Sensitivity Analysis for Solutions to Partial Differential Equations with Randomly Perturbed Data

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Outline

- Model problems: Poisson equation with
  - Randomly perturbed load and bnd condition
  - Randomly perturbed coefficient
- Fast methods for computing the distribution of the solution given the distribution of the data
- Error analysis and adaptivity
- Numerical example
- Summary and ideas on future work

The first model problem

Strong form: The Dirichlet Problem with multiple iid right hand sides, F^s, $s \in \Lambda$,

$$\begin{aligned} -\triangle U^s &= F^s \quad \text{in } \Omega, \\ U^s &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Weak form: Find $U^s \in V = H_0^1(\Omega)$ such that,

$$(\nabla U^s, \nabla v) = (F^s, v) \quad \text{for all } v \in V.$$

The corresponding adjoint problem

Strong form: Let $\psi \in L^2(\Omega)$,

$$-\triangle\phi = \psi \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \Gamma.$$

Weak form: Find $\phi \in V$ such that,

$$(\nabla w, \nabla\phi) = (w, \psi) \quad \text{for all } w \in V.$$

Note that the adjoint problem is *deterministic*.

Simple observation

Using Green's identity we get,

$$(U^s, \psi) = (\nabla U^s, \nabla \phi) = (F^s, \phi) \quad \text{for } s \in \Lambda.$$

- We can derive the distribution of (U, ψ) by just solving *one* PDE.
- This works as long as the differential operator is linear and deterministic i.e. for a wide range of problems.
- It also works for stochastic initial and boundary condition.

The second model problem

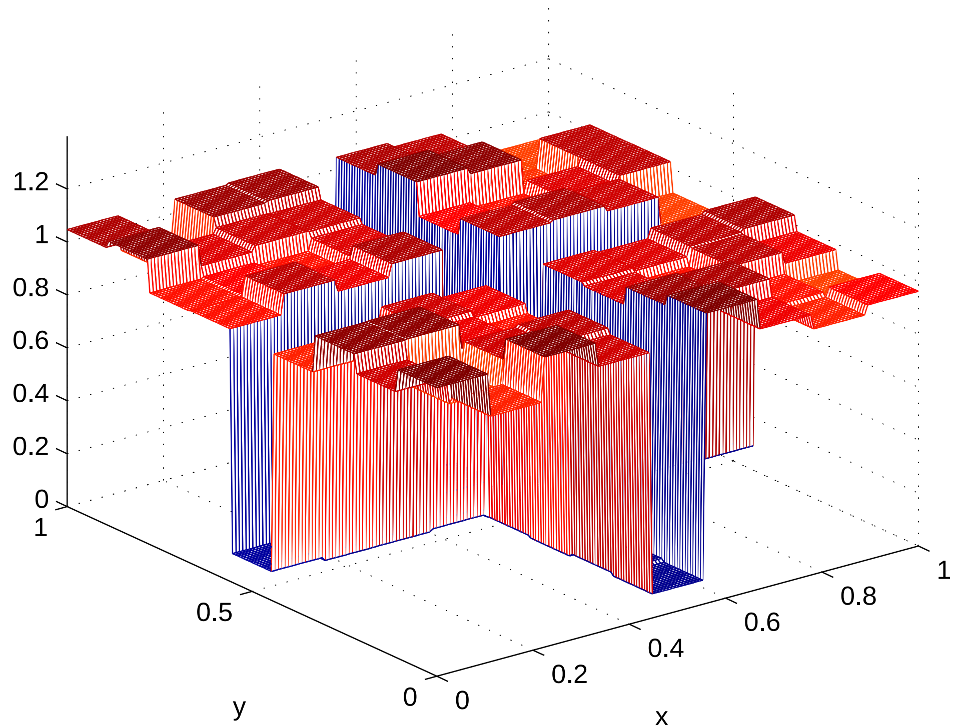
Strong form: The Poisson Equation with randomly perturbed diffusion coefficient, $\mathcal{A}^s = a + A^s > 0$, a deterministic, A^s *piecewise constant* random, $s \in \Lambda$, A^s are iid, $f \in L^2(\Omega)$ is deterministic,

