{ms}} \in V_H^{\mathrm{ms}}$

$$a(u_H^{\mathrm{ms}}, v) := \int_\Omega (A \nabla u_H^{\mathrm{ms}}) \cdot \nabla v \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x \quad \text{for all } v \in V_H^{\mathrm{ms}},$$

  fulfills, with C independent of A',

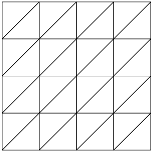
$$|||u_h - u_H^{\mathrm{ms}}||| \leq CH,$$

  but with $\dim(V_H^{\mathrm{ms}}) \ll \dim(V_h)$.
- Show that a basis for V_H^{ms} can be constructed by local parallel computations.
- Reuse the coarse representation in applications.
- Multiscale methods: VMS, MsFEM, HMM, GFEM, GMsFEM...

Outline

Orthogonal decomposition

- (coarse) FE mesh $\mathcal{T}$ with parameter $H > h$
- P1-FE space $V_H := \{v \in V \mid \forall T \in \mathcal{T}, v|_T \in P_1(T)\}$
- $\mathfrak{I}_\mathcal{T} : V \to V_H$ some interpolation operator



Decomposition

$$V = V_H \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \text{kernel}\, \mathfrak{I}_\mathcal{T} = \{v \in V \mid \mathfrak{I}_\mathcal{T} v = 0\}$$

- For each $v \in V_H$ define finescale projection $Qv \in V^{\mathrm{f}}$ by

$$a(Qv, w) = a(v, w) \quad \text{for all } w \in V^{\mathrm{f}}$$

***a*-Orthogonal Decomposition**

$$V = V_H^{\mathrm{ms}} \oplus V^{\mathrm{f}} \quad \text{with } V_H^{\mathrm{ms}} := (V_H - QV_H)$$

Ideal multiscale representation

Given the space V_H^{ms} we construct a Galerkin approximation:

Ideal method

Find $u_H^{\mathrm{ms}} \in V_H^{\mathrm{ms}}$ such that

$$a(u_H^{\mathrm{ms}}, v) = (f, v), \ \forall v \in V_H^{\mathrm{ms}}.$$

We have that $u - u_H^{\mathrm{ms}} = u_f \in V^{\mathrm{f}}$ since u_H^{ms} is the a-orthogonal projection of u onto V_H^{ms}. Therefore

$$|||u_f|||^2 = a(u, u_f) = (f, u_f) = (f, u_f - \mathfrak{I}_{\mathcal{T}} u_f) \leq \frac{C_{\mathfrak{I}_{\mathcal{T}}}}{\alpha^{1/2}} ||Hf||_{L^2(\Omega)} |||u_f|||.$$

For V_H^{ms} to be useful we need a discrete local basis.

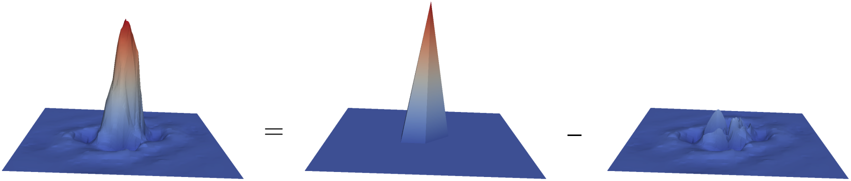
Localization of multiscale basis

- $\phi_x \in V_H$ denotes the classical nodal basis function
- $Q\phi_x \in V^{\mathrm{f}}$ denotes the finescale correction of ϕ_x

Generalized FE space

$$V_H^{\mathrm{ms}} = \operatorname{span}\{\phi_x - Q\phi_x\}$$

Example



$\phi_x - Q\phi_x \in V_H^{\mathrm{ms}}$ = $\phi_x \in V_H$ − $Q\phi_x \in V^{\mathrm{f}}$

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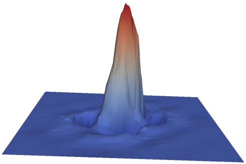

$\phi_x - Q\phi_x \in V_H^{\mathrm{ms}}$

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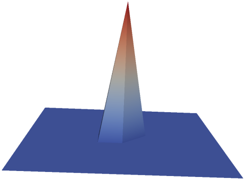

$\phi_x \in V_H$

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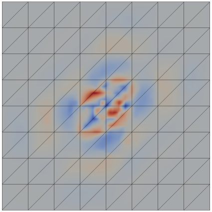


$Q\phi_x \in V^{\mathrm{f}}$

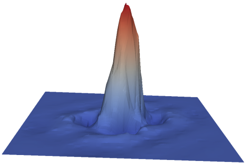
Localization of multiscale basis

- $\phi_x \in V_H$ denotes the classical nodal basis function
- $Q\phi_x \in V^{\mathrm{f}}$ denotes the finescale correction of ϕ_x

