Adaptive variational multiscale methods based on a posteriori error estimation

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The Model Problem

Poisson Equation.

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

where $a > 0$ bounded, and Ω is a domain in $\mathbf{R}^d$, $d = 1, 2, 3$.

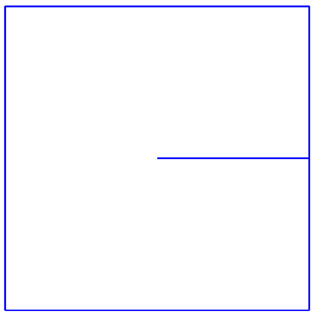
Weak form. Find $u \in H_0^1(\Omega)$ such that

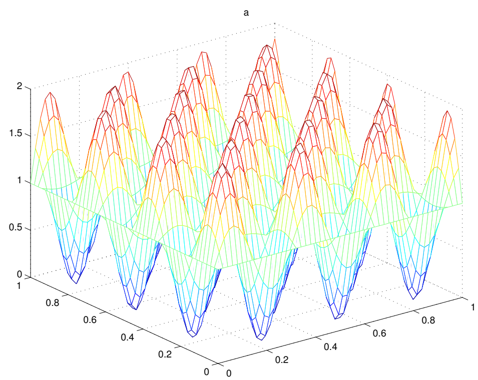
$$(a\nabla u, \nabla v) = \int_\Omega a\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx = (f, v)$$

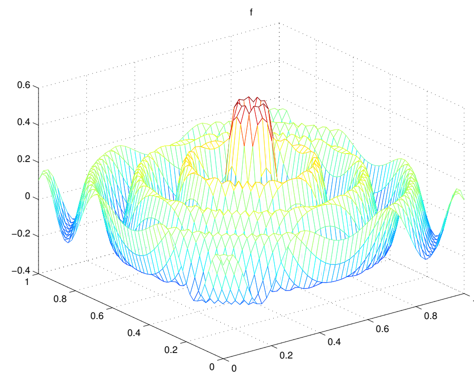
for all $v \in H_0^1(\Omega)$.

Multiscale Problems

Below are three examples of multiscale problems.







The first one represents difficulties in the domain (cracks, holes, ...) the second one oscillations in a and the third one oscillations in f.

Motivation

- Very important applications including materials, flow in porous media, ...
- The problems are very computationally challenging so error estimation and efficient algorithms are crucial.
- Attempts on using adaptive algorithms are not common in literature.

Variational Multiscale Method

We introduce two spaces V_c and V_f such that $V_c \oplus V_f = H_0^1(\Omega)$ i.e. $u \in H_0^1(\Omega)$ can be written as $u = u_c + u_f$.

- V_c is a finite dimensional approximation of $H_0^1(\Omega)$, (hat-functions).
- And V_f is the rest i.e. it captures the fine scale behavior.

We get the following equation

$$f = -\nabla \cdot a\nabla u = -\nabla \cdot a\nabla u_c - \nabla \cdot a\nabla u_f.$$

