Multiscale methods for elliptic and hyperbolic problems

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Outline and Papers

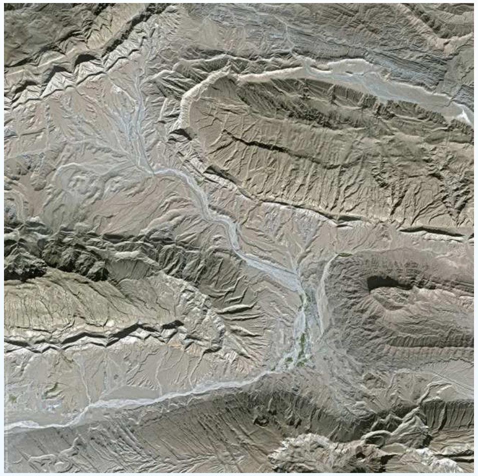
Outline

- Motivating example
- Derivation of multiscale method for an elliptic problem
- Convergence analysis
- Numerical examples
- Derivation of multiscale method for a hyperbolic problem
- Numerical examples
- Conclusions and future work

Papers

- A. Målqvist, *A priori error analysis of a multiscale method*
- M. G. Larson, A. Målqvist, and R. Söderlund, *A discontinuous Galerkin multiscale method for first order hyperbolic equations*

Motivating example: Secondary oil recovery

Find pressure p and water concentration s such that:

$$-\nabla \cdot k\lambda(s)\nabla p = q, \quad \dot{s} + \nabla \cdot [f(s)v] = g, \quad v = -k\lambda(s)\nabla p,$$

where k is permeability, $\lambda(s)$ the total mobility, v total velocity, f fractional flow, and g, q sink and source terms.

Motivation for a multiscale approach

These three ideas can be found in most multiscale methods:

- Split the problem into two or more scales.
- Localize the fine scale computations in space.
- Use the fine scale information to modify (improve) the coarse scale solution.

Why do we need it in this application?

- The permeability has multiscale features.
- The size of the system is huge, parallelism is needed.
- The diffusion coefficient changes in time, but only at the water front. We can reuse most of the modifications made on the fine scale.

Multiscale methods

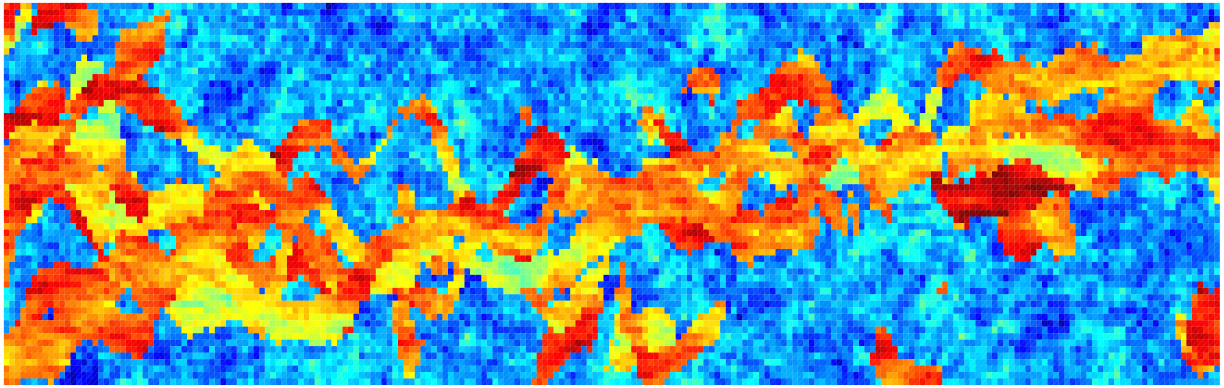
- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98
- Variational multiscale method: Hughes et al. 95, Larson-Målqvist 05, Nordbotten 09
- Multiscale finite element method: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06
- Multiscale finite volume method: Jenny et al. 03
- Heterogeneous multiscale method: Engquist-E 03, E-Ming-Zhang 04
- Equation free: Kevrekidis et al. 05
- ...

1. Convergence is typically only studied in the periodic case.
2. More work has been done for elliptic than hyperbolic problems.

A priori error analysis of a multiscale method

A. Målqvist

Model problem



We consider the strong form:

$$-\nabla \cdot \alpha \nabla u = f, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

The weak form reads: find $u \in \mathcal{V} := H_0^1(\Omega)$ such that,

$$\langle u, v \rangle := \int_\Omega \alpha \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx := l(v), \quad \text{for all } v \in \mathcal{V}.$$

We assume $f \in L^2(\Omega)$ and $0 < \alpha_0 \leq \alpha \in L^\infty(\Omega)$.

Derivation of multiscale methods

We let $\mathcal{T}_0$ be a (coarse) mesh of Ω and $\mathcal{T}_J$ be the mesh after J refinements. Let $\mathcal{V}_0 \subset \mathcal{V}_J \subset \mathcal{V}$ be corresponding spaces.

Reference solution $u_J \in \mathcal{V}_J$ fulfills $\langle u_J, w \rangle = l(w)$ for all $w \in \mathcal{V}_J$.

Let $\pi_0 : \mathcal{V} \cap C(\Omega) \to \mathcal{V}_0$ and $\mathcal{W}_J = \{w \in \mathcal{V}_J : \pi_0 w = 0\}$. Introduce the a-orthogonal map $I + T_J$ with $T_J : \mathcal{V}_0 \to \mathcal{W}_J$ fulfilling,

$$\langle v_0 + T_J v_0, w \rangle = 0, \quad \text{for all } v_0 \in \mathcal{V}_0, w \in \mathcal{W}_J.$$

We let $u_0 = \pi_0 u_J$ and write $u_J = u_0 + T_J u_0 + u_{l,J}$. Then $u_{l,J} = (1 - \pi_0 - T_J \pi_0) u_J \in \mathcal{W}_J$ solves,

$$\langle u_{l,J}, w \rangle = l(w), \quad \text{for all } w \in \mathcal{W}_J,$$

Find $u_0 \in \mathcal{V}_0$ s.t. $\langle u_0 + T_J u_0, v_0 \rangle = l(v_0) - \langle u_{l,J}, v_0 \rangle, \quad$ for all $v_0 \in \mathcal{V}_0$.

Three multiscale methods

VMS: $$\langle u_0 + T_J^{\text{vms}} u_0, v_0 \rangle = l(v_0) - \langle u_{l,J}^{\text{vms}}, v_0 \rangle,$$
$$\langle v_0 + T_J^{\text{vms}} v_0, v \rangle \approx 0,$$
$$\langle u_{l,J}^{\text{vms}}, v \rangle \approx l(v),$$

MsFEM: $$\langle u_0 + T_J^{\text{mfem}} u_0, v_0 + T_J^{\text{mfem}} v_0 \rangle = l(v_0 + T_J^{\text{mfem}} v_0),$$
$$\langle v_0 + T_J^{\text{mfem}} v_0, v \rangle \approx 0,$$

Sym-AVMS: $$\langle u_0 + T_J^k u_0, v_0 + T_J^k v_0 \rangle = l(v_0 + T_J^k v_0) - \langle u_{l,J}^k, v_0 + T_J^k v_0 \rangle,$$
$$\langle v_0 + T_J^k v_0, v \rangle \approx 0,$$
$$\langle u_{l,J}^k, v \rangle \approx l(v),$$

for all $v_0 \in \mathcal{V}_0$ and $v \in \mathcal{W}_J$. Note that $\langle v_0 + T_J v_0, w_J \rangle = 0$.

