A New Mixed Multiscale Method for Oil Reservoir Simulation

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The Model Problem

Poisson Equation on mixed form:

$$\begin{cases} \frac{1}{a}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ n \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma. \end{cases}$$

where $a > 0$ bounded, Ω is a domain in $\mathbf{R}^d$, $d = 1, 2, 3$, with boundary Γ, and f is a given function.

Applications

Elliptic problems of this kind needs to be solved in oil reservoir simulation.

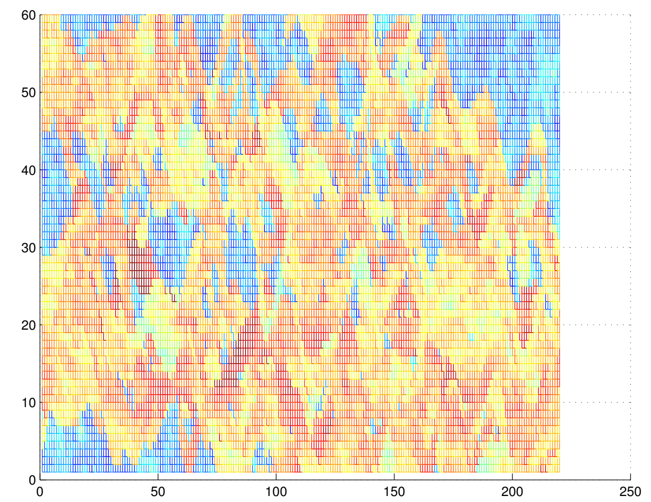


Figure 1: 2D slice of permeability (a) in oil reservoir (log scale) $\max a / \min a = 1.8e4$.

Why Multiscale Method?

- If we for the moment assume a to be periodic $a = a(x/\epsilon)$ we have (Hou),

$$\|\frac{1}{\sqrt{a}}(\boldsymbol{\sigma} - \boldsymbol{\Sigma})\| \leq C\frac{h}{\epsilon}.$$

- $h > \epsilon$ will give unreliable results even with exact quadrature.
- $h < \epsilon$ will be to computationally expensive to solve on a single mesh.
- Parallelized local problems must be solved.

Weak form

Find $\boldsymbol{\sigma} \in \boldsymbol{V} = \{\boldsymbol{v} \in H(\text{div};\Omega) : \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \Gamma\}$ and $u \in W = L^2(\Omega)/\mathbf{R}$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}, \boldsymbol{v}) + (u, \nabla \cdot \boldsymbol{v}) = 0, \\ -(\nabla \cdot \boldsymbol{\sigma}, w) = (f, w), \end{cases}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Here $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ scalar product for vector and scalar functions.

Coarse and Fine Scales

We introduce spaces $\boldsymbol{V}_c \oplus \boldsymbol{V}_f = \boldsymbol{V}$ and $W_c \oplus W_f = W$ (Hughes).

- $\boldsymbol{V}_c$ is a finite dimensional approximation of $H(\text{div}; \Omega)$. (finite element space e.g. Raviart-Thomas)
- W_c is an approximation of $L^2(\Omega)$. (e.g. piecewise constants).
- The degrees of freedom in these spaces should be possible to handle on a single computer.

Coarse and Fine Scales

Find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) + (u_f, \nabla \cdot \boldsymbol{v}_c) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) - (\nabla \cdot \boldsymbol{\sigma}_f, w_c) = (f, w_c) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) - (u_c, \nabla \cdot \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla \cdot \boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Coarse and Fine Scales