Generalized FE space

$$V_{H,k}^{\mathrm{ms}} = \operatorname{span}\{\phi_x - Q_k\phi_x\}$$

Example



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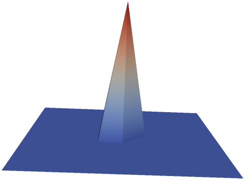

$\phi_x \in V_H$

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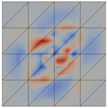


$Q_k\phi_x \in V^{\mathrm{f}}$

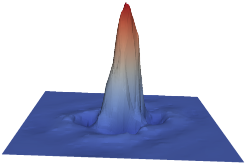
Localization of multiscale basis

- $\phi_x \in V_H$ denotes the classical nodal basis function
- $Q\phi_x \in V^{\mathrm{f}}$ denotes the finescale correction of ϕ_x

Generalized FE space

$$V_{H,k}^{\mathrm{ms},h} = \mathrm{span}\left\{\phi_x - Q_{k,h}\phi_x\right\}$$

Example



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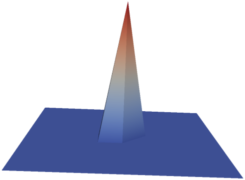

$\phi_x \in V_H$

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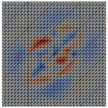

$Q_{k,h}\phi_x \in V^{\mathrm{f}}$

Localization of multiscale basis

- $\phi_x \in V_H$ denotes the classical nodal basis function
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Generalized FE space

$$V_{H,k}^{\mathrm{ms},h} = \mathrm{span}\,\{\phi_x - Q_{k,h}\phi_x\}$$

Localized Orthogonal Decomposition

Find $u_{H,k}^{\mathrm{ms},h} \in V_{H,k}^{\mathrm{ms},h}$ such that

$$a(u_{H,k}^{\mathrm{ms},h}, v) = (f, v), \quad \text{for all } v \in V_{H,k}^{\mathrm{ms},h}$$

A priori bound:

$$|||u_h - u_{H,k}^{\mathrm{ms},h}||| \leq CH$$

where $k = C_1(\beta/\alpha)\log(H^{-1})$ and C independent of A'.

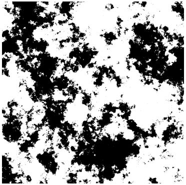
Outline

High contrast data (with Fredrik Hellman)

Poisson equation:

$$-\nabla \cdot A \nabla u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega.$$

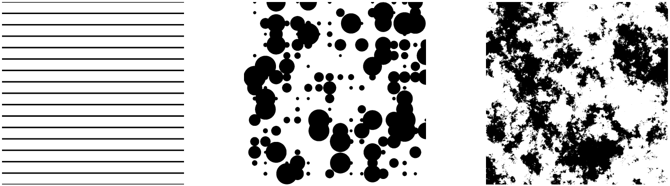
$A = 1$ in Ω_1 (black), $A = \alpha$ in Ω_α, $\alpha \ll 1$, and $f = \chi_{[1/4,3/4]^2}$.



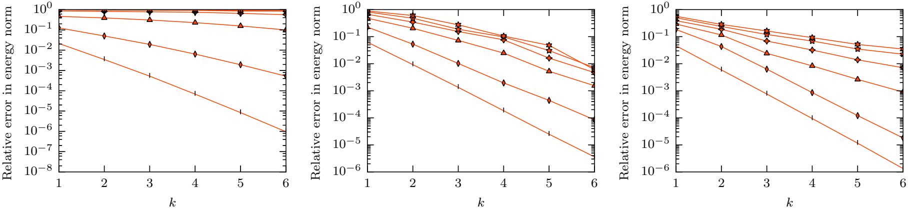
- High contrast data with channels leads to non-local behaviour.
- The decay rate of the basis functions determines the accuracy of LOD.
- The choice of interpolant $\mathfrak{I}_{\mathcal{T}} v = \sum_{x \in \mathcal{N}} \bar{v}_{\omega_x} \phi_x$ affects the decay.