$$\begin{aligned} -\nabla \cdot \mathcal{A}^s \nabla U^s &= f \quad \text{in } \Omega, \\ U^s &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Weak form: Find $U^s \in V = H_0^1(\Omega)$ such that,

$$(\mathcal{A}^s \nabla U^s, \nabla v) = (f, v) \quad \text{for all } v \in V.$$

Diffusion coefficient



The main idea

Remember A^s is piecewise constant. We will use *non-overlapping* DD with domains, d, where A^s is constant. On each domain we have,

$$
\begin{aligned}
U^{s,d} &= (K^a + A^{s,d}K)^{-1}b^s, \\
&= (I + A^{s,d}(K^a)^{-1}K)^{-1}(K^a)^{-1}b^s, \\
&\approx \sum_{t=0}^{T-1}(-A^{s,d}(K^a)^{-1}K)^t(K^a)^{-1}b^s,
\end{aligned}
$$

b^s is rhs plus boundary conditions.

The method

We have three parameters, I (iterations), D (domains), and T (terms in the approx. inv.).

for i from 1 to I **do**
  for d from 1 to D **do**
    Compute K and $(K^a)^{-1}$
    for s from 1 to S **do**
      Compute $b^s(f, \mathcal{A}^s, U^s_{(i-1)})$
    end for
    $U^{s,d}_{(i)} \approx \sum_{t=0}^{T-1}(-A^{s,d}(K^a)^{-1}K)^t(K^a)^{-1}b^s$
  end for
end for

Numerical and stochastic errors

1. Space discretization error h, (both examples).
2. Not converging in the domain decomposition algorithm I, (second example).
3. Finite number of terms to approximate the matrix inverse T (second example).
4. Finite number of realizations used in order to compute the desired stochastic quantity S (both examples).
5. Stochastic modeling error D (second example).

Error control

There are different possibilities.

- Stochastic moments of linear functionals of the solution, $|m(\{(U^s,\psi)\}_{s\in\Lambda}) - M_S(\{(U_h^s,\psi)\}_{s=1,\ldots,S})|$.
- The distribution function of linear functionals of the solution,
$$|F(x) - \tilde{F}_S(x)|,$$
where $F(x) = P(\{(U^s,\psi)\}_{s\in\Lambda} < x)$ and $\tilde{F}_S(x) = P(\{(U^s_{h,I,T},\psi)\}_{s=1}^S < x)$.

Example for second problem

Let $U^s_{h,I,T}$ be the approximate solution, $U^s_{0,\infty,\infty} = U^s$, and $e^s = U^s_{h,I,T} - U^s$. We have the following error representation formula,

$$
\begin{aligned}
(e^s, \psi) &= (e^s, -\nabla \cdot \mathcal{A}^s \nabla \Phi^s) \\
&= (f, \Phi^s) - (\mathcal{A}^s \nabla U^s_{h,\infty,\infty}, \nabla \Phi^s) \\
&\qquad + (\mathcal{A}^s \nabla (U^s_{h,\infty,T} - U^s_{h,I,T}), \nabla \Phi^s) \\
&\qquad + (\mathcal{A}^s \nabla (U^s_{h,\infty,\infty} - U^s_{h,\infty,T}), \nabla \Phi^s) \\
&= (e^s_I, \psi) + (e^s_{II}, \psi) + (e^s_{III}, \psi).
\end{aligned}
$$

Error estimates

Let $\max_s(|(e_I^s,\psi)| + |(e_{II}^s,\psi)| + |(e_{III}^s,\psi)|) \le e_M^\psi$, where $e_M^\psi = e_I^\psi + e_{II}^\psi + e_{III}^\psi$.

With probability $\int_{-\infty}^{\tau} e^{-t^2/2}\,dt/\sqrt{2\pi}$,

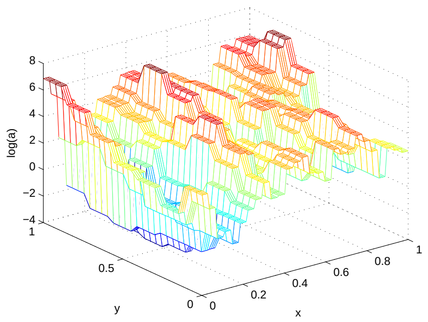
$$|F(x) - \tilde{F}_S(x)| \le \frac{\tau C}{\sqrt{S}} + e_M^\psi \cdot \tilde{F}'(x).$$

This is used to construct an adaptive algorithm.