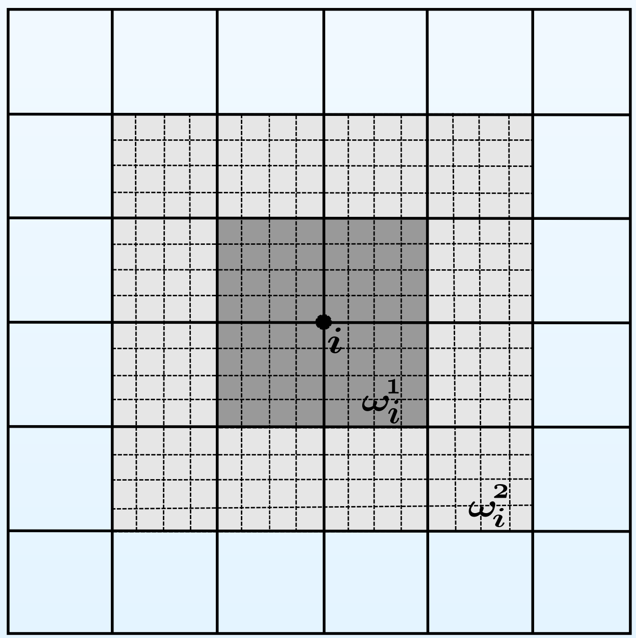
Approximation of T_J and $u_{l,J}$ in Sym-AVMS

We localize the fine scale equations. Let $\mathcal{V}_0 = \text{span}(\{\phi_i\})$ and,

$$\langle \phi_i + T_J \phi_i, v \rangle = 0, \quad \text{for all } w \in \mathcal{W}_J,$$

$$\langle u_{l,J,i}, v \rangle = l(\phi_i v), \quad \text{for all } w \in \mathcal{W}_J,$$

We introduce a patch ω_i^k around supp(ϕ_i):



Now let $\mathcal{W}_J(\omega_i^k) = \{v \in \mathcal{W}_J : \text{supp}(v) \subset \omega_i^k\}$.

Sym-AVMS

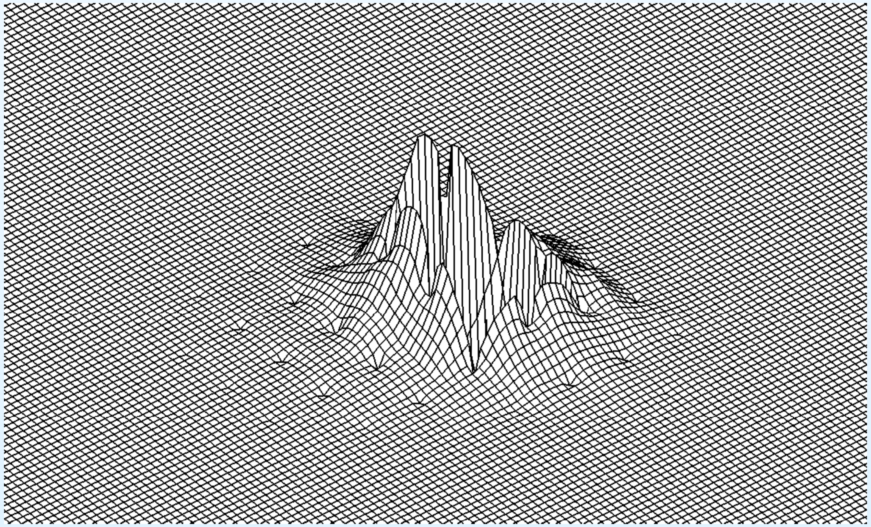
Let $T_J^k \phi_i \in \mathcal{W}_J(\omega_i^k)$ and $u_{l,J}^k \in \mathcal{W}_J(\omega_i^k)$ be given by,

$$\langle \phi_i + T_J^k \phi_i, v\rangle = 0, \quad \text{for all } w \in \mathcal{W}_J(\omega_i^k),$$

$$\langle u_{l,J,i}^k, v\rangle = l(\phi_i v), \quad \text{for all } w \in \mathcal{W}_J(\omega_i^k).$$

The method reads: Find $u_0^k \in \mathcal{V}_0$ such that

$$\langle u_0^k + T_J^k u_0^k, v_0 + T_J^k v_0\rangle = l(v_0 + T_J^k v_0) - \langle u_{l,J}^k, v_0 + T_J^k v_0\rangle, \ \forall v_0 \in \mathcal{V}_0.$$

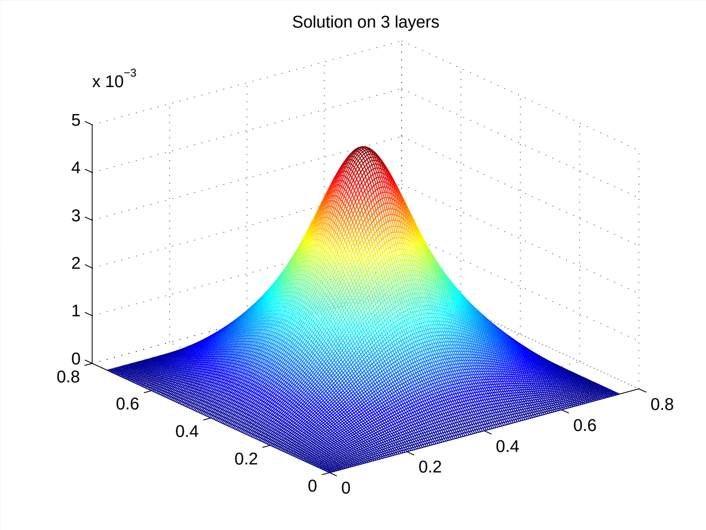


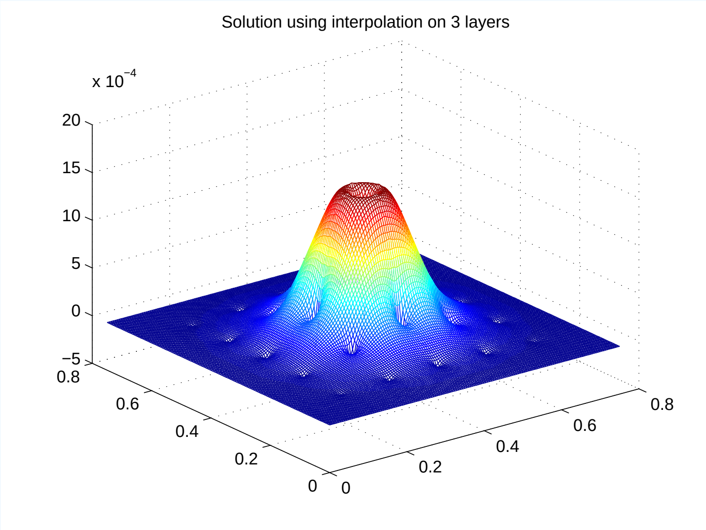
Observation about decay in $\mathcal{W}$ (Fourier)

Consider the Poisson equation,

$$-\Delta u = \phi_i \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where ϕ_i has local support in Ω. The weak form reads: find $u \in \mathcal{Z}$ such that, $\langle u, v\rangle = (\phi_i, v)$ for all $v \in \mathcal{Z}$.



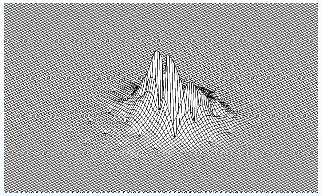


To the left $\mathcal{Z} = \mathcal{V}$ (log decay) and right $\mathcal{Z} = \mathcal{W}$ (exp decay).

Constraints are realized using Lagrangian multipliers.

Convergence analysis: basis functions $T_J\phi_i$

We sketch the convergence proof below. We start with the decay of $T_J\phi_i$.



Let $\{\chi_j\}_{j\in\mathcal{M}_J}$ be a hierarchical basis for $\mathcal{W}_J$. Let $\hat{A} = \langle\chi_l, \chi_j\rangle$, $l, j \in \mathcal{M}_J$. Further let $T_J\phi_i = \sum_{j\in\mathcal{M}_J} \alpha_j\chi_j$. We use CG with $\hat{\alpha}_0 = 0$ and right hand side $b_j = -\langle\phi_i, \chi_j\rangle$. We have,

$$|\alpha - \hat{\alpha}^m|_{\hat{A}} \leq 2\left(\frac{\sqrt{\kappa(\hat{A})} - 1}{\sqrt{\kappa(\hat{A})} + 1}\right)^m |\alpha|_{\hat{A}} := 2\rho^m|\alpha|_{\hat{A}},$$ where $|v|_A^2 = v^T A v$.

Note that $\sqrt{\kappa(\hat{A})} \sim J$ in 2D and $\sqrt{\kappa(\hat{A})} \sim 2^J$ in 3D.

