Multiscale methods for elliptic problems

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Outline and Papers

Outline

- Model problem, elliptic linear pde
- The variational multiscale method and related methods
- Derivation of proposed method with example
- A posteriori error estimates
- Adaptivity
- Application to oil reservoir simulation
- Convection dominated problem
- Future work

Papers

- M.G. Larson and A. Målqvist, *Adaptive Variational Multiscale Methods Based on A Posteriori Error Estimation: Energy Norm Estimates for Elliptic Problems*, CMAME 2007
- A. Målqvist, *Multiscale methods for elliptic problems* (in preparation)

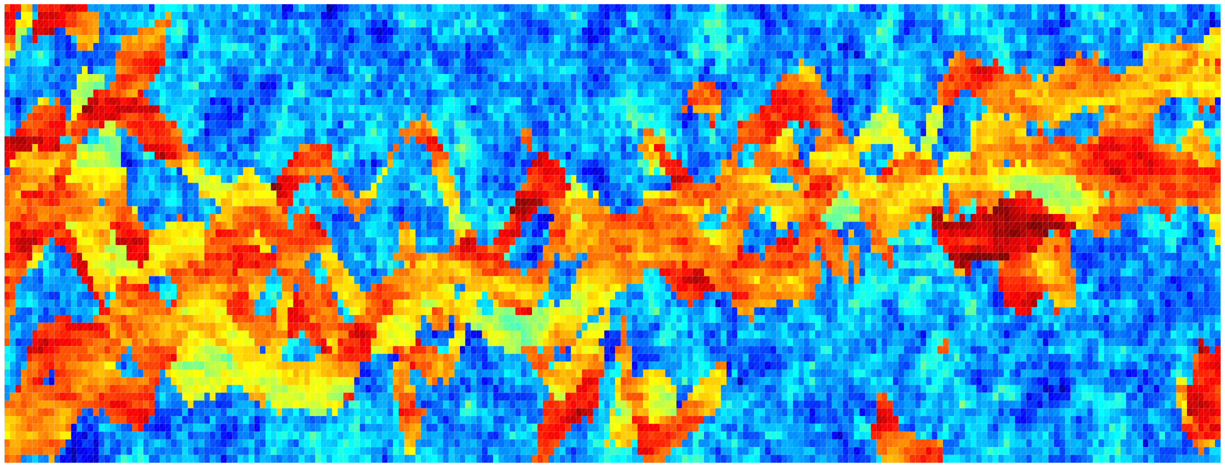
Model Problem

Elliptic equation: Find u such that

$$\mathcal{L}u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where $\mathcal{L}(a)$ is an elliptic differential operator with a coefficient $a(x)$ that has multiscale features, f is a given function, and Ω is a polygonal domain.



Motivation: A priori error bound for periodic case

If we consider the Poisson equation

$$\mathcal{L}u = -\nabla \cdot a\nabla u = f,$$

with periodic coefficient $a = a(x/\epsilon)$ solved using the finite element method on a mesh of size H we have (Hou-Wu-Cai),

$$\|\sqrt{a}\nabla(u - u_h)\| \leq C\frac{H}{\epsilon}\|f\|_{L^2(\Omega)},$$

- $\epsilon < H$ will give unreliable results even with exact quadrature.
- $\epsilon > H$ will be to computationally expensive to solve on a single mesh.

In this talk we consider a general positive and bounded a.

Some previous works and related methods

- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson-Målqvist 05, Nolen et al. 08, Nordbotten 09
- Multiscale finite element method: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06
- Multiscale finite volume method: Jenny et al. 03
- Heterogeneous multiscale method: Engquist-E 03
- Equation free: Kevrekidis et al. 05
- ...

Local approximations (in parallel) on a fine scale are used to modified the coarse scale equation.

Model problem

Convection-diffusion reaction: Find $u \in \mathcal{V}$ such that

$$\mathcal{L}u = -\nabla \cdot a\nabla u + \nabla \cdot (bu) + cu = f.$$

On weak form we have, find $u \in \mathcal{V}$ such that,

$$a(u, v) = (\mathcal{L}u, v) = (f, v) = l(v) \quad \text{for all } v \in \mathcal{V},$$