Numerical example: High contrast

High contrast data Three examples: $H = 2^{-4}$, $h = 2^{-10}$,



We let $\alpha = 10^{-1}, \dots, 10^{-6}$ and plot $|||u_h - u^{\mathrm{ms},h}_{H,k}|||$ vs. k, with $\mathfrak{I}^{SZ}_{\mathcal{T}}$,



Scott-Zhang type interpolation

Nodal variables:

Let $x \in \mathcal{N}$ be nodes of $\mathcal{T}$ and $\sigma_x \subset \Omega$ associated domains. We define a $L^2(\sigma_x)$-dual basis $\psi_x \in V_H$ fulfilling,

$$\int_{\sigma_x} \psi_x \phi_y = \delta_{xy}.$$

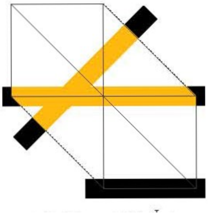
Let the nodal variable $N_x(v) = \int_{\sigma_x} \psi_x v$ and,

$$\mathfrak{I}_{\mathcal{T}}^{\sigma} v = \sum_{x \in \mathcal{N}} N_x(v) \phi_x.$$

- σ_x does not need to be full elements T or vertex patches $U_1(x)$.
- The stability of $|N_x(v)| \leq \|\psi_x\|_{L^2(\sigma_x)} \|v\|_{L^2(\sigma_x)}$ depends on the size and shape of σ_x and its distance to x.

Geometry dependent interpolation

- The interpolant $\mathfrak{I}_{\mathcal{T}} v = \sum_{x \in \mathcal{N}} \bar{v}_{\sigma_x} \phi_x$ defines V_f and V_H^{ms}.
- We need to force correctors to be small in the channels!



1. If $x \in \Omega_\alpha$ let $\sigma_x = \omega_x$, vertex patch
2. If $x \in \Omega_1$ let $\sigma_x \subset \omega_x \cap \Omega_1$, connected
3. We need sufficiently many nodes in Ω_1 (separation $\sim H$)

Then we can prove decay independent of α.

Numerical example: High contrast

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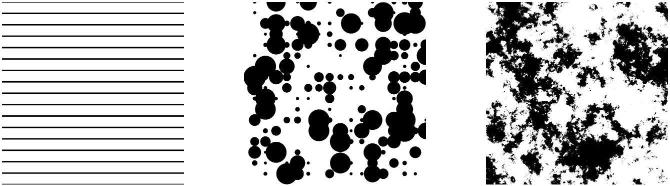

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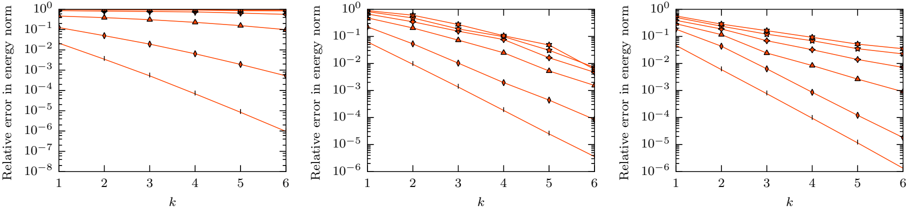

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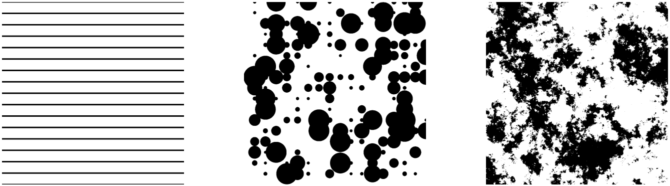

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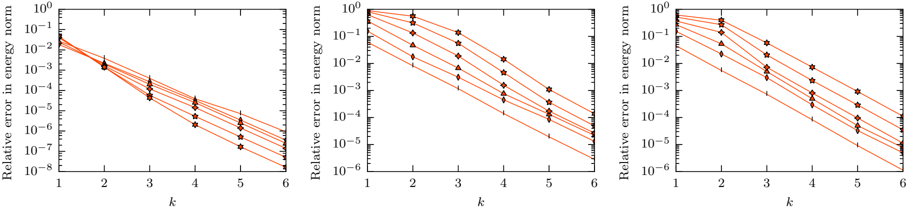
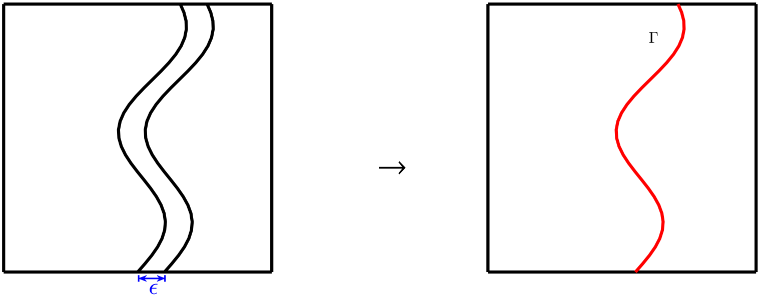


Outline

Interface model (ongoing with Siyang Wang)

Thin structures (so far) need to be resolved.