Variational Multiscale Method

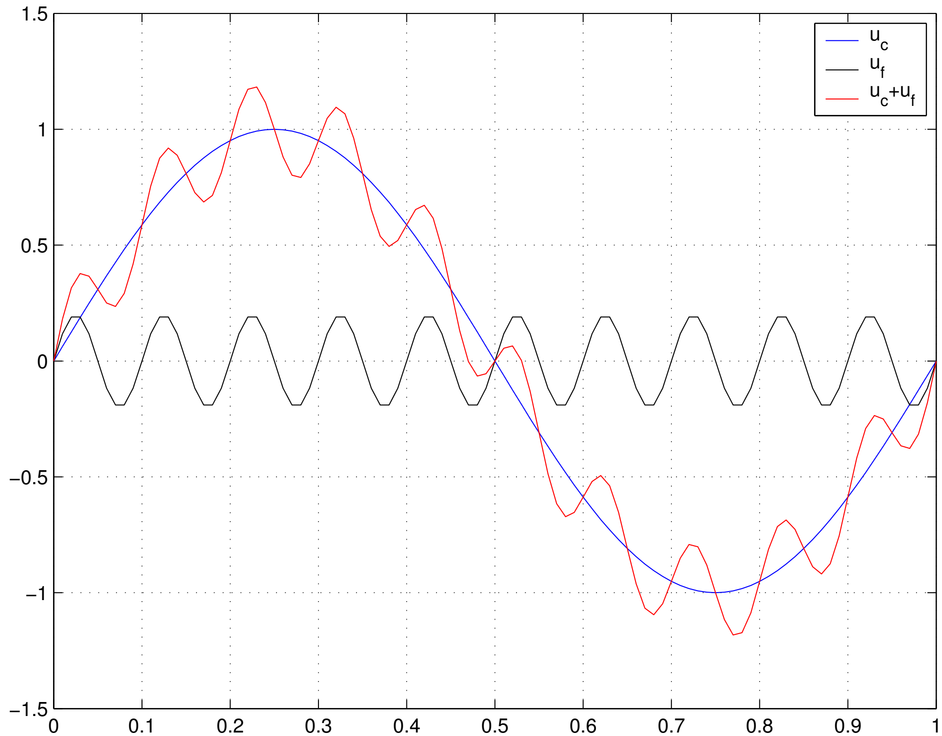


Figure 1: u_c, u_f, and $u_c + u_f$.

Variational Multiscale Method

Find $u_c \in V_c$ and $u_f \in V_f$ such that

$$
\begin{aligned}
(a\nabla u_c, \nabla v_c) + (a\nabla u_f, \nabla v_c) &= (f, v_c) \quad \text{for all } v_c \in V_c, \\
(a\nabla u_f, \nabla v_f) = (f, v_f) &- (a\nabla u_c, \nabla v_f) \\
&:= (R(u_c), v_f) \quad \text{for all } v_f \in V_f.
\end{aligned}
$$

Fine scale information is used to modify the coarse scale equation: Find $u_c \in V_c$ such that

$$
(a\nabla u_c, \nabla v_c) + (a\nabla \hat{A}_f^{-1} R(u_c), \nabla v_c) = (f, v_c) \;\; \forall v_c \in V_c.
$$

Our Basic Idea

- Discretization of V_f (analytical estimates are more common).
- Solve localized fine scale problems for each coarse node (or some coarse nodes) in parallel.
- Error estimation framework.
- Adaptive strategy for this setting.

Decouple Fine Scale Equations

Remember the fine scale equations:

$$(a\nabla u_f, \nabla v_f) = (R(u_c), v_f), \quad \text{for all } v_f \in V_f.$$

Include a partition of unity,

$$(a\nabla u_f, \nabla v_f) = (R(u_c), v_f) \;\; = \sum_{i=1}^{n} (R(u_c), \varphi_i v_f),$$

let $u_f = \sum_i^n u_{f,i}$ where
$(a\nabla u_{f,i}, \nabla v_f) = (R(u_c), \varphi_i v_f)$.

