Error Analysis of the Adaptive Variational Multiscale Method

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The Model Problem

The Dirichlet Problem.

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

where $a > 0$ is bounded.

Weak Form. Find $u \in \mathcal{V} = H_0^1(\Omega)$ such that

$$a(u, v) = (a\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in \mathcal{V}.$$

Example of a Basic Error Estimate

If we for the moment assume a to be periodic $a = a(x/\epsilon)$ we have (Hou),

$$\|e\|_a^2 = a(e,e) \leq C \left(\frac{h}{\epsilon}\right)^2 \|f\|^2.$$

- $h > \epsilon$ will give unreliable results even with exact quadrature.
- $h < \epsilon$ will often be to computationally expensive.

Conclusion of the Simple Estimate

We need to solve PDE:s on a scale that captures the oscillations but we can not afford to do it on the entire domain.

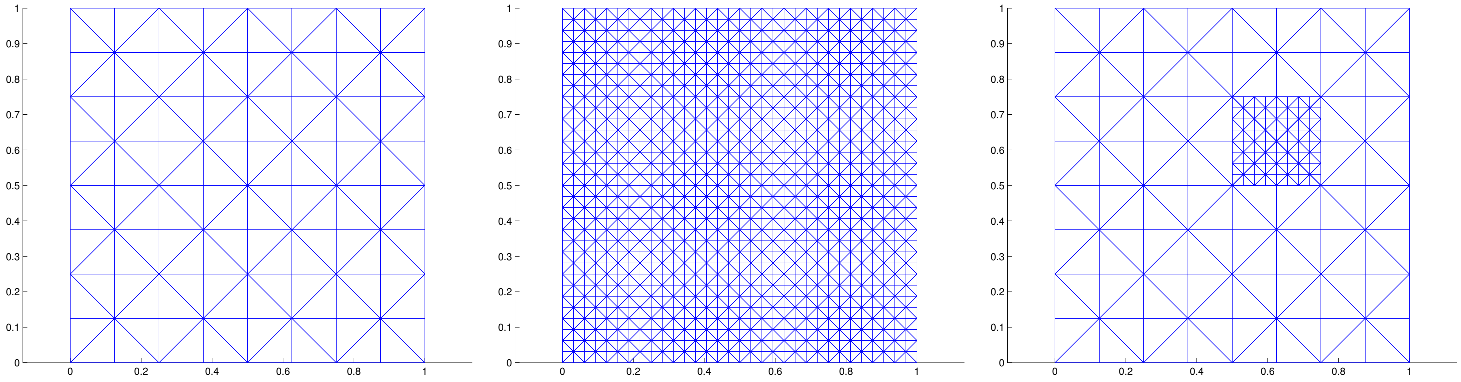


Figure 1: Coarse $H > \epsilon$ and fine $h < \epsilon$ mesh.

Since $\|e\|_a \sim \frac{h}{\epsilon}$, $\|e\|_a \sim H$ is reasonable.

First Step of Three: Spaces

Leaving the periodic setting. We let $\mathcal{V} = \mathcal{V}_c \bigoplus \mathcal{V}_f$.

- The goal of our calculation affects the choice of $\mathcal{V}_f$.
- Hierarchical basis gives an approximation of the nodal interpolant of u onto $\mathcal{V}_c$.
- The Scott-Zhang interpolant.
- A modified hierarchical basis that increases L^2 orthogonality between the scales.

Fine scale base functions needs to decay rapidly.