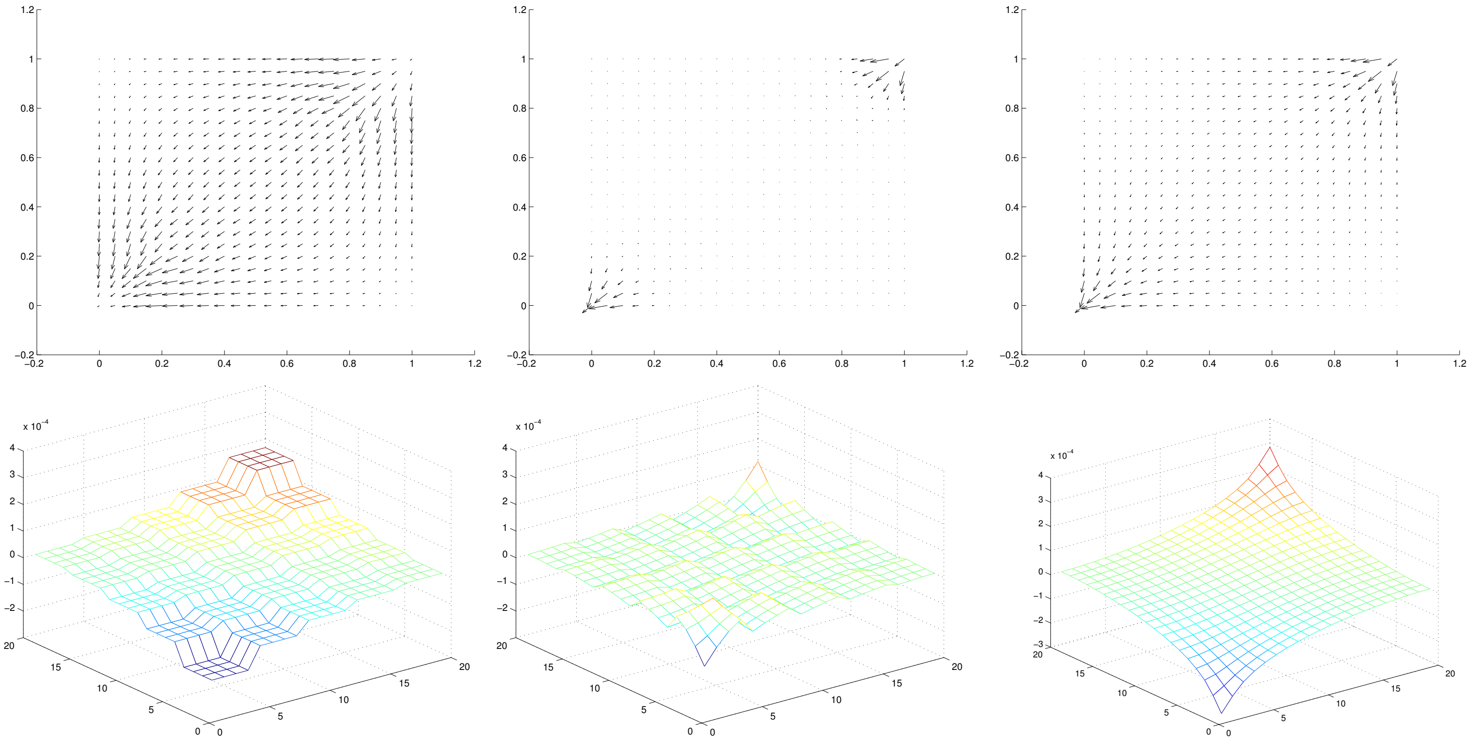


Figure 2: $\boldsymbol{\sigma}$ and u on coarse and fine scale.

Orthogonality

We use an hierarchical basis of Raviart-Thomas element for the flux and the piecewise constants for the pressure.

$$(w_c, \nabla \cdot \boldsymbol{v}_f) = \sum_K (w_c, \nabla \cdot \boldsymbol{v}_f)_K = \sum_K w_c^K \int_{\partial K} \boldsymbol{n} \cdot \boldsymbol{v}_f \, dx = 0$$

where w_c^K is the constant at coarse element K,

$$(w_f, \nabla \cdot \boldsymbol{v}_c) = \sum_K (w_f, \nabla \cdot \boldsymbol{v}_c)_K = \sum_K \nabla \cdot \boldsymbol{v}_c^K \int_K w_f \, dx = 0$$

Coarse and Fine Scales

Find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) + (u_f, \nabla \cdot \boldsymbol{v}_c) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) - (\nabla \cdot \boldsymbol{\sigma}_f, w_c) = (f, w_c) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) - (u_c, \nabla \cdot \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) + (\nabla \cdot \boldsymbol{\sigma}_c, w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Modified set of Equations

Find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) = 0 \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) = (f, w_c) \\ (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f) \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f) \end{cases}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Decoupling of Fine Scale Equations

We start by introducing two partitions of unity, $\sum_i \boldsymbol{\phi}_i = \boldsymbol{I}$ and $\sum_i \psi_i = 1$ where $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{\phi}_i \in \boldsymbol{V}_c$ coarse Raviart-Thomas base function, and $\psi_i \in W_c$ coarse piecewise constant base functions.

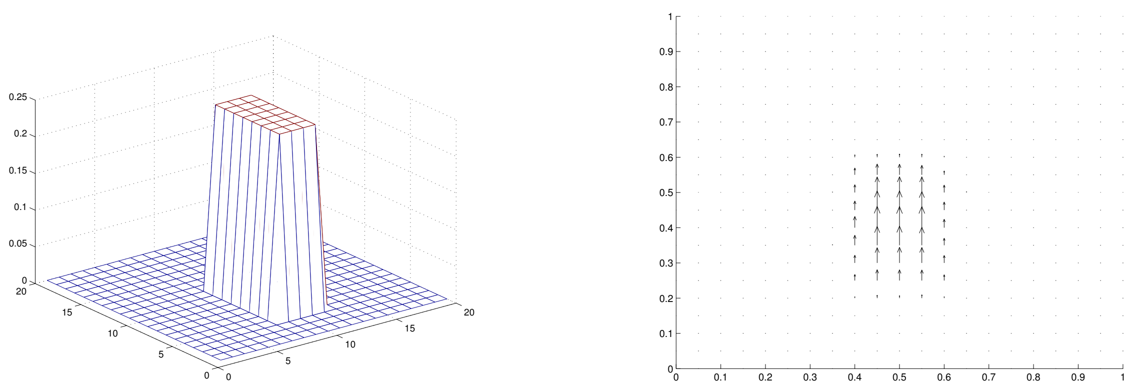


Figure 3: ψ_i, and $\boldsymbol{\phi}_i$.