Convergence analysis: local solutions $T_J\phi_i$

We have $T_J\phi_i = \sum_{j\in\mathcal{M}_J} \alpha_j\chi_j$, with corresponding vector α, where $\mathcal{M}_J$ is the set non-coarse nodes on level J.

Since b_j has support on a coarse 1-ring and the HB only spreads information within ω_i^k in $2k$ iterations we have,

$$|\alpha_{\Omega\backslash\omega^k}|^2 = \sum_{j\in\mathcal{M}_{\mathcal{J}}(\Omega\backslash\omega_i^k)} |\alpha_j|^2 = \sum_{j\in\mathcal{M}_{\mathcal{J}}(\Omega\backslash\omega_i^k)} |\alpha_j - \hat{\alpha}_j^{2k}|^2 \leq |\alpha - \hat{\alpha}^{2k}|^2,$$

where $\alpha_{\Omega\backslash\omega^k}$ only contains the node values outside ω_i^k.

Furthermore $|\alpha_{\Omega\backslash\omega^k}|_{\hat{A}}^2 \leq C|\alpha - \hat{\alpha}^{2k}|_{\hat{A}}^2 \leq C\rho^{4k}|\alpha|_{\hat{A}}^2$ which means that the coefficients in α decays away from node i and more precisely $\|T_J\phi_i\|_{\Omega\backslash\omega_i^k} \leq C\rho^{2k}\|T_J\phi_i\|$, with $\|v\|_\omega^2 = \langle v, v\rangle_\omega$.

Convergence analysis: local solutions $T_J^k\phi_i \to T_J\phi_i$

Now let $T_J^k\phi_i = \sum_{j\in\mathcal{M}_J(\omega_i^k)} \alpha_j^k\chi_j$.

We have $\langle T_J\phi_i - T_J^k\phi_i, w\rangle = 0$ for all $w \in \mathcal{W}_J(\omega_i^k)$.

Now let $w = \sum_{j\in\mathcal{M}_J(\omega_i^k)}(\alpha_j - \alpha_j^k)\chi_k \in \mathcal{W}_J(\omega_i^k)$, with corresponding vectors α_{ω^k} and α^k. We get,

$$\begin{aligned}|\alpha - \alpha^k|_{\hat{A}}^2 &= (\alpha - {\color{red}\alpha_{\omega_k}})^T\hat{A}(\alpha - \alpha^k)\\ &= \alpha_{\Omega\setminus\omega^k}^T\hat{A}(\alpha - \alpha^k)\\ &\leq |\alpha_{\Omega\setminus\omega^k}|_{\hat{A}}|\alpha - \alpha^k|_{\hat{A}},\end{aligned}$$

But now $|\alpha - \alpha^k|_{\hat{A}} \leq C\rho^{2k}|\alpha|_{\hat{A}}$ or,

$\|T_J\phi_i - T_J^k\phi_i\| \leq C\rho^{2k}\|T_J\phi_i\|$ and $\|u_{l,J,i} - u_{l,J,i}^k\| \leq C\rho^{2k}\|u_{l,J,i}\|$.

Convergence analysis: system

Let $\mathcal{V}_{0,J} = \text{span}(\{\phi_i + T_J\phi_i\})$, $\mathcal{V}_{0,J}^k = \text{span}(\{\phi_i + T_J^k\phi_i\})$. We introduce projections $P_J : \mathcal{V} \to \mathcal{V}_{0,J}$ and $P_J^k : \mathcal{V} \to \mathcal{V}_{0,J}^k$ such that,

$$\langle P_J(u_J - u_{l,J}), v\rangle = \langle u_J - u_{l,J}, v\rangle, \quad \text{for all } v \in \mathcal{V}_{0,J},$$
$$\langle P_J^k(u_J - u_{l,J}^k), v\rangle = \langle u_J - u_{l,J}^k, v\rangle, \quad \text{for all } v \in \mathcal{V}_{0,J}^k.$$

Since $\langle u_0 + T_J u_0, v_0 + T_J v_0\rangle = (f, v_0 + T_J v_0) - \langle u_{l,J}, v_0 + T_J v_0\rangle$.

We have $u_J^k = P_J^k(u_J - u_{l,J}^k) + u_{l,J}^k$ and $u_J = P_J(u_J - u_{l,J}) + u_{l,J}$.

Algebraic manipulation gives

$$u_J - P_J^k(u_J - u_{l,J}^k) - u_{l,J}^k = (1 - P_J^k)P_J u_J + P_J^k(P_J u_J + u_{l,J} - u_J)$$
$$+ (1 - P_J^k)(u_{l,J} - u_{l,J}^k) = (1 - P_J^k)P_J u_J + \sum_i (1 - P_J^k)(u_{l,J,i} - u_{l,J,i}^k).$$

Convergence analysis: main result

Lemma 1 *It holds* $|||(1 - P_J^k)P_J u_J||| \leq C\|u_J\|_{L^\infty(\Omega)}\rho^{2k}/h_0$.

Let $w = \sum v_i(T_J\phi_i - T_J^k\phi_i)$, with $v_i = \pi_0 u_J(x_i)$. Since P_J^k is a projection $|||P_J u_J - P_J^k P_J u_J||| \leq |||w||| \leq C\|u_J\|_{L^\infty(\Omega)}\rho^{2k}/h_0$.

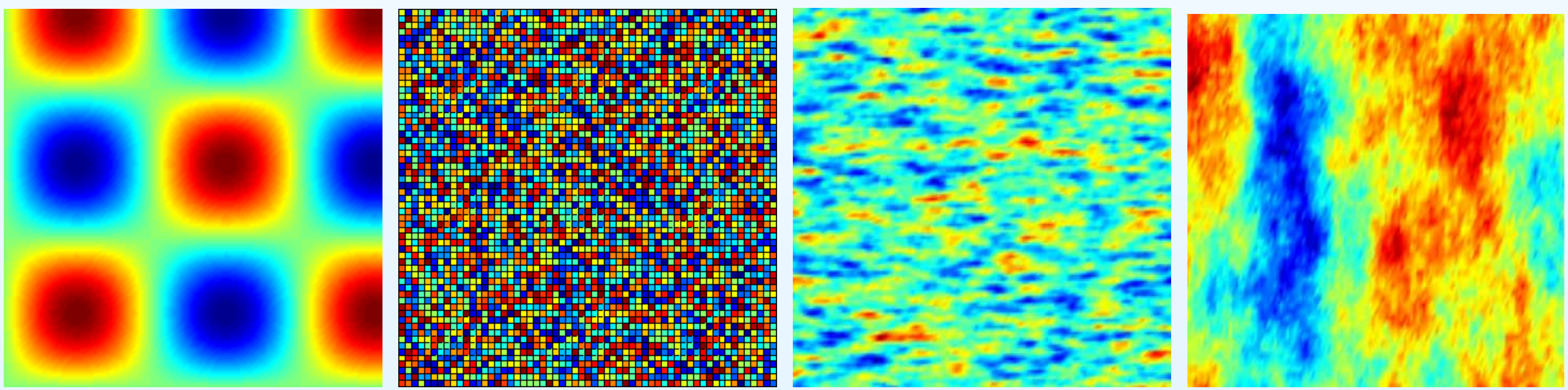
Theorem 2 *Let* u_J *be the reference solution and* u_J^k *the Sym-AVMS approximation. Then,*

$$|||u_J - u_J^k||| \leq C\left(\|u_J\|_{L^\infty(\Omega)}/h_0 + \|f\|_{L^2(\Omega)}\right)\rho^{2k},$$

where $\rho = \frac{\sqrt{\kappa(\hat{A})}-1}{\sqrt{\kappa(\hat{A})}+1}$ *and* $\sqrt{\kappa(\hat{A})} \sim J$ *in 2D and* $\sqrt{\kappa(\hat{A})} \sim 2^J$ *in 3D.*