First Step: Spaces

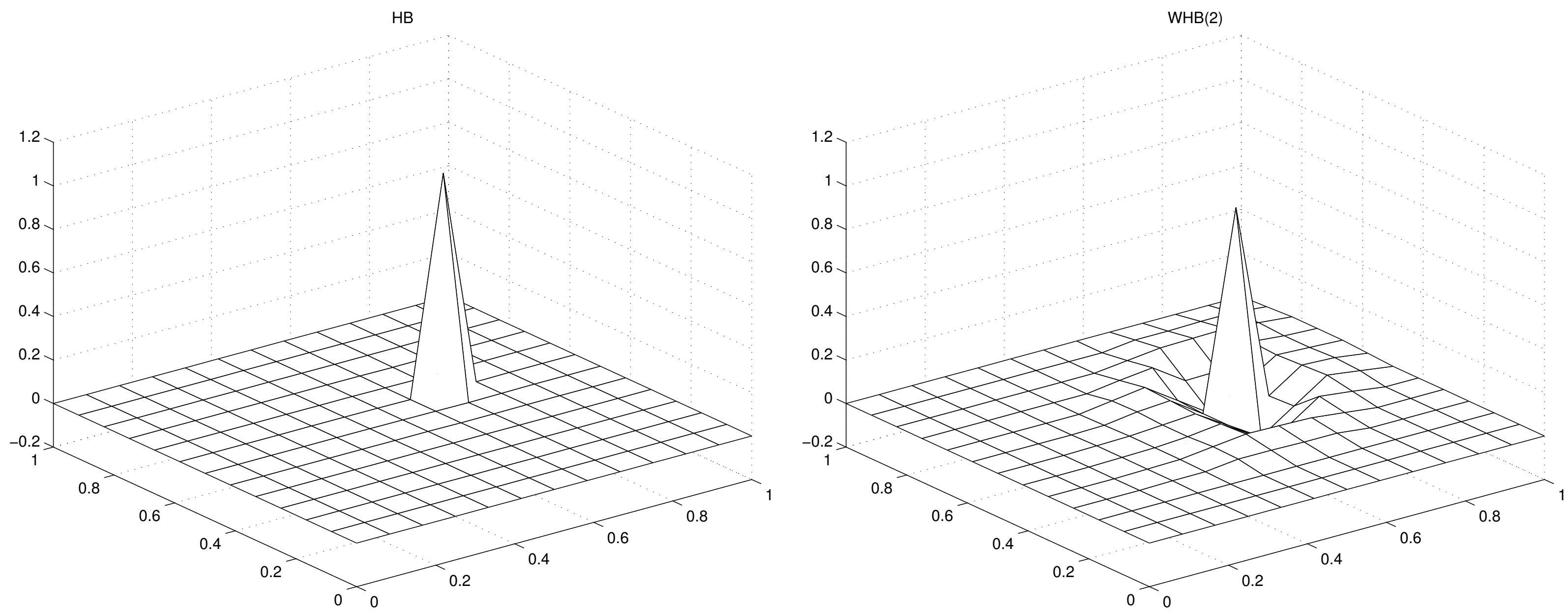


Figure 2: Different choices of fine scale base functions.

Second Step: VMM and Decoupling

We let $u = u_c + u_f$ and $v = v_c + v_f$ in the weak form (Hughes): Find $u_c \in \mathcal{V}_c$ and $u_f \in \mathcal{V}_f$ such that

$$\begin{aligned} a(u_c, v_c) + a(u_f, v_c) &= (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(u_f, v_f) &= (f, v_f) - a(u_c, v_f) \\ &:= (R(u_c), v_f) \quad \text{for all } v_f \in \mathcal{V}_f. \end{aligned}$$

Let $\{\varphi_i\}_{i \in \mathcal{N}}$ be a partition of unity and define $u_{f,i} \in \mathcal{V}_f$,

$$a(u_{f,i}, v_f) = (R(u_c), \varphi_i v_f) \quad \text{for all } v_f \in \mathcal{V}_f.$$