Decoupling of Fine Scale Equations

We introduce $\boldsymbol{\sigma}_{f,i} \in \boldsymbol{V}_f$ and $u_{f,i} \in W_f$ such that $\boldsymbol{\sigma}_c = \sum_i \boldsymbol{\sigma}_c^i \phi_i$, $\boldsymbol{\sigma}_c^i \in \mathbf{R}$, u_c, $\boldsymbol{\sigma}_f = \sum_i \boldsymbol{\sigma}_{f,i}$, and $u_f = \sum_i u_{f,i}$ solves:

$$\left\{ \begin{aligned} (\tfrac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_c) + (\tfrac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_c) + (u_c, \nabla \cdot \boldsymbol{v}_c) &= 0, \\ -(\nabla \cdot \boldsymbol{\sigma}_c, w_c) &= (f, w_c), \\ (\tfrac{1}{a}\boldsymbol{\sigma}_{f,i}, \boldsymbol{v}_f) + (u_{f,i}, \nabla \cdot \boldsymbol{v}_f) &= -(\tfrac{1}{a}\boldsymbol{\sigma}_c^i \boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\sigma}_{f,i}, w_f) &= (f, w_f \psi_i), \end{aligned} \right.$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

Local Solutions

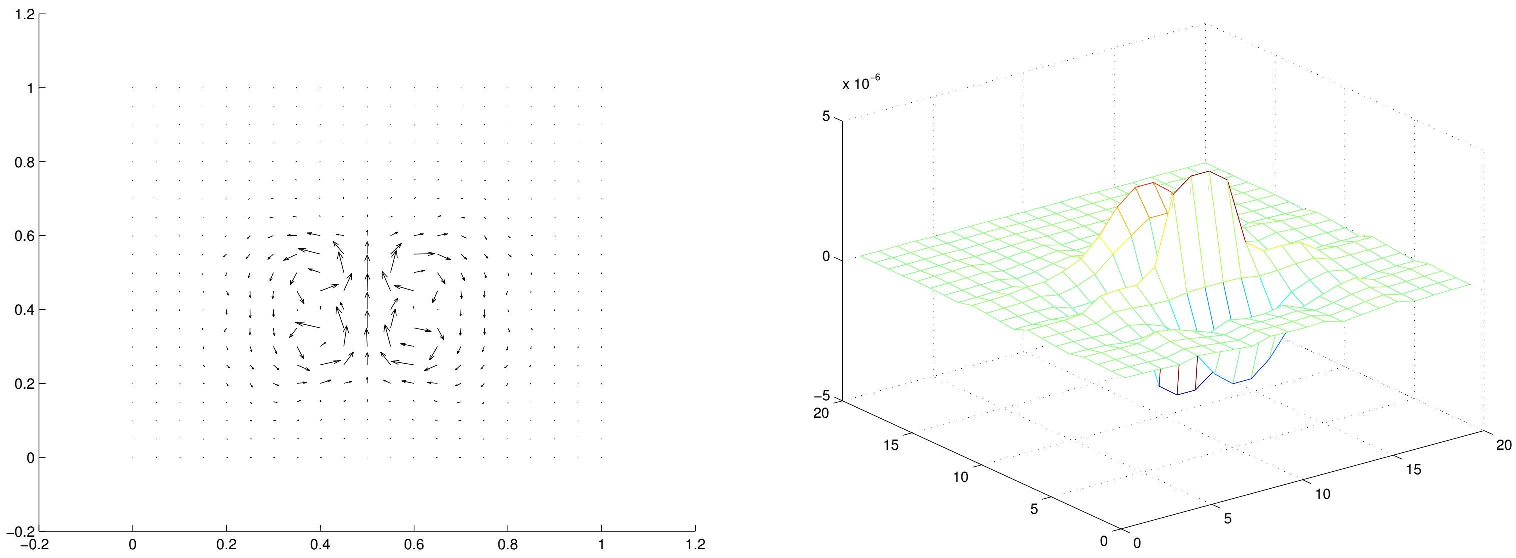


Figure 4: The local solutions $\boldsymbol{\sigma}_{f,i}$ and $u_{f,i}$

In this simple example $a = 1$.

Motivation for Introducing Patches

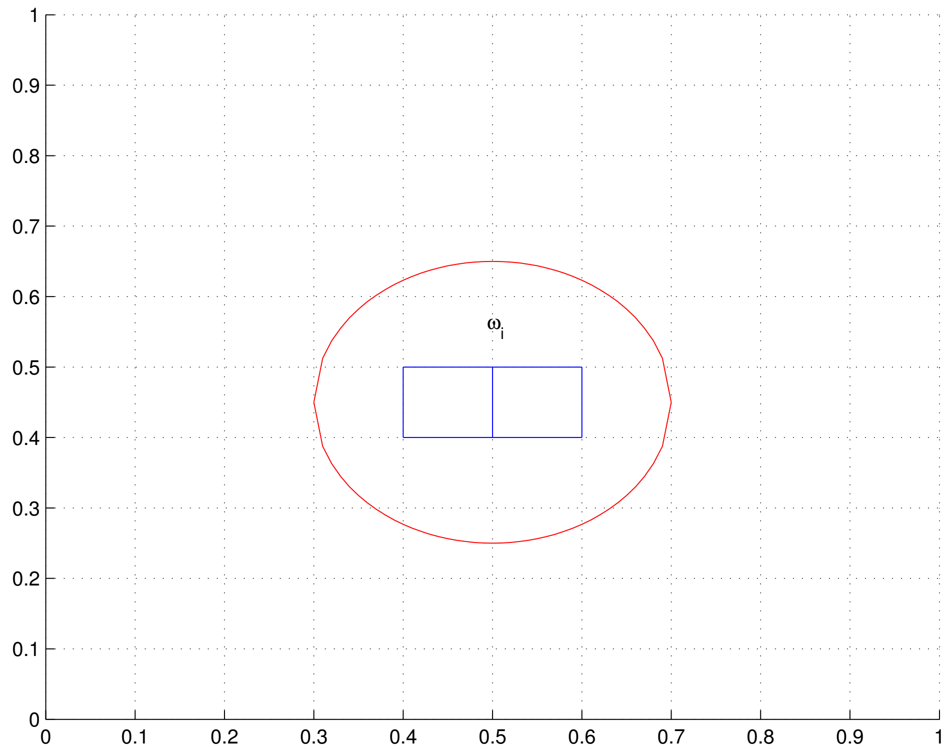
- The right hand side has support on $supp(\boldsymbol{\phi}_i) = supp(\psi_i)$.
- The equations are solved in a slice space where solutions decay rapidly.