Numerical examples

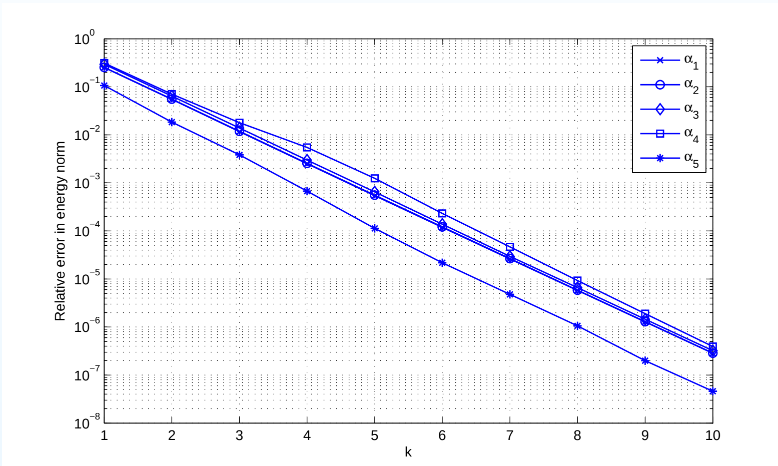
$$\begin{cases} \alpha_1(x,y) = 1, \\ \alpha_2(x,y) = 1 + 0.5 \cdot sin(8x)sin(8y), \\ \alpha_3(x,y) = 0.1 + 0.9 * \texttt{rand}, \quad (x,y) \in \tau, \text{ for all } \tau \in \mathcal{T}_1, \\ \alpha_4(x,y) = a_{\text{GSLIB}}(i,j), \text{ for } \frac{i-1}{120} \leq x < \frac{i}{120}, \ \frac{j-1}{120} \leq y < \frac{j}{120}, \\ \alpha_5(x,y) = a_{\text{SPE}}(i,j), \text{ for } \frac{i-1}{120} \leq x < \frac{i}{120}, \ \frac{j-1}{120} \leq y < \frac{j}{120}, \end{cases}$$

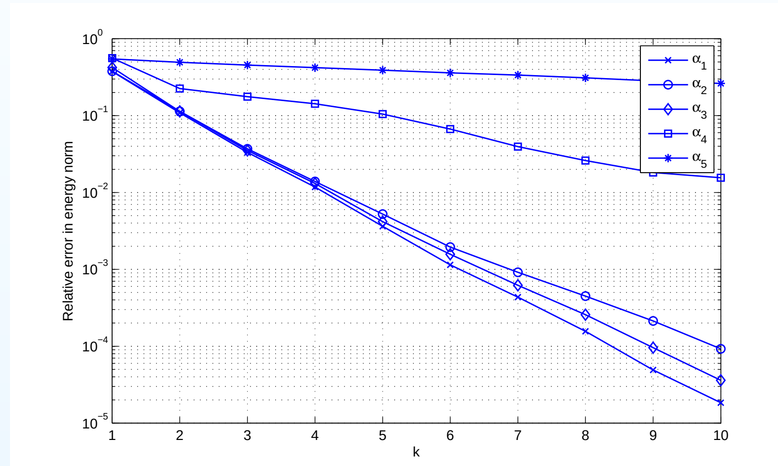


We let $f = \chi_{\text{inj}} - \chi_{\text{prod}}$, with $\text{supp}(\chi_{\text{inj}}) = [0, 1/60] \times [0, 1/60]$, and $\text{supp}(\chi_{\text{prod}}) = [1 - 1/60, 1] \times [1 - 1/60, 1]$.

Convergence of local solution $T_J^k \phi_i$

We let $i = 435$, $J = 3$, and $h_0 = 1/30$, using rectangular mesh.





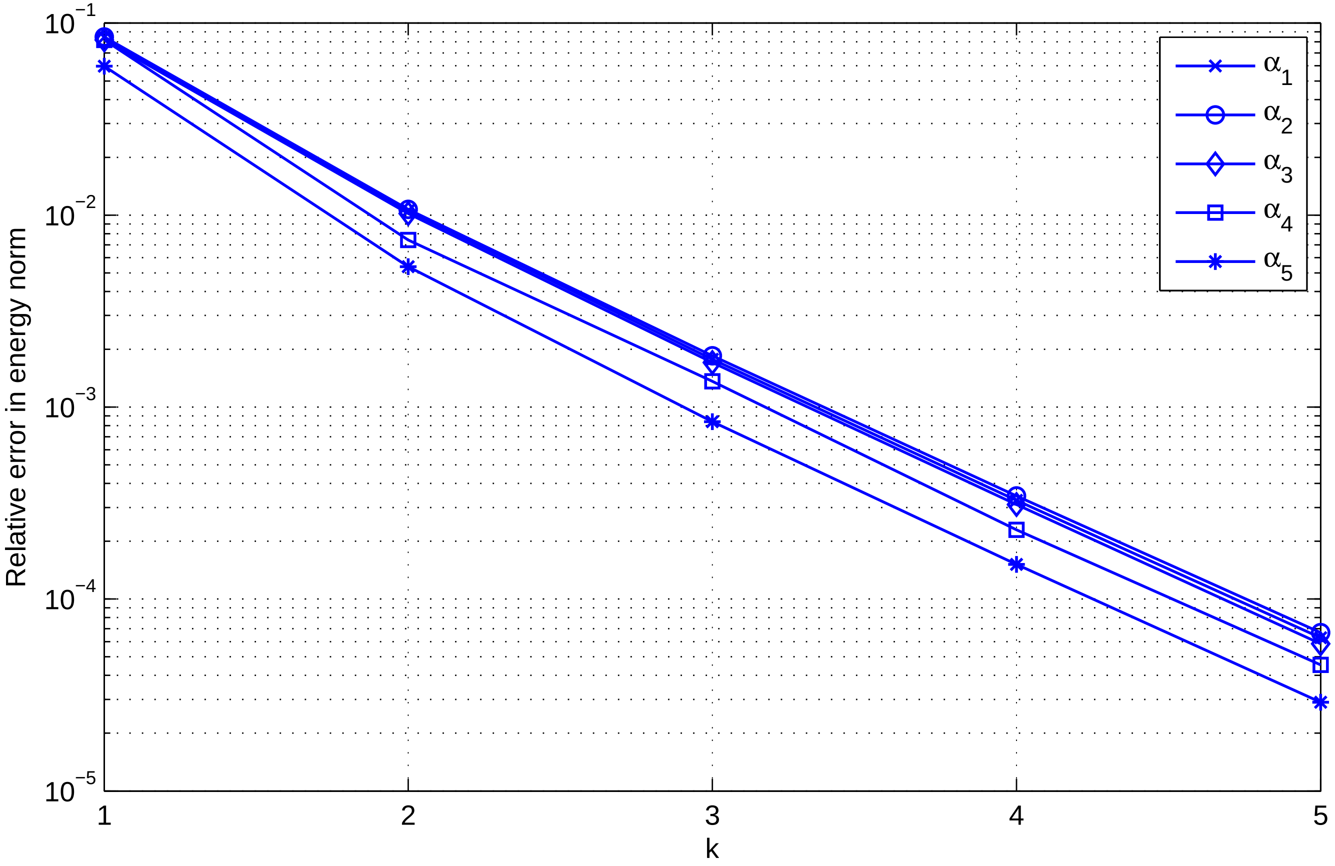
Relative error in energy norm (left). We get exponential convergence in k.

Corresponding error using $2k$ cg iterations (right) $\Rightarrow$ slower convergence for high condition numbers.

Preconditioner that works in the argument?

Convergence of global solution

Again $J = 3$ and $h_0 = 1/30$. We plot the error $u_J - u_J^k$ in energy norm (relative).