Second Step: VMM and Decoupling

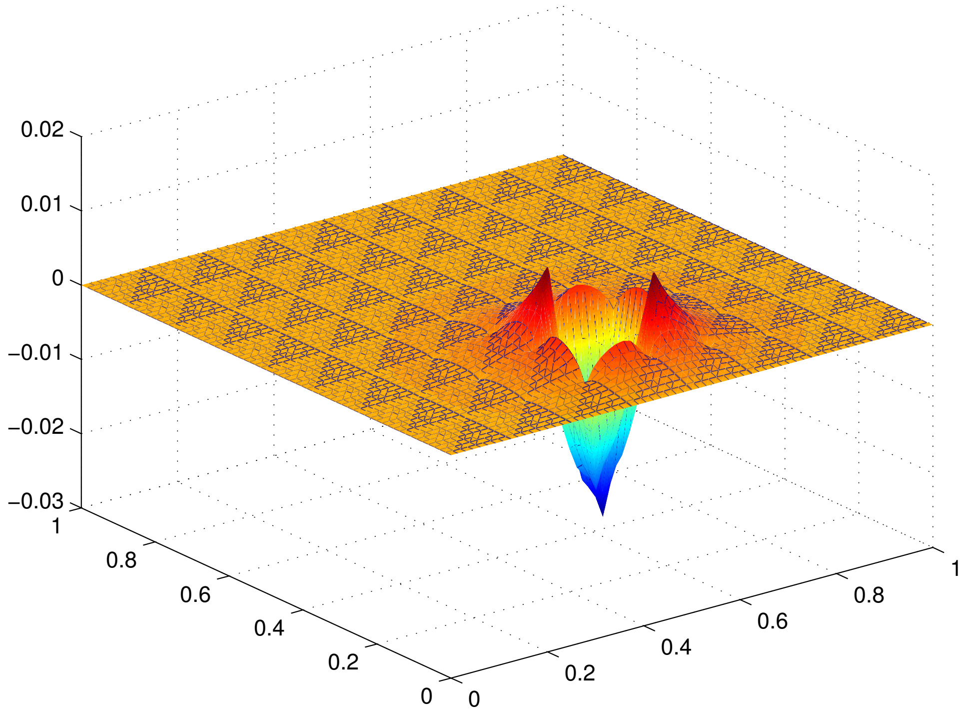


Figure 3: The fine scale solution $u_{f,i}$.

Third Step: Approximation on Patches

We finally solve the fine scale problems approximately on patches ω_i: Find $U_c \in \mathcal{V}_c$ and $U_{f,i} \in \mathcal{V}_f^h(\omega_i)$ such that

$$a(U_c, v_c) + a(U_f, v_c) = (f, v_c) \quad \text{for all } v_c \in V_c,$$
$$a(U_{f,i}, v_f) = (R(U_c), \varphi_i v_f) \quad \text{for all } v_f \in V_f^h(\omega_i).$$

Fine scale problems can be solved for all coarse nodes $\mathcal{N}$ or some $\mathcal{F}$ where $\mathcal{N} = \mathcal{C} \cup \mathcal{F}$.

Third Step: Approximation on Patches

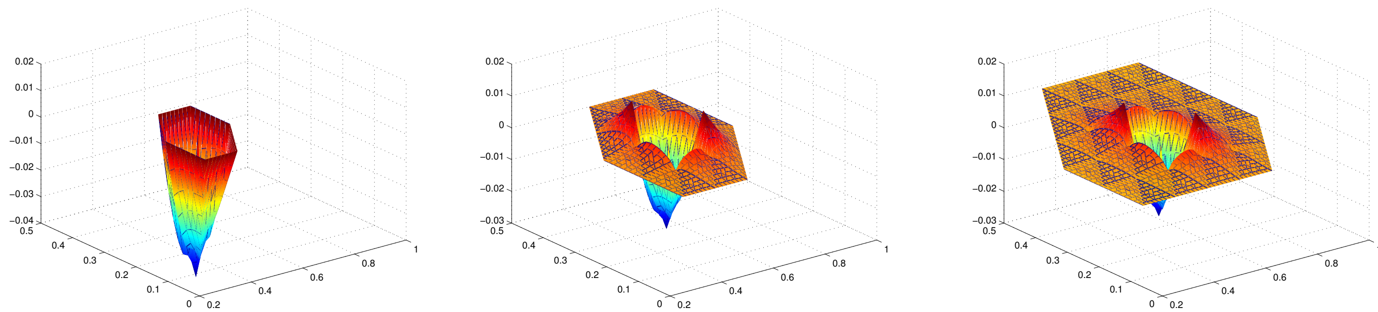


Figure 4: The fine scale solution $U_{f,i}$ for different patches ω_i.

One, two, and three layer stars. We get good agreement with the truth mesh solution.