As $\epsilon \to 0$ we have convergence (with rate) to an interface problem.

$$\begin{aligned} -\nabla \cdot A \nabla u &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \\ [u] &= 0, \quad \text{on } \Gamma \\ -\nabla_\Gamma \cdot A_\Gamma \nabla_\Gamma u &= f_\Gamma - [n \cdot A \nabla u], \quad \text{on } \Gamma. \end{aligned}$$

Weak form and LOD

On weak form we have: find $u \in V = H_0^1(\Omega) \cap H^1(\Gamma)$ such that

$$a(u, v) := \int_\Omega A \nabla u \cdot \nabla v \, dx + \int_\Gamma A_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds = \int_\Omega f v \, dx + \int_\Gamma f_\Gamma v \, ds,$$

for all $v \in V = H_0^1(\Omega) \cap H^1(\Gamma)$. We note that $a(\cdot, \cdot)$ is a scalar product on V.

Localized Orthogonal Decomposition

Given an interpolant $\mathfrak{I}_{\mathcal{T}} : V \to V_H$ we can formulate the LOD method: find $u_H^{\text{ms}} \in V_H^{\text{ms}}$ such that

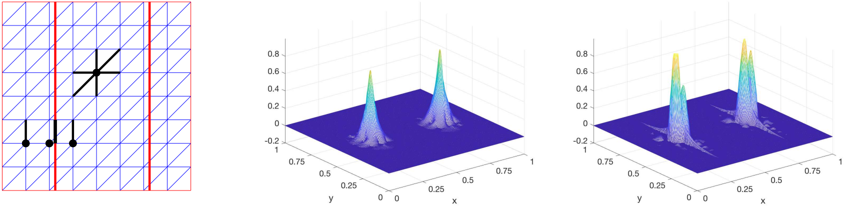
$$a(u_H^{\text{ms}}, v) = \int_\Omega f v \, dx + \int_\Gamma f_\Gamma v \, ds, \quad \text{for all } v \in V_H^{\text{ms}}.$$

Edge based Scott-Zhang interpolation

Fulfills convergence result we need

$$\|v - \mathfrak{I}_{\mathcal{T}} v\|_{L^2(\Omega)} + \|v - \mathfrak{I}_{\mathcal{T}} v\|_{L^2(\Gamma)} \leq CH \left(\|v\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Gamma)} \right)^{1/2}.$$

Decay follows if we take averages on the interface.



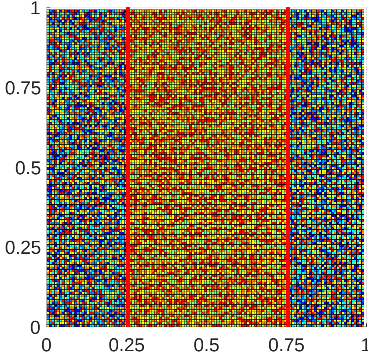
The figure shows domain of integration and edge based versus element based interpolation.

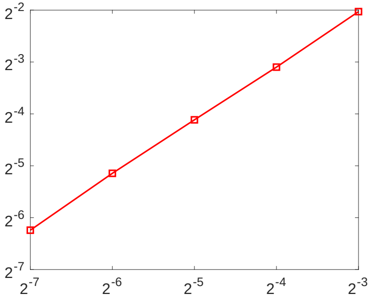
- Fine scale discretization: *A simple finite element method for elliptic bulk problems with embedded surfaces* Burman et.al.

Numerical example

Let $A_\Gamma = 1$ and A be random between 0.1 and 1 (left and right) and between 0.5 and 1 (middle), $f = 1$, $f_\Gamma = 5$.

Data varies on 2^{-7} and $h = 2^{-9}$, $k = \log(H^{-1})$.





Error in energy norm vs H.

Outline

Applications of LOD

Stationary/eigenvalue problems

- Gross-Pitaevskii, (Henning-M.-Peterseim), 2014.
- Helmholtz, (Ohlberger-Verfürth, Gallistl-Peterseim), 2015-2017.
- Reduced basis, (Abdulle-Henning), 2015.
- Linear and quadratic eigenvalue problems, (M.-Peterseim), 2015-2016.
- Elasticity, (Henning-Persson), 2016.
- Fractional Laplacians, (Brown et.al.), 2018.
- Stochastic diffusion, (Gallistl-Peterseim), 2018.

Time-dependent problems

- Heat equation and thermoelasticity, (M. & Persson), 2017-2018.
- Wave equation, (Abdulle & Henning), 2017.

Comments and conclusion

- Multiscale methods are useful when solving many similar problems.
- Thin high conductivity channels are challenging and important.
- Global fine scale connections are equally problematic for iterative methods (Multigrid, DD).
- We treat this by splitting the space with carefully chosen interpolants.
- LOD works well for problems with interfaces using edge based interpolation.
- Future work include interior interfaces and wave propagation in fractured porous media.