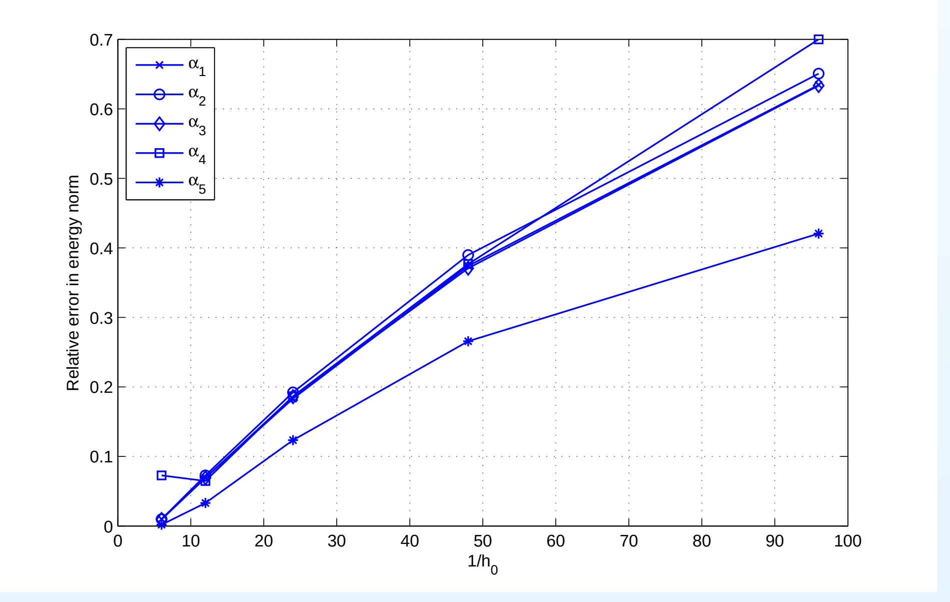


How does the error depend on h_0?

Remember

$$\|u_J - u_J^k\| \leq C \left(\|u_J\|_{L^\infty(\Omega)} / h_0 + \|f\|_{L^2(\Omega)} \right) \rho^{2k},$$

We let $J = 2$ and $k = 3$.



The bound is probably not sharp in terms of h_0.

Summary of this paper

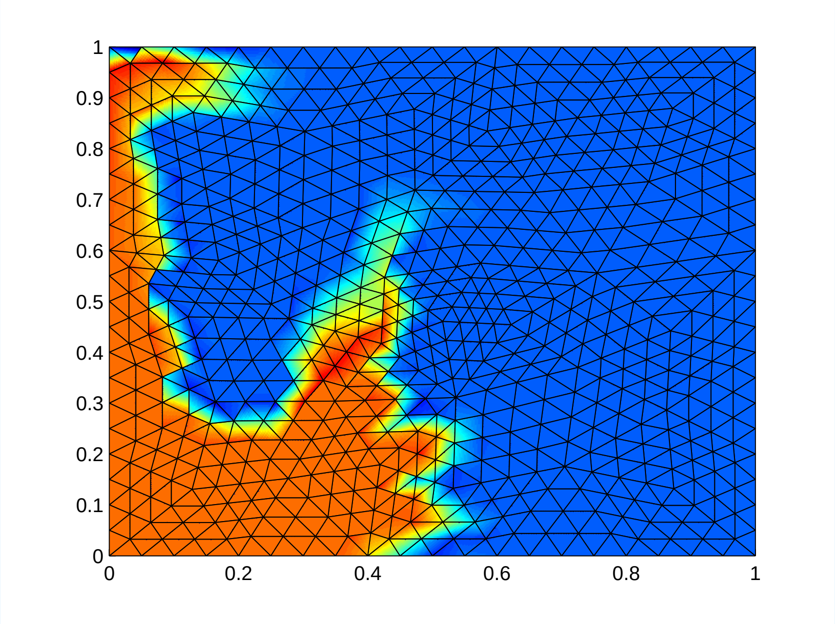
1. We prove an *a priori* error bound and thereby convergence as $k \to \infty$ for Sym-AVMS, for fix h_0 and J.
2. The bound reveals that for fix h_0 and J we get *exponential decay* in the number of layers k.
3. Numerics experiments confirms this and furthermore reveals that a very small value $k \sim 2$ is needed for 2D examples in practise.
4. The numerics indicates that for high ratios $\frac{\max_x \alpha(x)}{\min_x \alpha(x)}$ direct computation of the linear systems on the patches is preferable to using a few iterations of cg.
5. There are still improvements needed in the analysis in the case when $\frac{\max_x \alpha(x)}{\min_x \alpha(x)}$ is large and in the dependency on h_0. Preconditioner and/or wavelet basis might resolve this.

A discontinuous Galerkin multiscale method for first order hyperbolic equations

M. G. Larson, A. Målqvist, and R. Söderlund

Model problem

We let $I = (0, T]$ and Ω be a domain with boundary $\partial\Omega = \Gamma_- \cup \Gamma_+$, where $\Gamma_- = \{x \in \partial\Omega : n \cdot \sigma < 0\}$ is inflow.



Strong form: Given $u(0) = u_0$ find $u(t)$, $t \in I$ such that

$$\dot{u} + \nabla \cdot (\sigma u) = f, \quad \text{in } \Omega \times I,$$

$$u = g \quad \text{on } \Gamma_- \times I, \quad n \cdot \nabla u = 0 \quad \text{on } \Gamma_+ \times I.$$

Discontinuous Galerkin (dG1) formulation

We let $\mathcal{T}_J$ be a mesh for Ω and $\mathcal{V}_J = \{v : v|_K \in P_1(K), \forall K \in \mathcal{T}_J\}$. Further let $\mathcal{E}_I \cup \mathcal{E}_\Gamma$ be the set of edges, $[v] = v^+ - v^-$, $\langle v \rangle = (v^+ + v^-)/2$, and the upwind value

$$\tilde{v} = v^+, \quad n \cdot \sigma > 0,$$

$$\tilde{v} = v^-, \quad n \cdot \sigma < 0,$$

$$\tilde{v} = \langle v \rangle, \quad n \cdot \sigma = 0.$$

Find $u_J(t) \in \mathcal{V}_J$ such that,

$$(\dot{u}_J, v) + q(u_J, v) := \sum_{K \in \mathcal{T}_J} (\dot{u}_J, v)_K - \sum_{K \in \mathcal{T}_J} (u_J, \sigma \cdot \nabla v)_K + \sum_{E \in \mathcal{E} \setminus \Gamma_-} (n \cdot \sigma \tilde{u}_J, [v])_E$$

$$= (f, v) - \sum_{E \in \Gamma_-} (n \cdot \sigma, g v)_E := l(v), \ \forall v \in \mathcal{V}_J, \ t \in I.$$

Brezzi, Marini, and Süli, *DG methods for first-order hyperbolic systems,* 2004.

Variational multiscale framework

Let $0 = t_0 < t_1 < \cdots < t_N = T$. Given u_J^{n-1} we get u_J^n as the solution to,

$$(\dot{u}_J, v) + q(u_J, v) = l(v), \quad \forall v \in \mathcal{V}_J,\ t \in (t_{n-1}, t_n],$$
$$(u_J(t_{n-1}), v) = (u_J^{n-1}, v), \quad \forall v \in \mathcal{V}_J.$$

We let $\mathcal{T}_0$ be a coarse mesh with space $\mathcal{V}_0$ and let $\mathcal{V}_J = \mathcal{V}_0 \oplus \mathcal{W}_J$ using $L^2(\Omega)$ orthogonal split.

Find $u_0 \in \mathcal{V}_0$ and $w_J = u_J - u_0 \in \mathcal{W}_J$ such that,

$$(\dot{u}_0 + \dot{w}_J, v_0 + v_J) + q(u_0 + w_J, v_0 + v_J) = l(v_0 + v_J),$$
$$(u_0(t_{n-1}) + w_J(t_{n-1}), v_0 + v_J) = (u_0^{n-1} + w_J^{n-1}, v_0 + v_J),$$

for all $v_0 \in \mathcal{V}_0$, $v_J \in \mathcal{W}_J$, and $t \in (t_{n-1}, t_n]$.

Coarse and fine scale equations

We consider the non-symmetric version of AVMS and split the coarse and the fine scale using the L^2-orthogonality.

$$\begin{aligned}
(\dot{u}_0, v_0) + q(u_0 + w_J, v_0) &= l(v_0), \\
(u_0(t_{n-1}), v_0) &= (u_0^{n-1}, v_0), \\
(\dot{w}_J, v) + q(w_J, v) &= l(v) - q(u_0, v), \\
(w_J(t_{n-1}), v) &= (u_J^{n-1}, v),
\end{aligned}$$

for all $v_0 \in \mathcal{V}_0$, $v \in \mathcal{W}_J$, and $t \in (t_{n-1}, t_n]$.

Again we split the fine scale contribution into different parts. Let $w_J = u_{l,J} + T_J u_0 + u_{0,J}$, where $u_{l,J}$ is associated with $l(v)$, $T_J u_0$ with $q(u_0, v)$, and $u_{0,J}$ with (u_J^{n-1}, v).