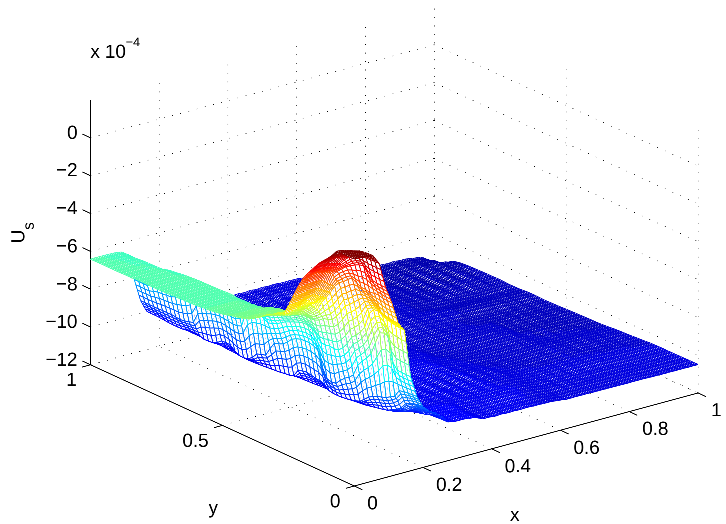
Example: oil reservoir simulation

$$
\begin{aligned}
-\nabla \cdot \mathcal{A}^s \nabla U^s &= f \quad \text{in } \Omega, \\
\mathcal{A}^s \partial_n U^s &= 0 \quad \text{on } \Gamma_N, \\
U^s &= 0 \quad \text{on } \Gamma_D,
\end{aligned}
$$

where $\Gamma_N \cup \Gamma_D = \Gamma$, U^s is pressure, and $\mathcal{A}^s$ is permeability. We let $\psi = 1$.