Error Equation

We introduce the coarse and fine scale error $e_c = u_c - U_c$, $e_{f,i} = u_{f,i} - U_{f,i}$, and $e = e_c + \sum_{i\in\mathcal{N}} e_{f,i}$. We have the following orthogonality properties:

$$a(e_c, v_c) + a(e_f, v_c) = 0, \quad \text{for all } v_c \in \mathcal{V}_c$$

and

$$a(e_{f,i}, v_f) + a(e_c, \varphi_i v_f) = 0, \quad \text{for all } v_f \in \mathcal{V}_f^h(\omega_i).$$

Energy Norm Estimate

$$
\begin{aligned}
\|e\|_a^2 &= a(e,e) = a(e, e - \pi_c e) \\
&= (f, e - \pi_c e) - a(U_c, e - \pi_c e) - a(U_f, e - \pi_c e) \\
&= \sum_{i \in \mathcal{N}} (f, \varphi_i (e - \pi_c e)) - a(U_c, \varphi_i (e - \pi_c e)) \\
&\qquad - a(U_{f,i}, e - \pi_c e)
\end{aligned}
$$

Remember

$$a(U_{f,i}, v_f) + a(U_c, \varphi_i v_f) - (f, \varphi_i v_f) = 0, \text{ for all } v_f \in \mathcal{V}_f^h(\omega_i).$$

Energy Norm Estimate

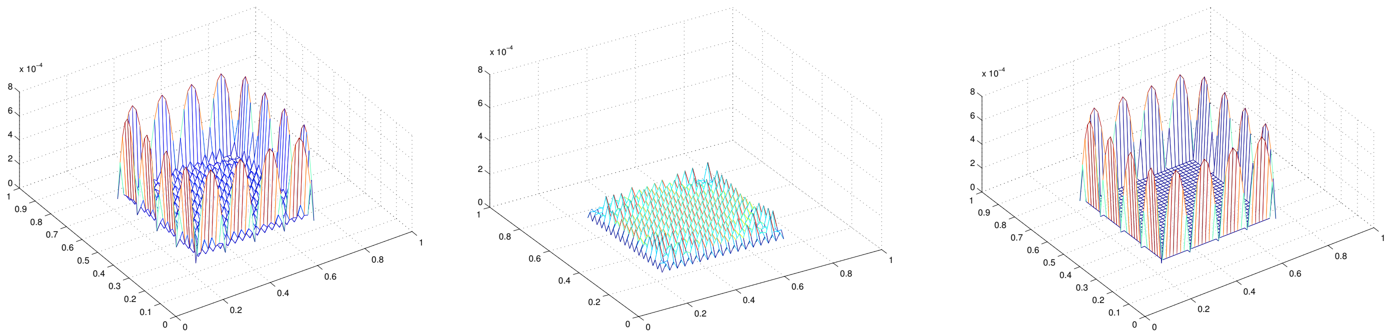


Figure 5: We study $z = e - \pi_c e$. We have $z - \pi_f^0 z$ to the left, $z - \pi_f z$ in the middle and $\pi_f z - \pi_f^0 z$ to the right.

Energy Norm Estimate

We work with these two parts separately. We start with $z - \pi_f z$ where $z = e - \pi_c e$,

$$\sum_{i \in \mathcal{N}} (f, \varphi_i(z - \pi_f z)) - a(U_c, \varphi_i(z - \pi_f z))$$

$$- a(U_{f,i}, z - \pi_f z)$$

$$\leq \left(\sum_{i \in \mathcal{N}} C_a \| h \mathcal{R}_i(U_{f,i}) \|^2_{\omega_i} \right)^{1/2} \| e \|_a .$$

Energy Norm Estimate

Next we study the boundary part $\pi_f z - \pi_f^0 z$,

$$\sum_{i\in\mathcal{N}} (f, \varphi_i(\pi_f z - \pi_f^0 z)) - a(U_c, \varphi_i(\pi_f z - \pi_f^0 z))$$
$$- a(U_{f,i}, \pi_f z - \pi_f^0 z)$$
$$\leq \left(\sum_{i\in\mathcal{N}} C_a \|\sqrt{H}\Sigma(U_{f,i})\|^2_{\partial\omega_i} \right)^{1/2} \|e\|_a.$$

$\Sigma(U_{f,i})$ is a variational approximation of $\partial_n U_{f,i}$.

Energy Norm Estimate

$$\|e\|_a^2 \le \sum_{i\in\mathcal{N}} C_a \left(\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2 + \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2 \right)$$

If we just solve local problems on part of the domain $\mathcal{F}$ we get,

$$\|e\|_a^2 \le \sum_{i\in\mathcal{C}} C_a \|H\mathcal{R}(U_c)\|_{\omega_i}^2 + \sum_{i\in\mathcal{F}} C_a \left(\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2 + \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2 \right)$$

Energy Norm Estimate

The boundary part $\|\sqrt{H}\Sigma(U_{f,i})\|^2_{\partial\omega_i}$, where $\Sigma(U(f,i))$ is an approximation of $\partial_n U_{f,i}$ decays rapidly on $\partial\omega_i$.