Split of fine scale equations and coarse scale equation

Let $\mathcal{V}_0 = \text{span}(\{\phi_i\})$ and let $u_{l,J,i}, T_J\phi_i, u_{0,J,i} \in \mathcal{W}_J$ solve

$$(\dot{u}_{l,J,i}, v_J) + q(u_{l,J,i}, v_J) = l(\phi_i v_J),$$

$$(u_{l,J,i}(t_{n-1}), v_J) = 0,$$

$$(\dot{T_J\phi_i}, v_J) + q(T_J\phi_i, v_J) = -q(\phi_i, v_J),$$

$$(T_J\phi_i(t_{n-1}), v_J) = 0,$$

$$(\dot{u}_{0,J,i}, v_J) + q(u_{0,J,i}, v_J) = 0,$$

$$(u_{0,J,i}(t_{n-1}), v_J) = (\phi_i u_J^{n-1}, v_J),$$

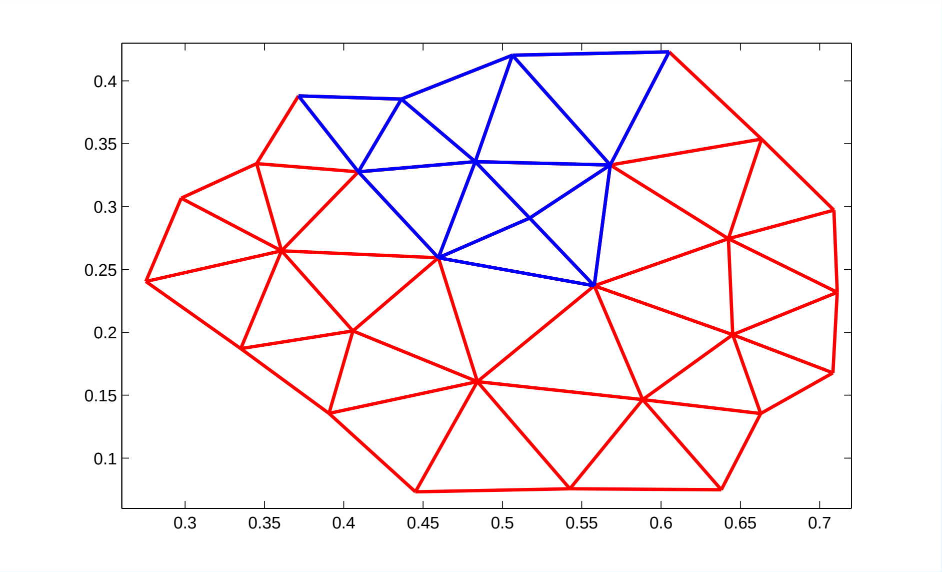
for all $v_0 \in \mathcal{V}_0$, $v_J \in \mathcal{W}_J$, and $t \in (t_{n-1}, t_n]$. Let $u_0 = \sum_i \alpha_i \phi_i$ and $T_J u_0 = \sum_i \alpha_i T_J \phi_i$. We get the following coarse scale system,

$$(\dot{u}_0, v_0) + q(u_0 + T_J u_0, v_0) = l(v_0) - q(u_{l,J} + u_{0,J}, v_0),$$

$$(u_0(t_{n-1}), v_0) = (u_0^{n-1}, v_0).$$

Solving fine scale problems on patches

We now consider two types of patches.



We solve localized fine scale problems in $\mathcal{W}_J(\omega_i^k)$, let u_0^k solve,

$$(\dot{u}_0^k, v_0) + q(u_0^k + T_J^k u_0^k, v_0) = l(v_0) - q(u_{l,J}^k + u_{0,J}^k, v_0),$$

$$(u_0^k(t_{n-1}), v_0) = (u_0^{k,n-1}, v_0).$$

We only consider uniform fine time step for all equations.

Sketch of algorithm

for $i = 1$ to N **do**

Compute $T_J^k \phi_i$ and $u_{l,J,i}^k$ on patches, given data σ, f, Ω.

Compute to moments $q(T_J^k \phi_i, \phi_j)$ and $q(u_{l,J,i}^k, \phi_j)$.

end for

while do $t_n < T$

for $i = 1$ to N **do**

Compute $u_{0,J,i}^{k,n}$ given $u_{0,J,i}^{k,n-1}$, $T_J^k \phi_i$, $u_{l,J,i}^k$, and $u_0^{k,n-1}$.

Compute moments $q(u_{0,J,i}^{k,n}, \phi_j)$.

end for

Compute $u_0^{k,n}$ given $q(T_J^k \phi_i, \phi_j)$, $q(u_{l,J}^k + u_{0,J}^{k,n}, \phi_j)$, and $u_0^{k,n-1}$.

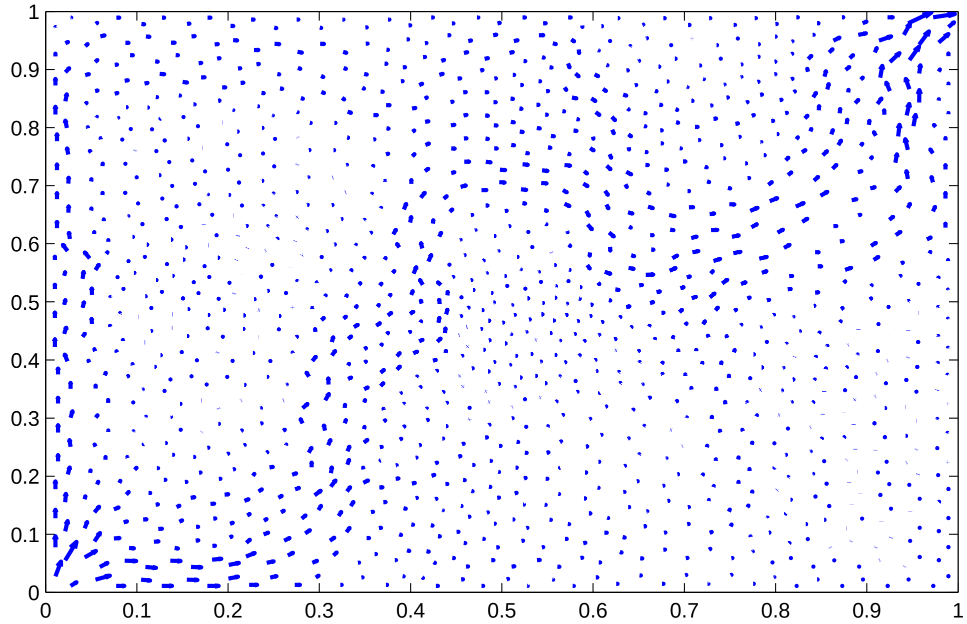
end while

Numerical examples

We let $\Omega = [0,1] \times [0,1]$ consider two different problems,

(Case 1) $\quad \Gamma_- = [0,1] \times \{0\}, \quad \sigma = [0,1], \quad g = 0, \quad f = 1,$

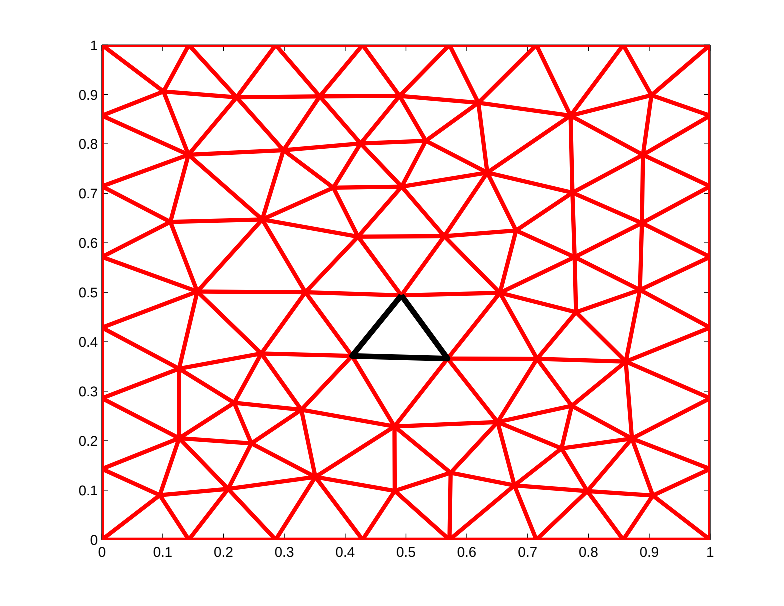
(Case 2) $\quad \Gamma_- = \emptyset, \quad \sigma = \sigma_{\text{SPE}}, \quad f = \chi_{\text{inj}} - \chi_{\text{prod}}.$