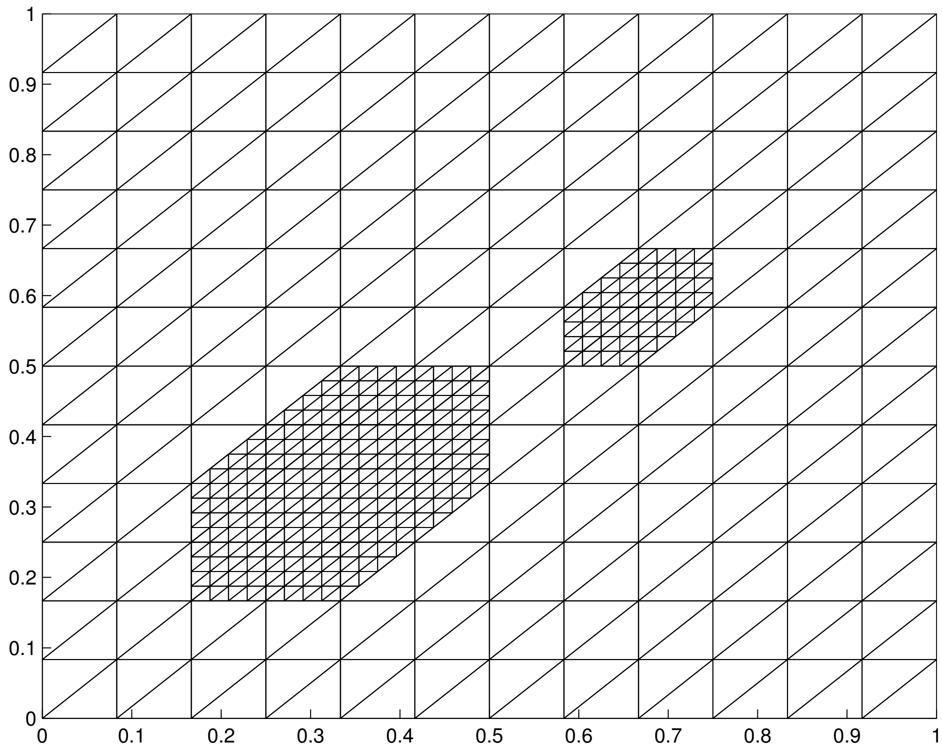
Approximate Solution

Since φ_i has support on a star S_i^1 in node i we solve the fine scale equations approximately on ω_i with $U_{f,i} = 0$ on $\partial\omega_i$.

Find $U_c \in V_c$ and $U_f = \sum_i^n U_{f,i}$ where $U_{f,i} \in V_f^h(\omega_i)$ such that

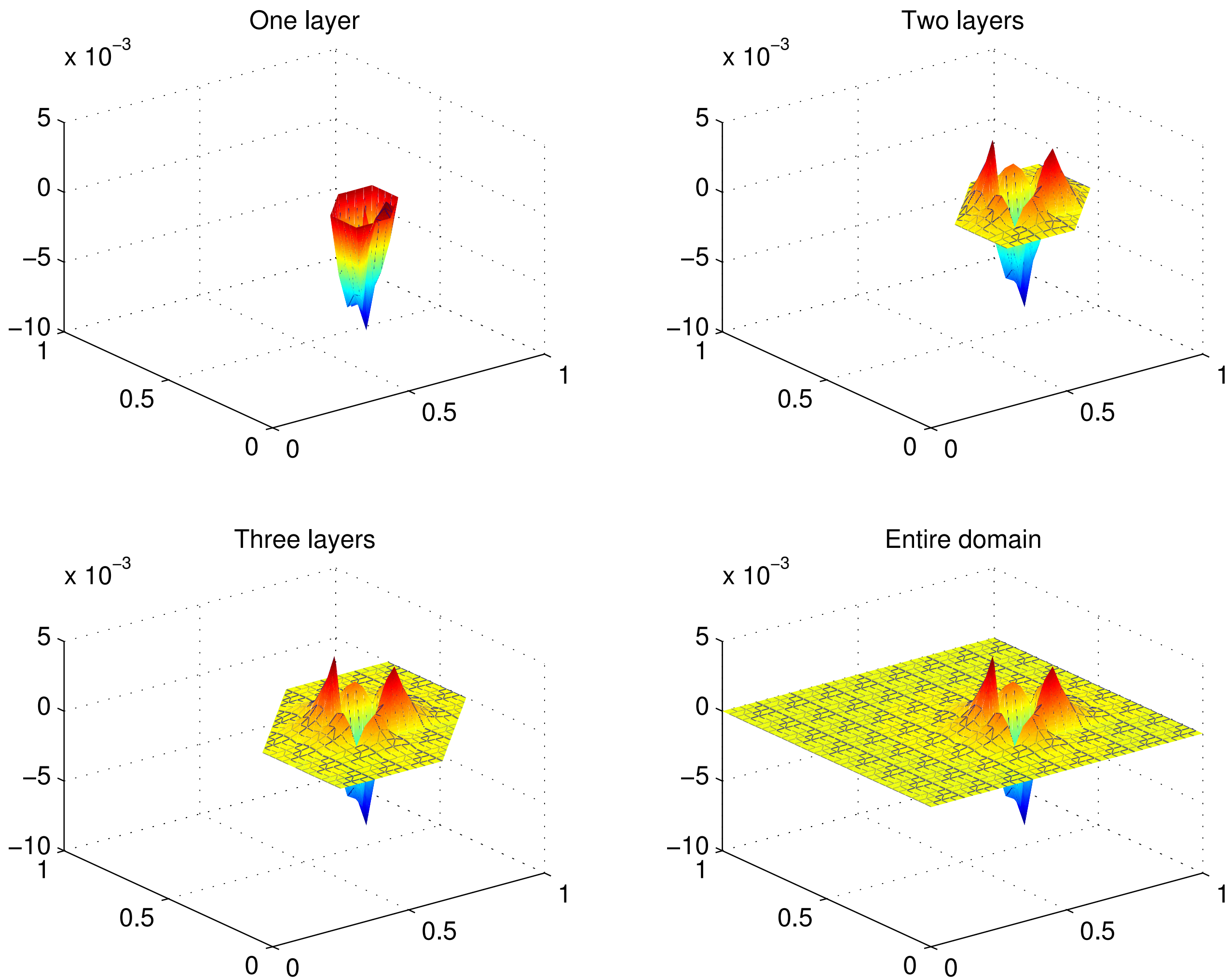
$$\begin{aligned}(a\nabla U_c, \nabla v_c) + (a\nabla U_f, \nabla v_c) &= (f, v_c) \quad \text{for all } v_c \in V_c,\\ (a\nabla U_{f,i}, \nabla v_f) &= (R(U_c), \varphi_i v_f) \quad \text{for all } v_f \in V_f^h(\omega_i).\end{aligned}$$