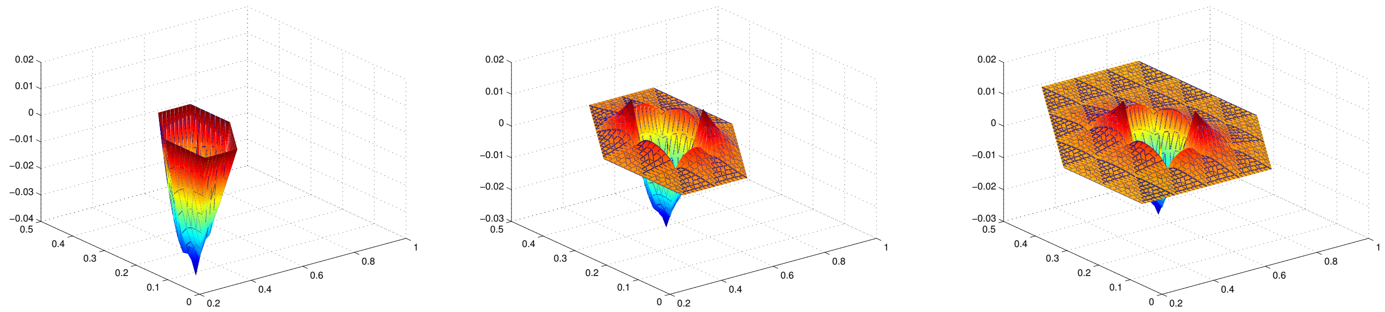


Figure 6: One two and three layer stars.

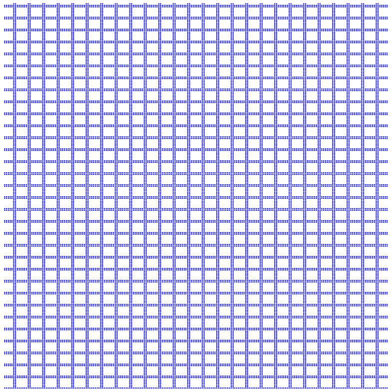
Adaptive Algorithm

$$\|e\|_a^2 \leq \sum_{i \in \mathcal{N}} C_a \left(\|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2 + \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2 \right)$$

1. Start with given r and L where $h = H/2^r$.
2. Calculate U using AVMM.
3. $E_r^i = \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}^2$ $E_L^i = \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i}^2$.
4. Stop if E_h^i and E_L^i are small enough else if $E_h^i > E_L^i$ let $r_{\text{new}} := 2r$ and if $E_L^i > E_h^i$ let $L_{\text{new}} = 2L$ end return to 2.

Adaptive Algorithm

Example using the Adaptive Algorithm,



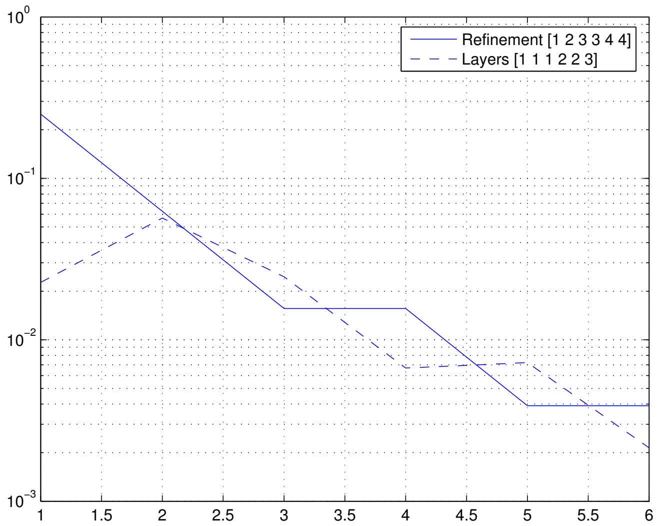


Figure 7: Periodic a (0.05 blue 1 white) with $\epsilon = H$ and 129×129 coarse nodes.