$$\int_E \boldsymbol{n} \cdot \boldsymbol{\sigma}_{f,i} \, dx = 0$$

and

$$\int_K u_{f,i} \, dx = 0.$$

The Patch



The patch ω_i typically consists of coarse elements but could have any geometry.

Solving Local Neumann Problems

Find $\boldsymbol{\Sigma}_c = \sum_i \boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i \in \boldsymbol{V}_H$, $\boldsymbol{\Sigma}_{f,i} \in \boldsymbol{V}_h(\omega_i)$, $U_c \in W_H$, and $U_{f,i} \in W_h(\omega_i)$ such that

$$\left\{ \begin{aligned} (\tfrac{1}{a}\boldsymbol{\Sigma}_c, \boldsymbol{v}_c) + (\tfrac{1}{a}\boldsymbol{\Sigma}_f, \boldsymbol{v}_c) + (U_c, \nabla \cdot \boldsymbol{v}_c) &= 0, \\ -(\nabla \cdot \boldsymbol{\Sigma}_c, w_c) &= (f, w_c), \\ (\tfrac{1}{a}\boldsymbol{\Sigma}_{f,i}, \boldsymbol{v}_f) + (U_{f,i}, \nabla \cdot \boldsymbol{v}_f) &= -(\tfrac{1}{a}\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\Sigma}_{f,i}, w_f) &= (f, w_f \psi_i), \end{aligned} \right.$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_H$, $\boldsymbol{v}_f \in \boldsymbol{V}_h(\omega_i)$, $w_c \in W_H$, and $w_f \in W_h(\omega_i)$.

Example of Local Solutions U

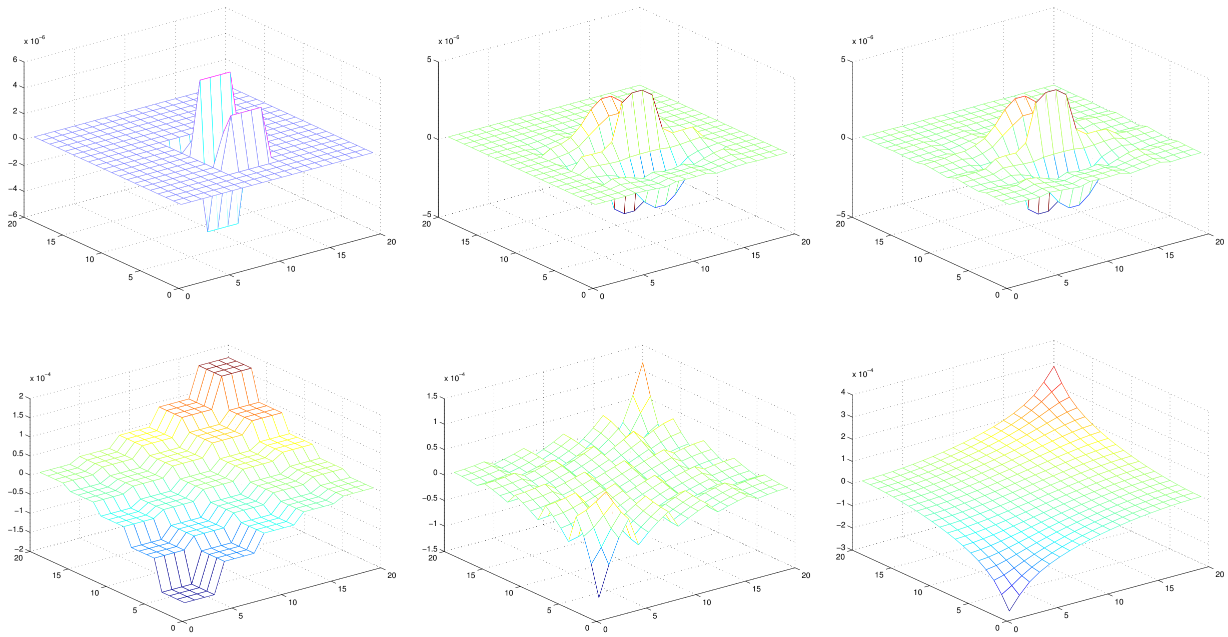


Figure 5: Above: 1, 2, and 3 layer patches, below: U_c, U_f, and, U using 3 layers of coarse elements.