Refinement and Layers



One and two layer stars.

Localized Fine Scale Solution



Energy Norm Estimate

$$\|\sqrt{a}\nabla e\| \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|_{\omega_i} + \sum_{i\in\mathcal{F}} C_i \left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i}\right)$$

- The first term is coarse mesh error.
- The second term is the normal derivative of the fine scale solutions on $\partial\omega_i$.
- The third term is fine scale error.

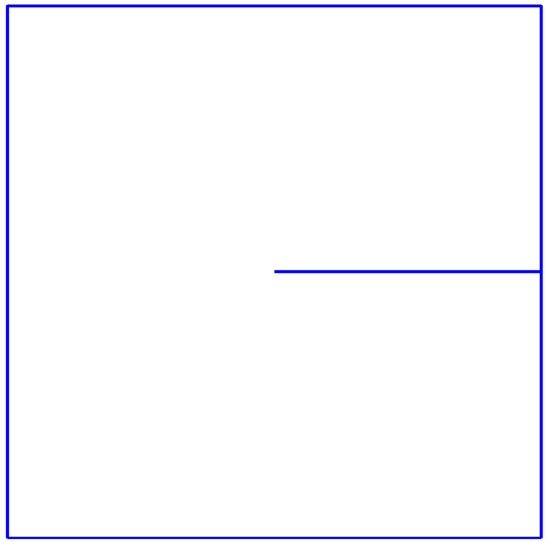
Adaptive Strategy

$$\|\sqrt{a}\nabla e\| \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|_{\omega_i} + \sum_{i\in\mathcal{F}} C_i \left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i} \right)$$

- We calculate these for each $i \in \{\text{coarse fine}\}$.
- Large values $i \in$ coarse $\rightarrow$ more local problems.
- Large values $i \in$ fine $\rightarrow$ more layers or smaller h.