Linear Functional Error (e, ψ)

We introduce the dual problem: find ϕ such that

$$a(v, \phi_c) + a(v, \phi_f) = (v, \psi), \quad \text{for all } v \in \mathcal{V}.$$

From the orthogonality showed earlier on we have $a(e, \phi_c) = 0$. We get

$$\begin{aligned}(e, \psi) &= a(e, \phi) = a(e, \phi_f) = (f, \phi_f) - a(U, \phi_f) \\ &= \sum_{i \in \mathcal{C}} (f, \varphi_i \phi_f) - a(U_c, \varphi_i \phi_f) \\ &\quad + \sum_{i \in \mathcal{F}} (f, \varphi_i \phi_f) - a(U_c, \varphi_i \phi_f) - a(U_{f,i}, \phi_f).\end{aligned}$$

Linear Functional Error (e, ψ)

We need to calculate an approximation of ϕ_f numerically.

- If possible ϕ_f can be calculated by a global calculation on a mesh where $h < h_\phi < H$.
- Or ϕ_f can be calculated by AVMM. We need to keep track on neighboring patches to form ϕ_f locally.

Linear Functional Error (e, ψ)

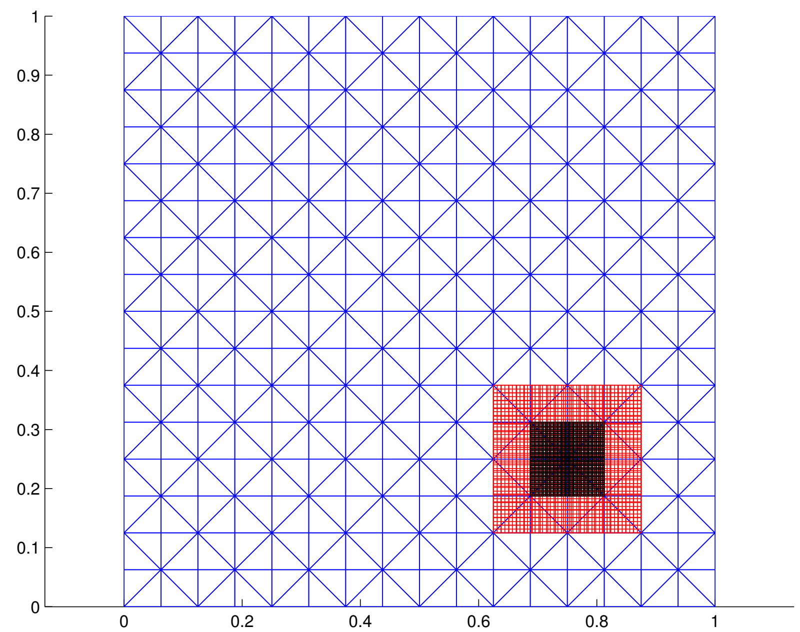


Figure 8: The dual solution on the red patches affects the error calculated on the black patch.

Linear Functional Error (e, ψ)

We consider an example where we seek a very accurate solution in part of the domain.

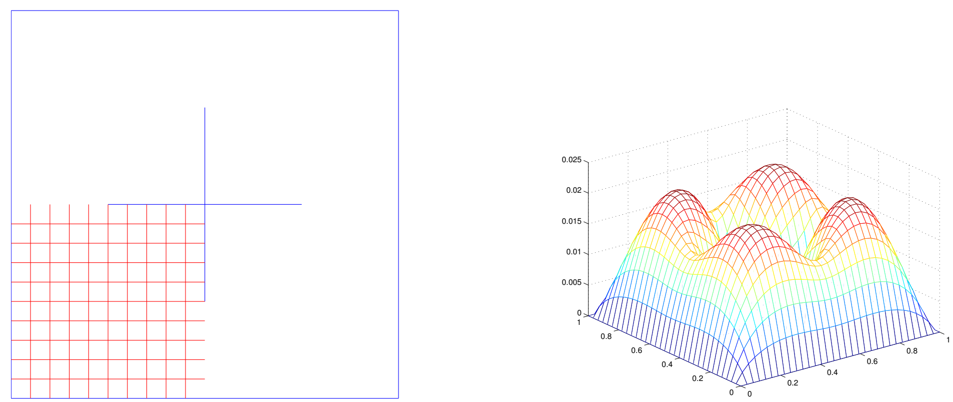


Figure 9: Geometry to the left with interesting region marked $\psi = I_{\{0 \le x, y \le 0.5\}}$. Solution to the right, $a = f = 1$.