Example of Local Solutions Σ

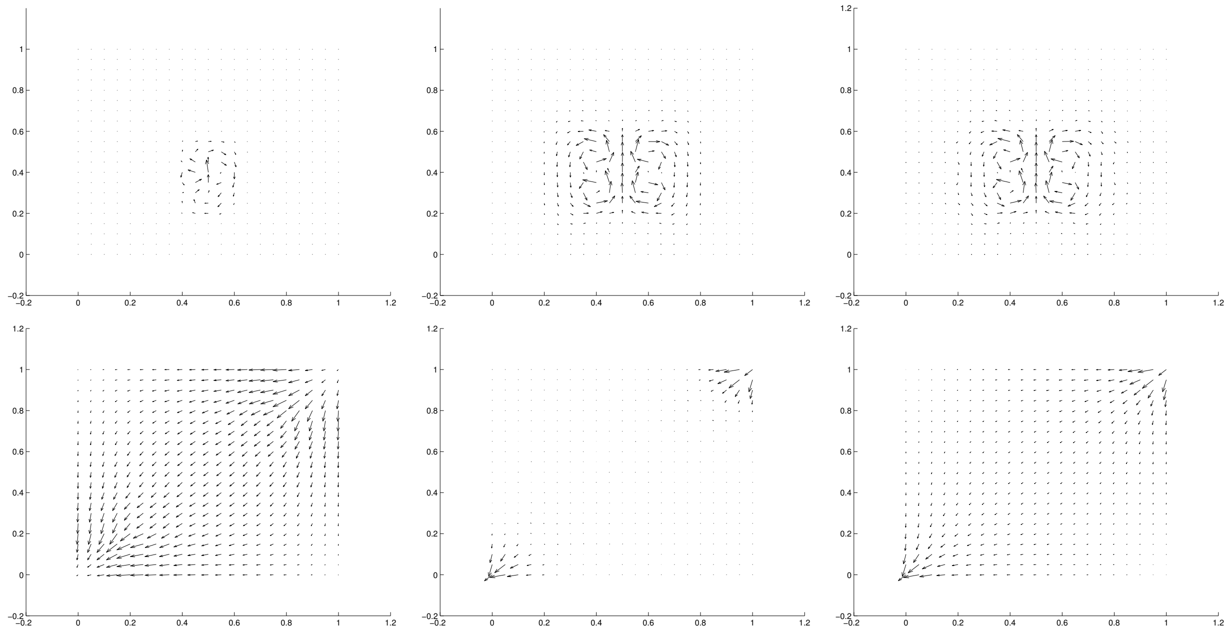
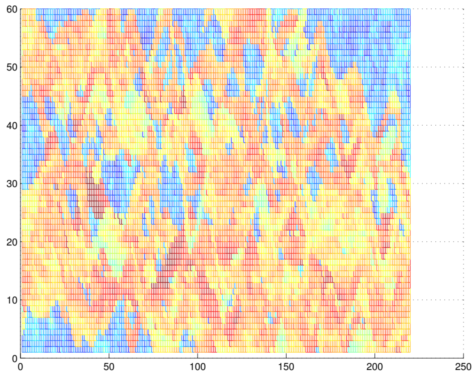


Figure 6: Above: 1, 2, and 3 layer patches, below: Σ_c, Σ_f, and, Σ using 3 layers of coarse elements.

Oil Reservoir Simulation

In the figure we see the bottom layer of the Upper Ness formation.



We let $f = 1$ in the lower left corner and $f = -1$ in the upper right corner. $\max a / \min a = 1.8e4$.