Poisson equation on mixed form: Find $\{u_1, u_2\} \in \mathcal{V}$ such that

$$-\nabla \cdot u_2 = f \text{ and } u_2 = a\nabla u_1$$

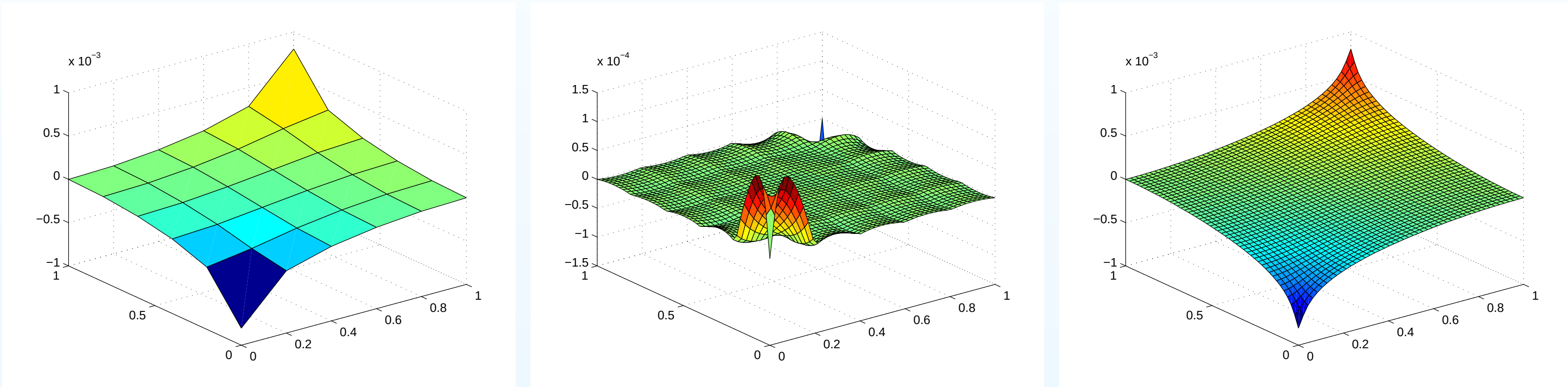
or on weak form,

$$a(u, v) = (\frac{1}{a}u_2, v_2) + (u_1, \nabla \cdot v_2) - (\nabla \cdot u_2, v_1) = (f, v_1) = l(v),$$

for all $\{v_1, v_2\} \in \mathcal{V}$.

Variational Multiscale Method: Hughes et. al. $95, 98$

We split the space $\boxed{\mathcal{V}_c \oplus \mathcal{V}_f = \mathcal{V} \quad \left(= H_0^1(\Omega)\right)}$



- $\mathcal{V}_c$ is a finite dimensional approximation of $\mathcal{V}$. (finite element space)
- $\mathcal{V}_f$ can be chosen in different ways
  - Hierarchical basis
  - $L^2(\Omega)$-orthogonal to $\mathcal{V}_c$
  - Wavelet modified hierarchical basis

The Proposed Multiscale Method

Starting from the weak form: find $u \in \mathcal{V}$ such that

$$a(u, v) = l(v) \quad \text{for all } v \in \mathcal{V},$$

we want to decouple the fine scale part from the rest of the problem. Let $\mathcal{T}, \mathcal{T}^* : \mathcal{V}_c \to \mathcal{V}_f$ such that

$$a(\mathcal{T} v_c, v_f) = -a(v_c, v_f), \quad \text{for all } v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f$$
$$a(v_f, \mathcal{T}^* v_c) = -a(v_f, v_c), \quad \text{for all } v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f$$

Using these operators we plug in $u = u_c + \mathcal{T} u_c + u_f \in \mathcal{V}_c \bigoplus \mathcal{V}_f$ and $v = v_c + \mathcal{T}^* v_c + v_f$ in the weak form,

$$a(u_c + \mathcal{T} u_c, v_c + \mathcal{T}^* v_c) = l(v_c + \mathcal{T}^* v_c) \quad \text{for all } v_c \in \mathcal{V}_c,$$
$$a(u_f, v_f) = l(v_f) \quad \text{for all } v_f \in \mathcal{V}_f.$$

Coarse Scale Equations

Find $u_c \in \mathcal{V}_c$ such that

$$a(u_c + \mathcal{T}u_c, v_c + \mathcal{T}^* v_c) = l(v_c + \mathcal{T}^* v_c) - a(u_f, v_c + \mathcal{T}^* v_c)$$

for all $v_c \in \mathcal{V}_c$, note that $a(u_f, v_f) = l(v_f)$ for all $v_f \in \mathcal{V}_f$.

- If a is symmetric we get $\mathcal{T}^* v_c = \mathcal{T} v_c$ i.e. a symmetric formulation
- In standard VMS $\mathcal{T}^* v_c = 0$ and u_f is included in the coarse scale. The computation of $\mathcal{T}u_c + u_f \in \mathcal{V}_f$ is decoupled and done analytically on each coarse element using homogeneous Dirichlet boundary conditions.
- In MsFEM $u_f = 0$ and therefore not present in the coarse scale equations. Here the computation of $v_c + \mathcal{T} v_c$ is decoupled and solved numerically on each element (or larger domains) using approximate boundary conditions.