Numerical Examples

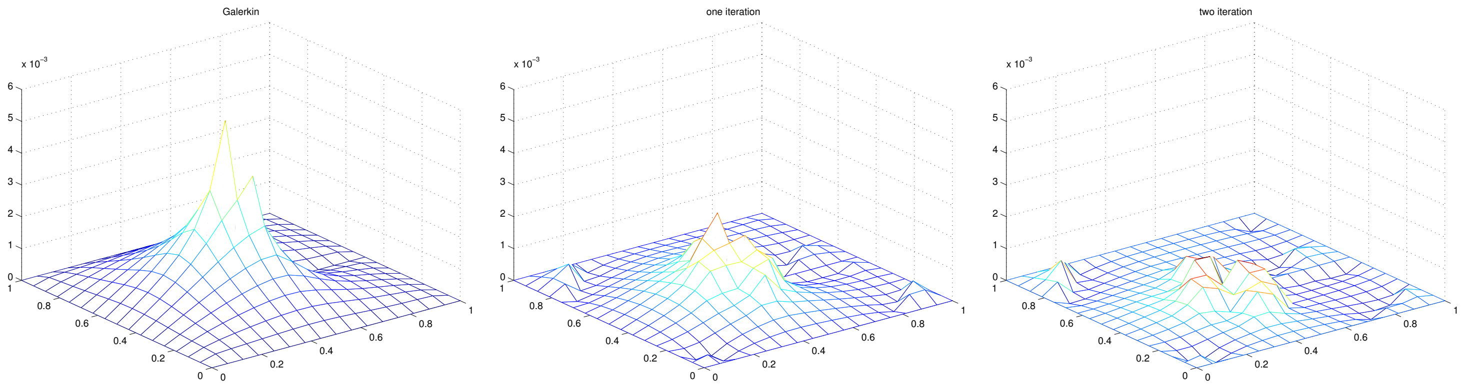
We start with a unit square containing a crack.



We let the coefficient $a = 1$ and solve, $-\triangle u = f$ with $u = 0$ on the boundary including the crack.