Oil Reservoir Simulation

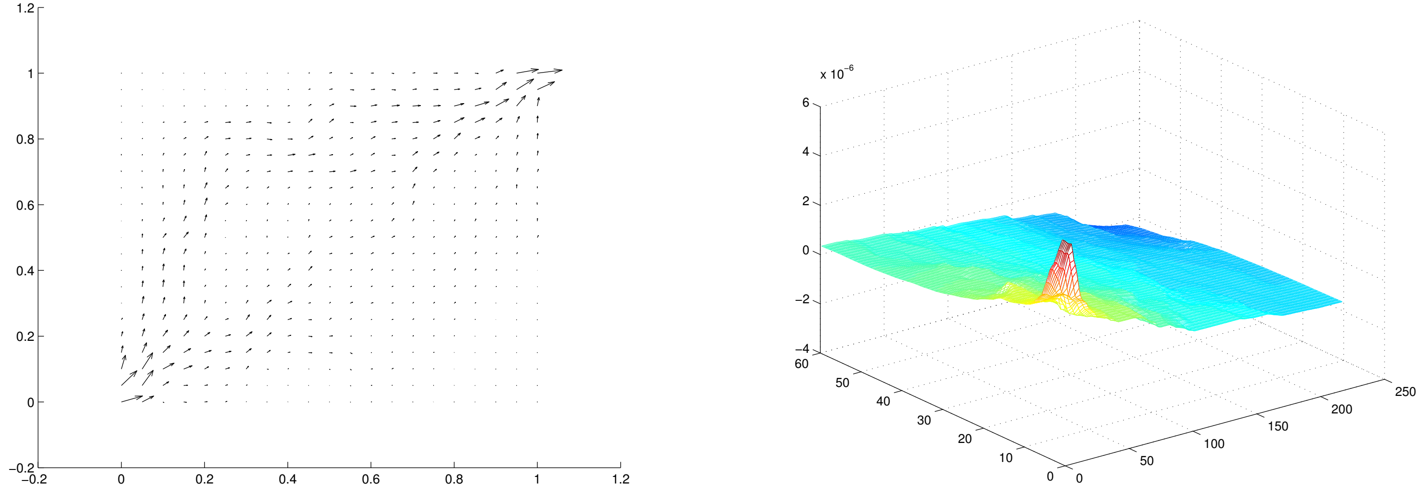


Figure 7: To the left we see the flux and to the right the pressure. We use 220×60 elements for the reference solution.

Oil Reservoir Simulation

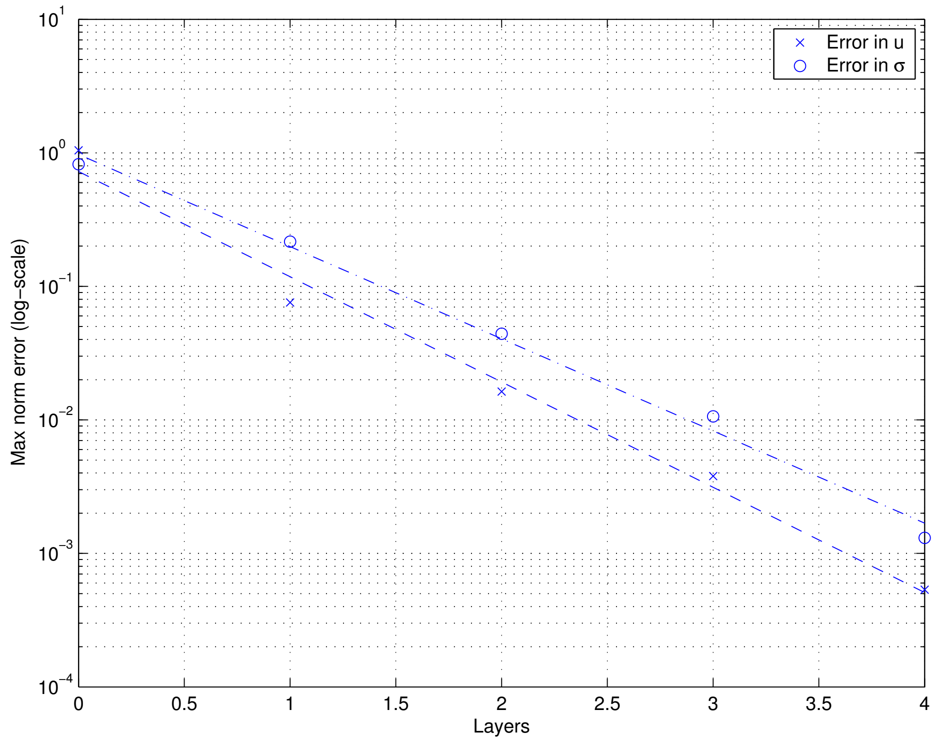


Figure 8: 55×15 coarse elements and $h = H/4$.

Conclusions

- We see exponential decay in max norm error compared with reference solution for increasing the number of layers.
- Slightly slower convergence for the flux but still exponential.
- Natural question: Is it necessary to solve local problems with same resolution and accuracy on the entire domain?

Energy Norm Estimate $\|v\|_a^2 = (\frac{1}{a}v, v)$

Next we present an estimate of the error.

$$\|\boldsymbol{\sigma} - \boldsymbol{\Sigma}\|_a^2 \leq \sum_i C_a \|\frac{1}{a}(\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U_{f,i}^*\|_{\omega_i}^2$$

$$+ \sum_i C_a \|h(f\psi_i + \nabla \cdot (\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|_{\omega_i}^2.$$

$$+ \sum_i C_a \|\frac{1}{2\sqrt{h}} U_{f,i}^*\|_{\partial\omega_i \setminus \Gamma}^2$$

U^* is a post processed version of U.

Adaptive Strategy

- Calculate Σ.
- Calculate the error indicators on each patch,

$$X_i(h) = \|\frac{1}{a}(\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) - \nabla U_{f,i}^*\|_{\omega_i}^2$$

$$Y_i(h) = \|h(f\psi_i + \nabla \cdot (\boldsymbol{\Sigma}_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|_{\omega_i}^2$$

$$Z_i(L) = \|\frac{1}{2\sqrt{h}} U_{f,i}^*\|_{\partial\omega_i \setminus \Gamma}^2$$