Linear Functional Error (e, ψ)

We solve the dual problem with $\psi = I_{\{0 \le x, y \le 0.5\}}$.

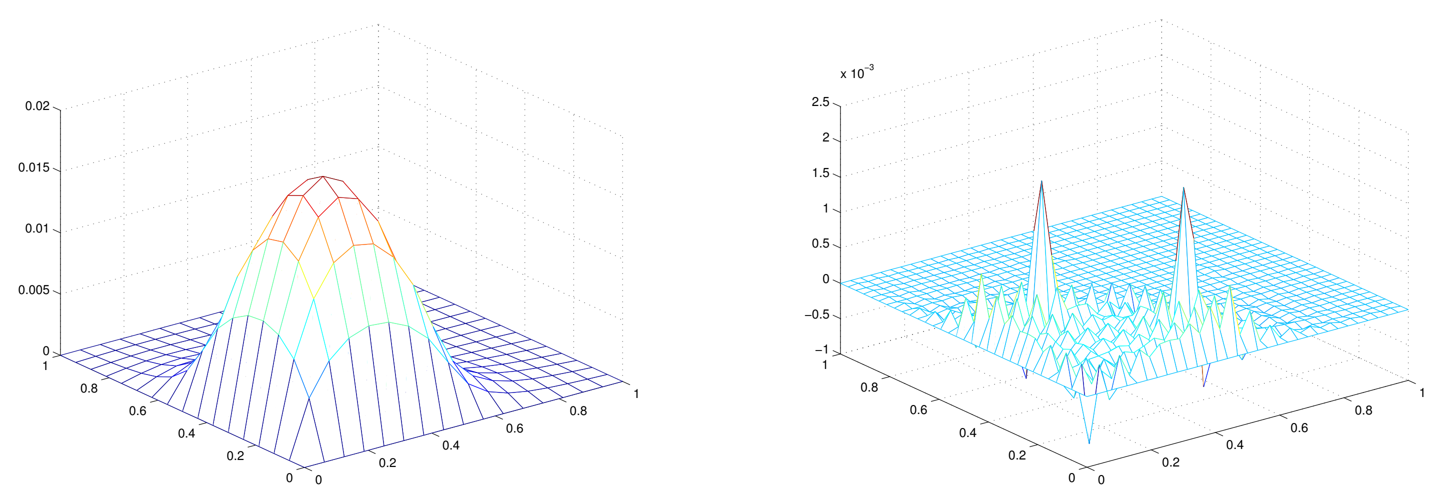


Figure 10: Dual solution ϕ to the left and ϕ_f to the right.

Linear Functional Error (e, ψ)

We remember the error estimate,

$$(e, \psi) = \sum_{i \in \mathcal{C}} (f, \varphi_i \phi_f) - a(U_c, \varphi_i \phi_f)$$
$$+ \sum_{i \in \mathcal{F}} (f, \varphi_i \phi_f) - a(U_c, \varphi_i \phi_f) - a(U_{f,i}, \phi_f).$$

1. Start with all nodes in $\mathcal{C}$. Calculate U and ϕ_f.
2. Calculate error estimators, solve local problems or increase number of layers.
3. Stop if the error is small enough else go to 2.

Linear Functional Error (e, ψ)

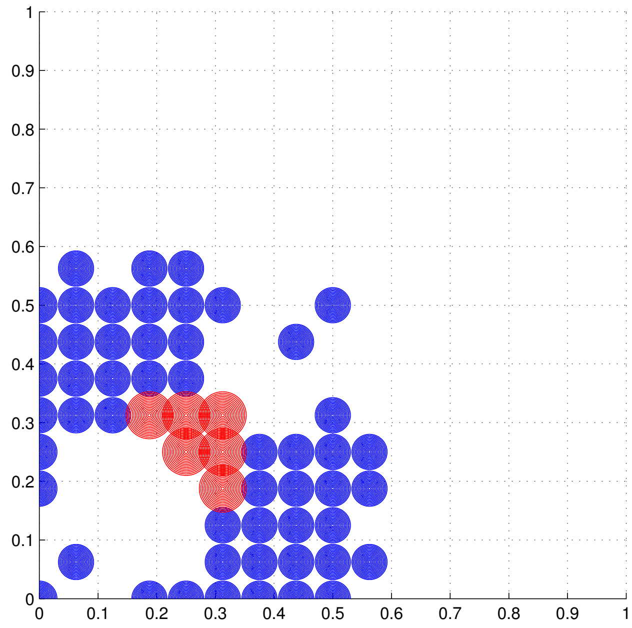


Figure 11: Local problems are solved using two and three layer stars in marked nodes.