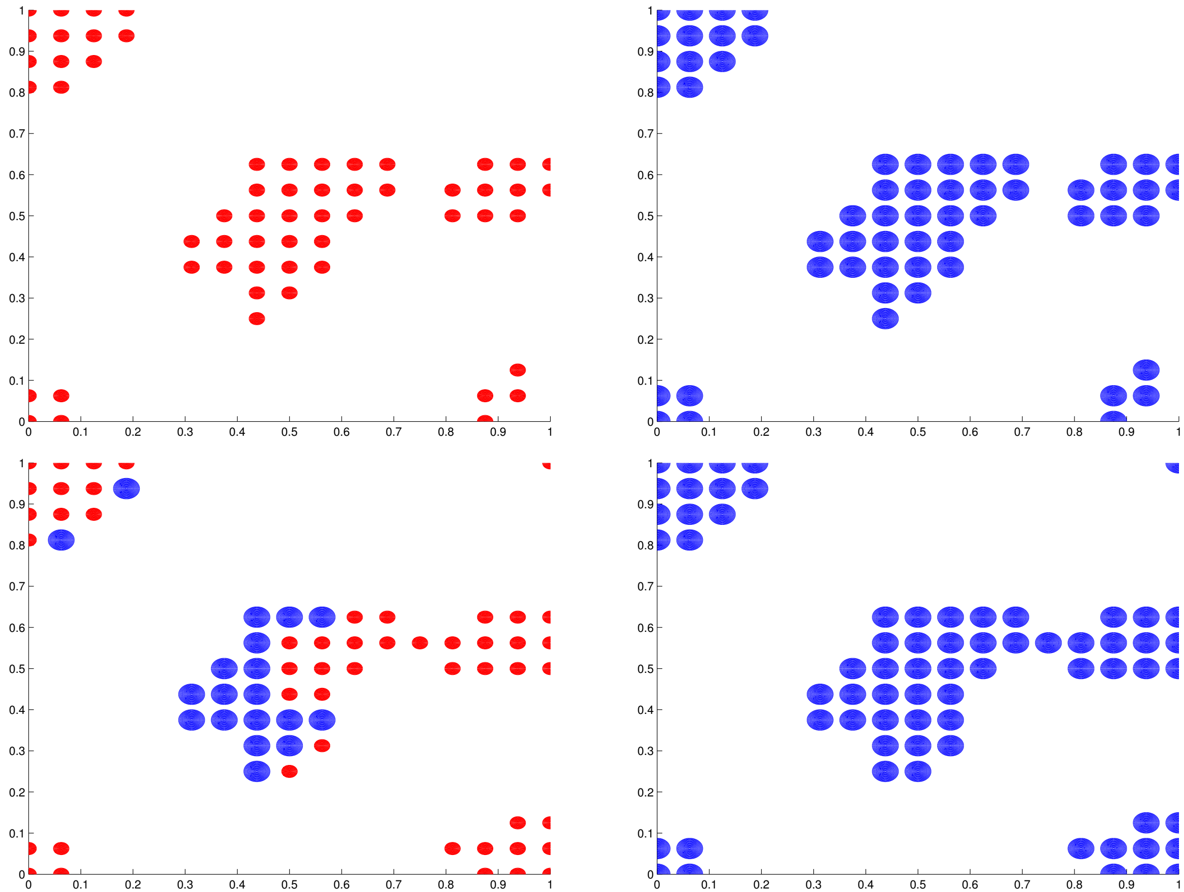
Numerical Examples

We solve the problem by using the adaptive algorithm.



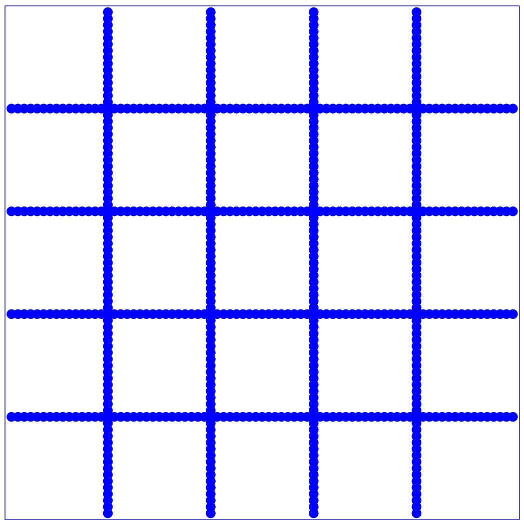
We plot the difference between our solution and a reference solution.