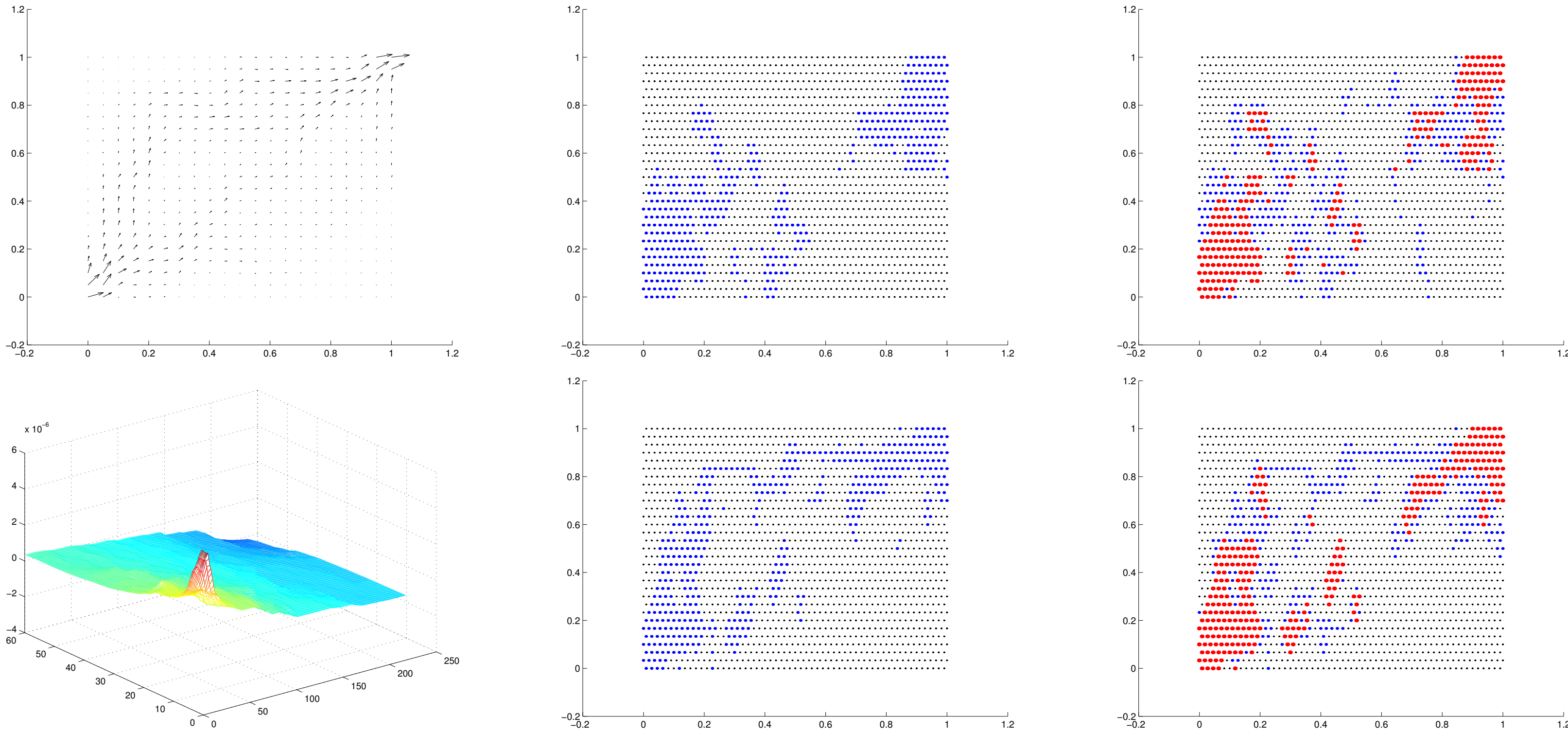
Adaptive Strategy

- If indicators $X_i(h)$ or $Y_i(h)$ are big on a patch we decrease h.
- If indicator $Z_i(L)$ is big we increase the size of the patch.
- Go back to the first step or stop if the solution is good enough.

Back to the Numerical Example

- We start with one layer patches with one refinement in all local problems.
- We calculate the error estimators.
- Then we increase 25% of the patches with one layer and refine the mesh on 25% according to the estimators.
- We repeat this process one more time.