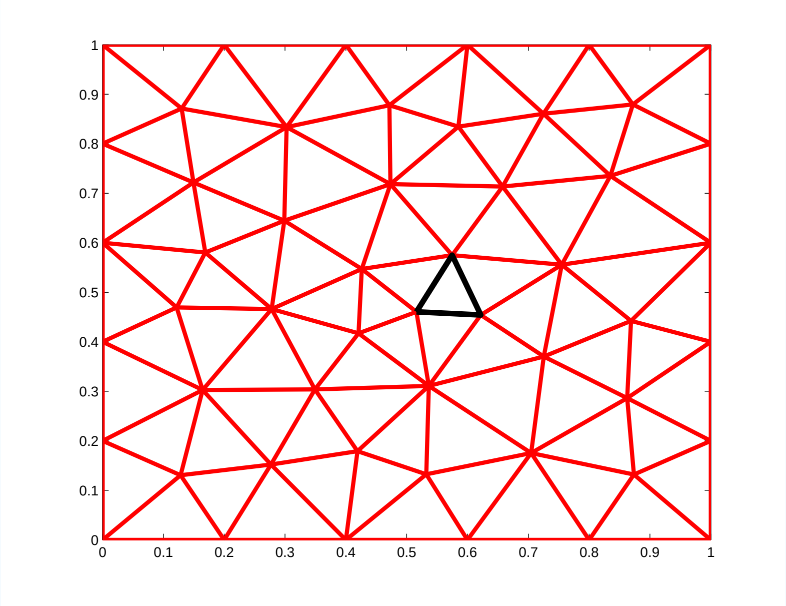


We use backward Euler for the time discretization.

Convergence of local solutions

We consider $T_J^k \phi_i(t_n)$, where ϕ_i has support in the black triangles, $u_{l,J,i}^k$ and $u_{0,J,i}^k$ behave in a very similar way.

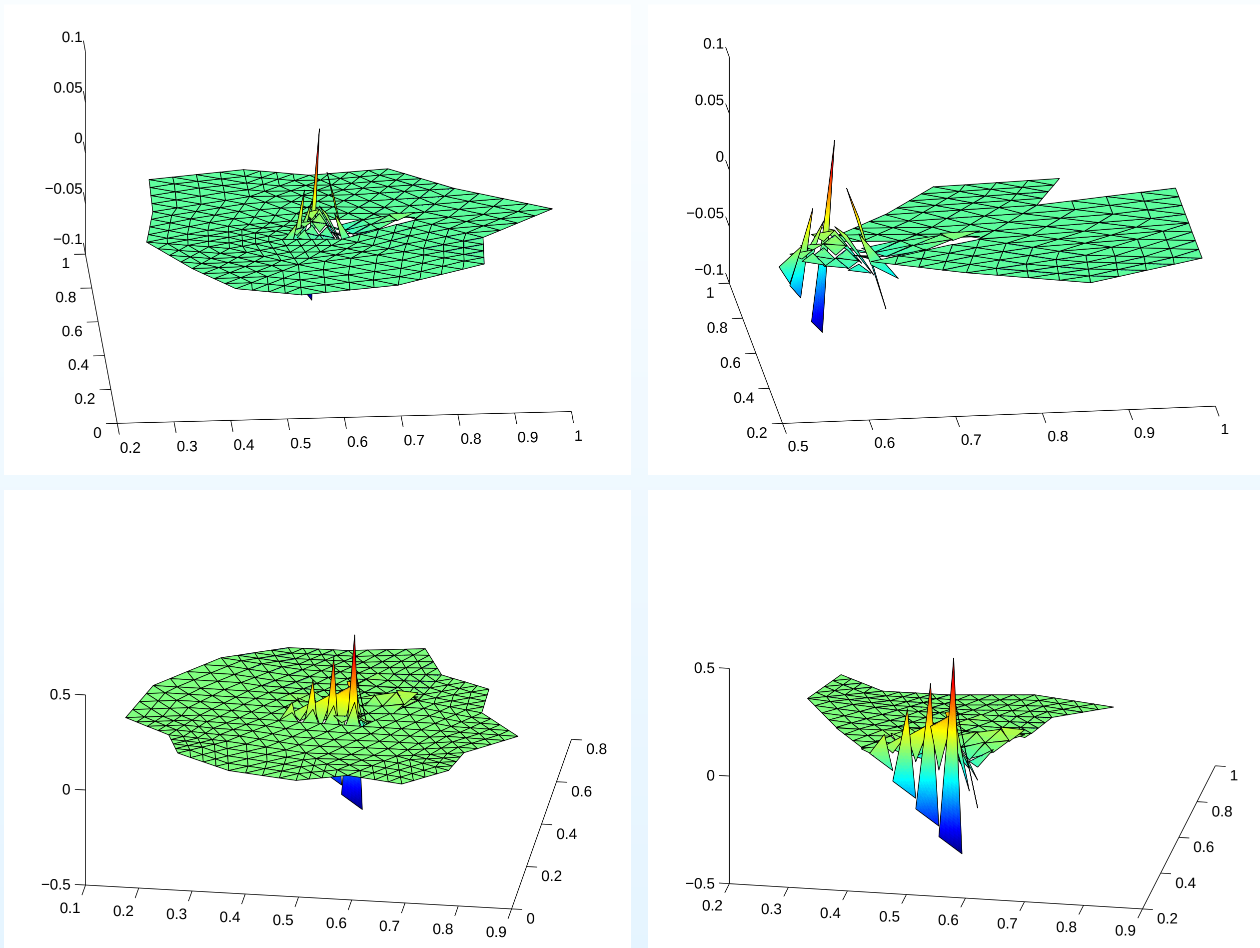




- Case 1 (left) dt $= 0.005$, $i = 216$.
- Case 2 (right) dt $= 1$, $i = 210$.

We now plot $T_J^k \phi_i$ using both symmetric and directed patches.

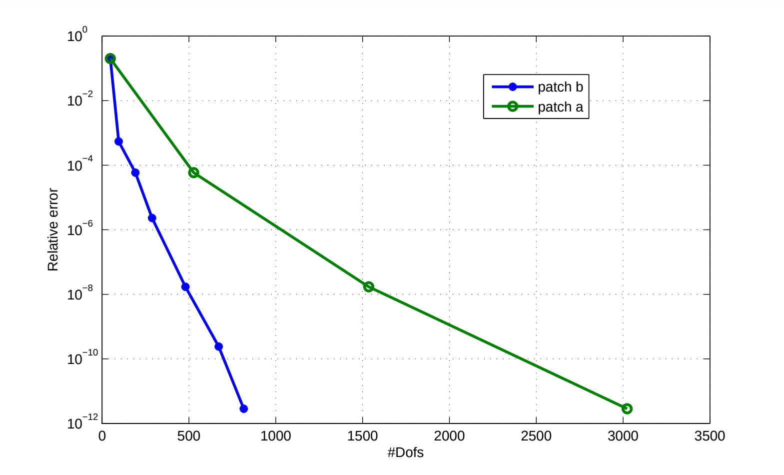
Local solutions

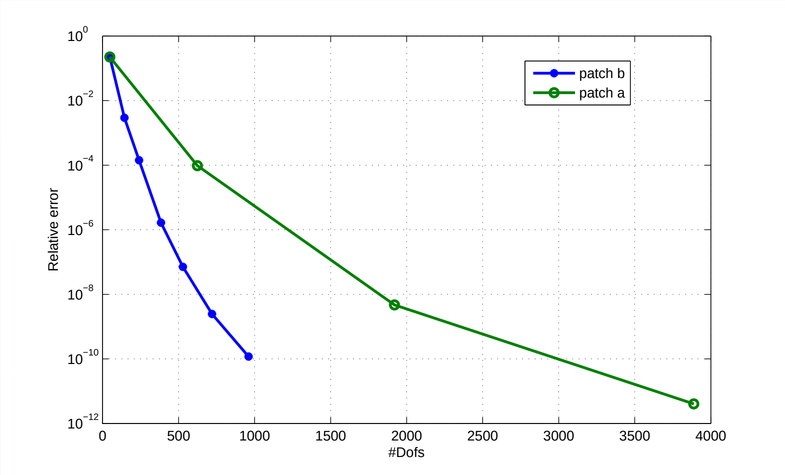


Case 1 above and case 2 below.