Linear Functional Error (e, ψ)

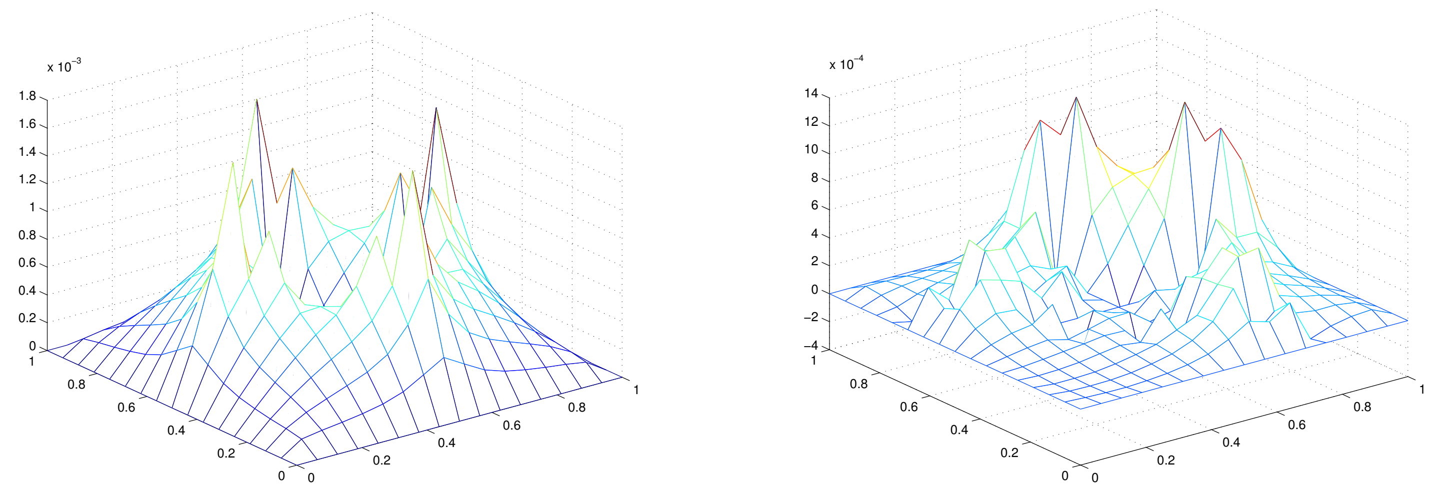


Figure 12: Error compared to reference mesh for standard Galerkin (left) and two iterations of AVMM as described above (right).

Standard A Posteriori Error Estimates

For duality based algorithms we know that:

- The dual solution ϕ need to be approximated but not in $\mathcal{V}_c$.
- Regular refinement or higher order methods allocate lots of memory.

Instead we solve the dual problem by AVMM in each coarse node,

$$(e, \psi) = \sum_{i=1}^{n} (R(U), \Phi_{f,i}) + a(U, \phi_f - \Phi_f).$$

Standard A Posteriori Error Estimates

The second term can be estimated in the following way,

$$
\begin{aligned}
a(e,(\phi_f - \Phi_f)) &\leq \|e\|_a \, \|\phi_f - \Phi_f\|_a \\
&\leq \|e\|_a \, \|\phi - (\Phi_c + \Phi_f)\|_a .
\end{aligned}
$$

Both these terms can be estimated.

We can use standard Galerkin on the primal and AVMM on the dual and have control on the error committed by not solving the dual exactly.

Ongoing Projects on This Subject

- Adaptivity for the local problems.
- More tests on how the split between $\mathcal{V}_c$ and $\mathcal{V}_f$ affects the algorithm.
- More layers than two.
- 3D implementation.