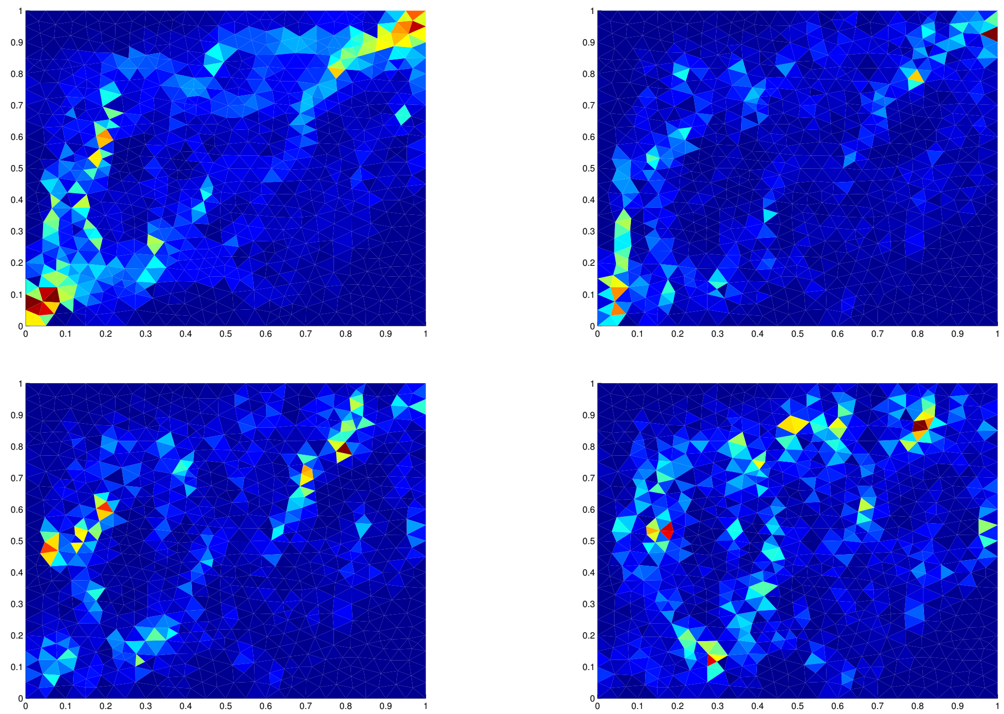
Number of Layers and Refinements



Black circle is one, blue is two and red is three layers/refinements.

Reactive Error in Energy Norm

Galerkin 105.6% and one iteration 15.8%.



Two iterations 10.1% and three iterations 7.6%.

Conclusions

- The error indicators finds critical areas.
- More computational effort in these areas decreases the global error quickly.
- To get an equally good approximation without adaptivity we need to use three refinements on two layer patches.
- In the example above we still have 70% of the patches using one layer and one refinement.

Applications

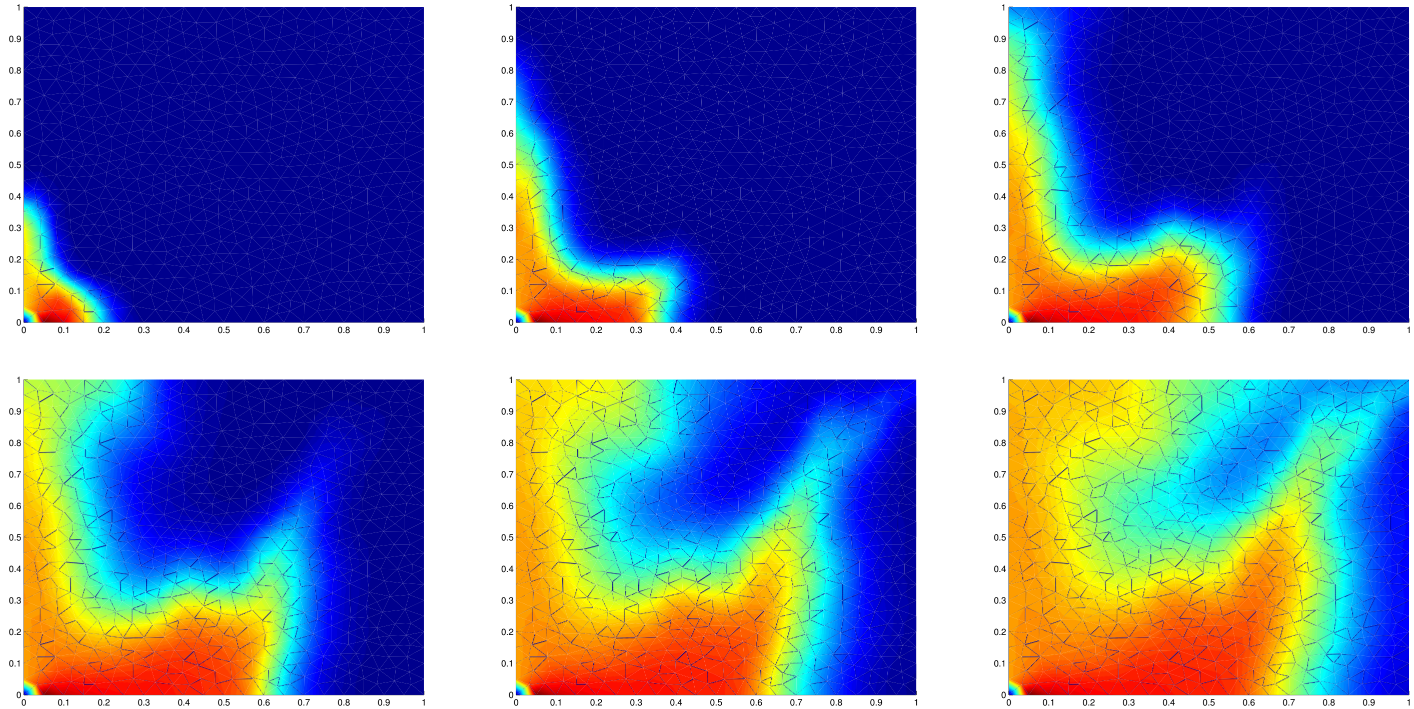
Given a good approximation of $\boldsymbol{\sigma}$ we can solve the following equation to simulate the water concentration in the well.

$$\begin{cases} \dot{c} + \nabla \cdot (\boldsymbol{\sigma} c) - \epsilon \triangle c = g \quad \text{in } \Omega \times (0, T], \\ \partial_n c = 0 \quad \text{on } \Gamma, \\ c = c_0 \quad \text{for } t = 0, \end{cases}$$

We use cg1-cg1 with sd to solve the equation since ϵ is very small.

Applications

Water concentration at different times.



σ taken from top layer in Tarbert formation.

Outlook

- Implementation in 3D.
- More scales.
- Multiscale approach for the transport problem.