Numerical Examples

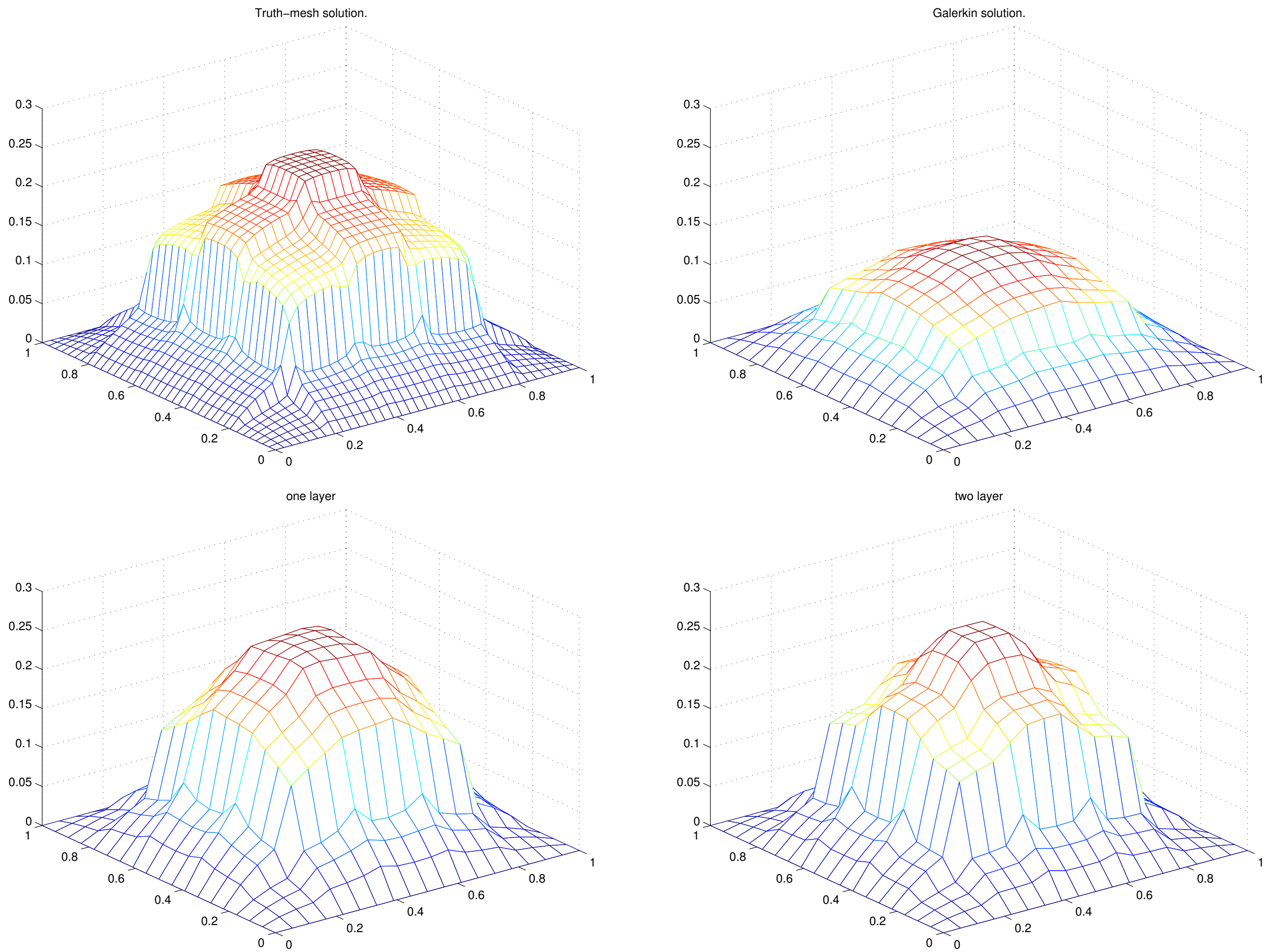


Numerical Examples

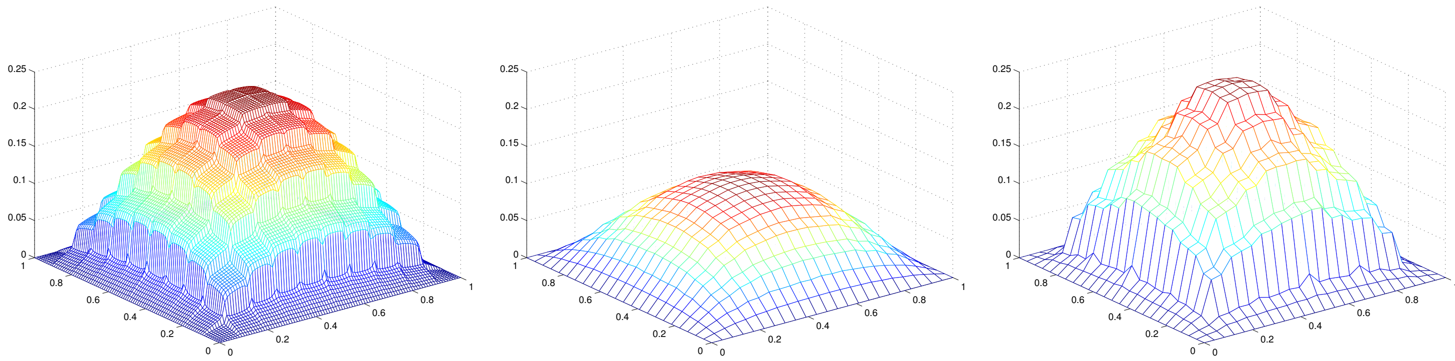
In this example we study a discontinuous coefficient a in $-\nabla \cdot a\nabla u = f$. $a = 1$ (white) and $a = 0.05$ (blue).



Numerical Examples



Numerical Examples



The number of layers seems to depend on the fine scale structure rather that the domain size.

Outlook

- Extended numerical tests in both 2D and 3D.
- Mixed formulation.
- Other equations (convection-diffusion, ...).
- More scales.
- Comparing results with classical Homogenization theory.