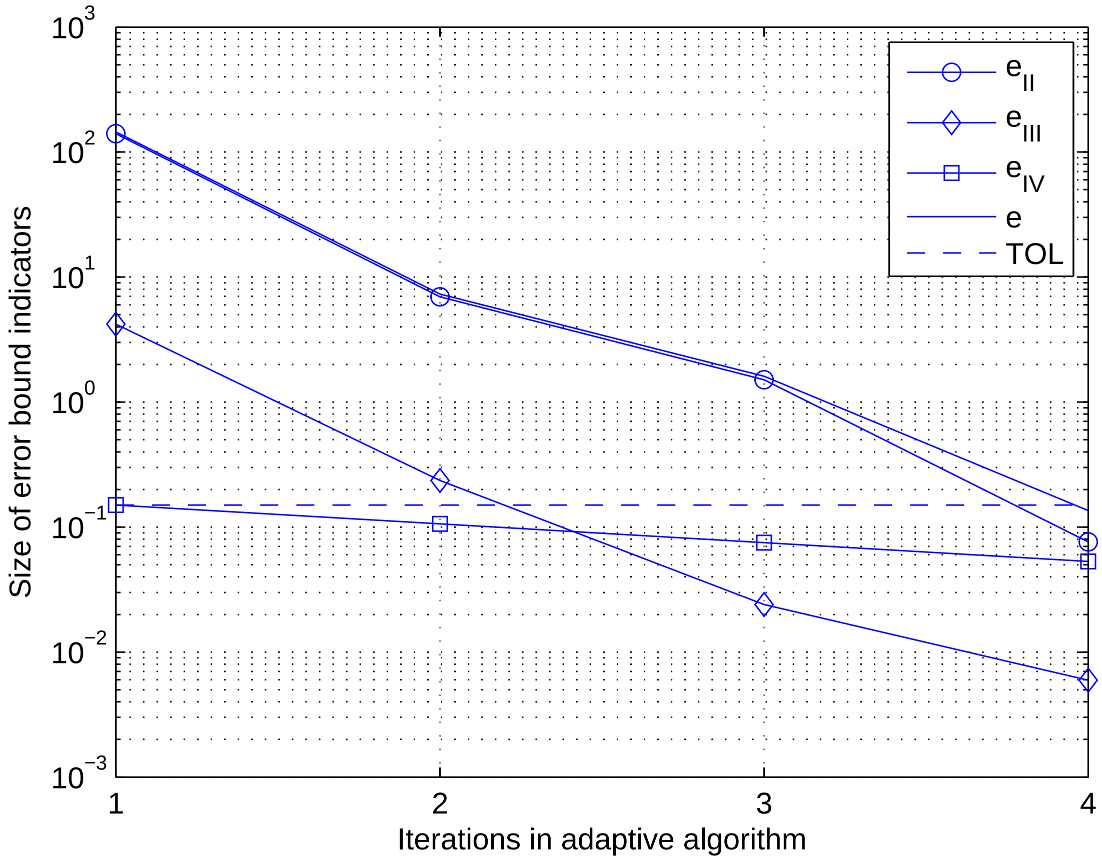


Solution to one realization



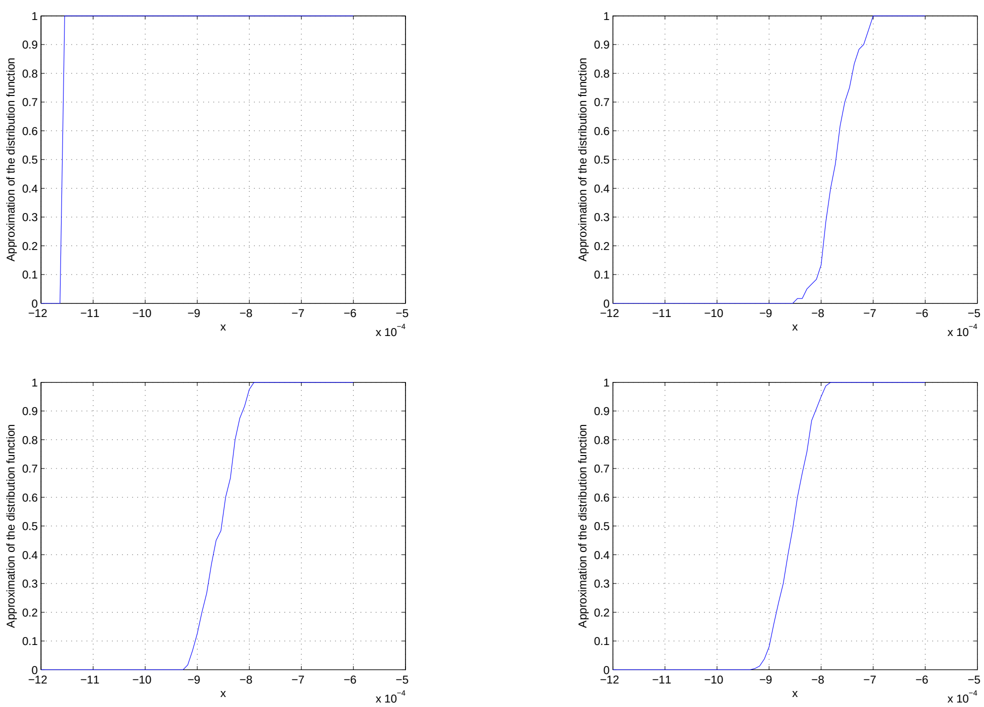
We use the adaptive algorithm, h is constant, starting with $I = 100$, $T = 1$, and $S = 30$ and ending up with $I = 800$, $T = 4$, and $S = 240$.

Error indicators



Converging distribution function

We plot the approximation to $F(x)$ after each iteration.



Summary and future work

`http://www.math.colostate.edu/~estep`.

- First paper: Random data in load, boundary condition, and initial condition.
- Second paper: Random coefficient.

In preparation and future project.

- Third paper: Adaptive modeling of random coefficient (the fifth error type).
- *Fourth paper: Application to other engineering problems.*