Convergence local solutions

Relative error compared to reference solution in H^1-norm for symmetric (green) and directed (blue) patches. Case 1 (left) and case 2 (right). Note x-axis is dofs in local problems.



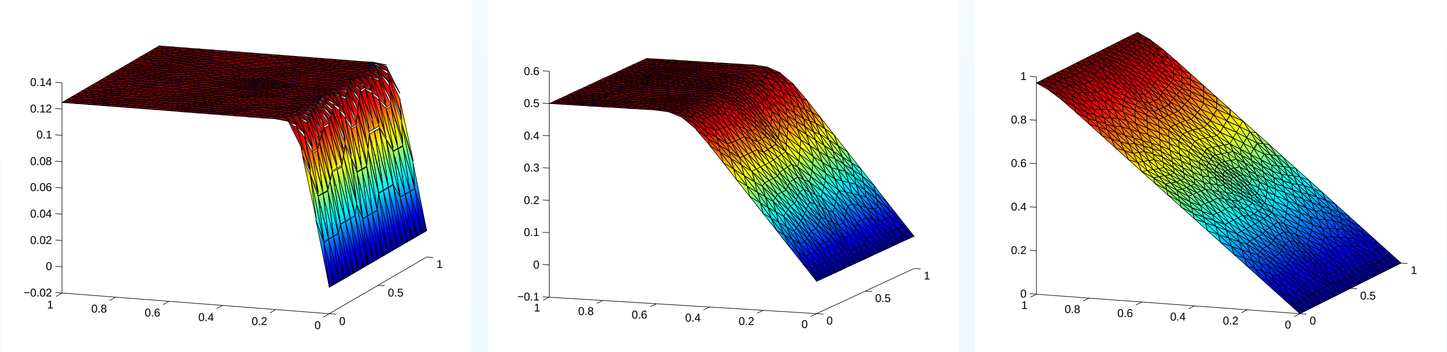


- The time step (dt) will affect how much $T_J^k \phi_i$ spreads out over the domain.
- We would like to have a coarse time step (dt) which leads to a reasonable size of the patches and then a fine time step used to compute $T_J^k \phi_i(t_{n-1} + \text{dt})$.

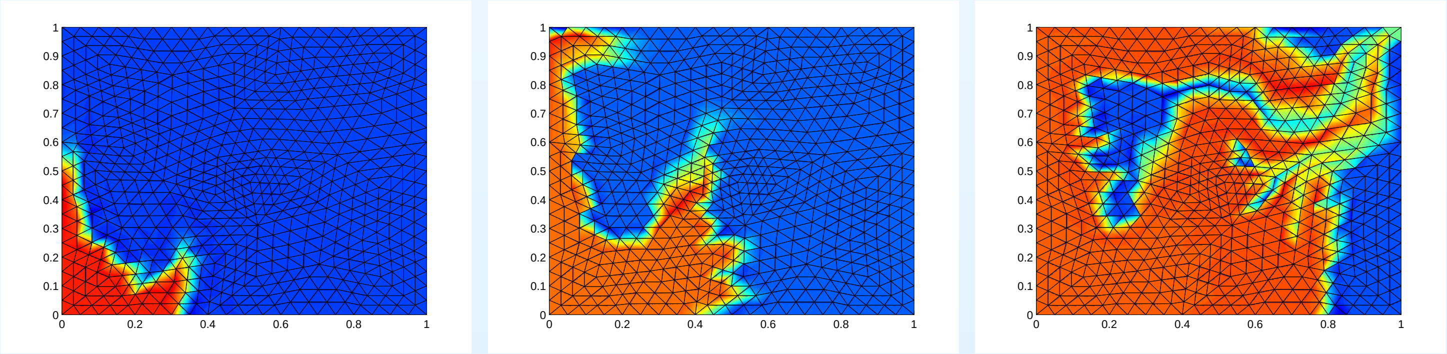
Reference solutions

We first plot reference solutions for the two problems:

Case 1:

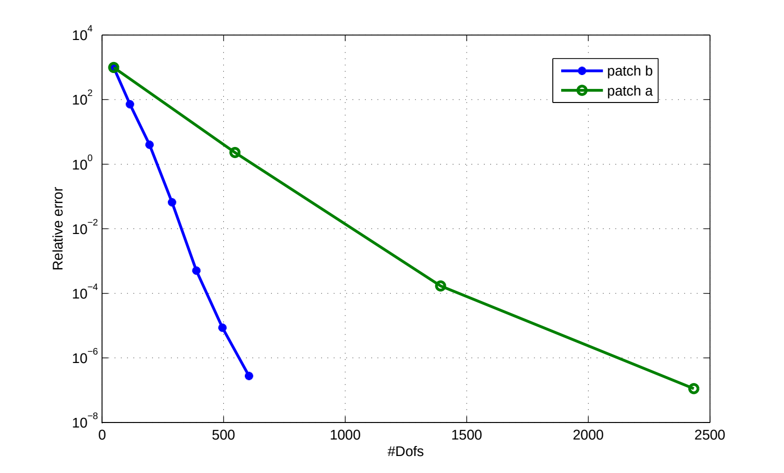


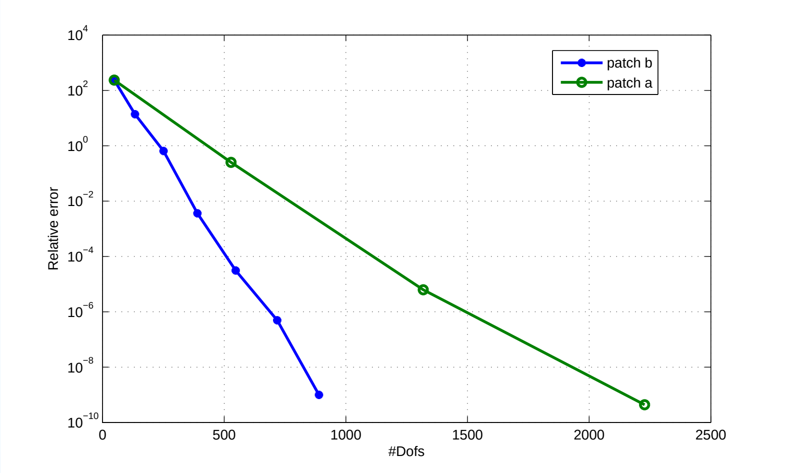
Case 2:



Convergence of global solution

Relative error compared to reference solution in H^1-norm at final time.





Case 1 (left) and case 2(right). We again see that the directed patches are much more efficient and that we get exponential decay in (average) degrees of freedom in the local sub problems.

The large relative error for small patches is the result of accumulation over time.

Summary of this paper

1. We have extended the AVMS framework to hyperbolic problems.
2. We have implemented an efficient way of choosing the shape of the patches.
3. We provide numerical examples that show very promising results.
4. A lot of work is still needed for the time discretization. Both simple and more complicated problems.
5. We have not yet done any error analysis. A posteriori error estimates would give us a possibility to refine and increase the size of the patches adaptively. We would also be able to choose the time step adaptively.

Other recent results and future directions

We have also studied

- a posteriori error estimation for Poisson equation, CG, DG, RT
- adaptive algorithms for local mesh patch size refinement
- convection dominated stationary problems

Future projects include

- improving the convergence result
- adaptive algorithm for the hyperbolic method
- convergence of adaptive algorithms
- solving the coupled system using RT and DG
- multiscale in time
- implement AMVS on parallel machines, 3D