Approximation of Fine Scale Solutions

- Let $\tilde{\mathcal{T}}, \tilde{\mathcal{T}}^*$ be a computable approximations of $\mathcal{T}, \mathcal{T}^*$
- Let U_f be a computable approximation of u_f

We get the method: find $U_c \in \mathcal{V}_c$ such that

$$\boxed{a(U_c + \tilde{\mathcal{T}} U_c, v_c + \tilde{\mathcal{T}}^* v_c) = l(v_c + \tilde{\mathcal{T}}^* v_c) - a(U_f, v_c + \tilde{\mathcal{T}}^* v_c)}$$

for all $v_c \in \mathcal{V}_c$. On matrix form this leads to,

$$\boxed{KU_c = b}$$

Given U_c, U_f, and $\tilde{\mathcal{T}}$, U can be computed.

When $a(U_f, v_c + \tilde{\mathcal{T}}^* v_c)$ is included, the error is orthogonal to $v_c + \tilde{\mathcal{T}}^* v_c$ i.e. $a(u - U_c - \tilde{\mathcal{T}} U_c - U_f, v_c + \tilde{\mathcal{T}}^* v_c) = 0$.

Construction of $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}}^*$

Recall that $u_c = \sum_i u_c^i \phi_i$ with $\{\phi_i\}$ a basis in $\mathcal{V}_c$ and let

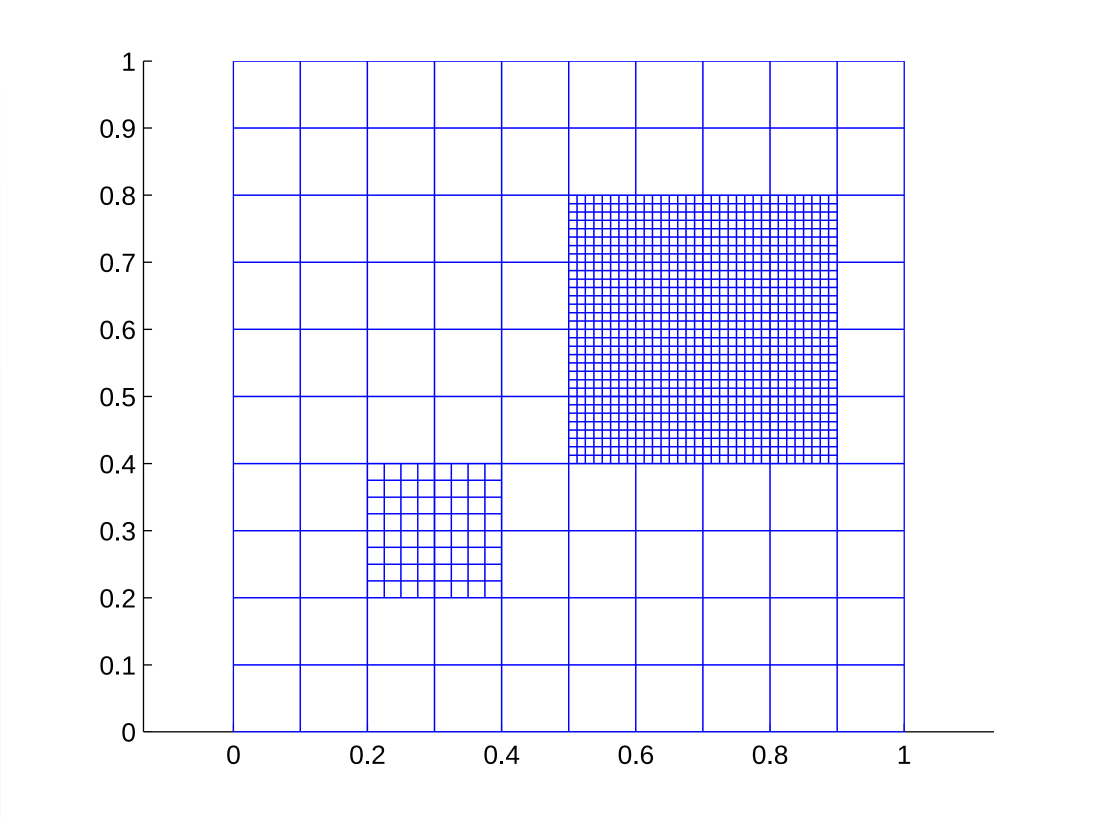
$$a(\mathcal{T}\phi_i, v_f) = -a(\phi_i, v_f) \quad \text{for all } v_f \in \mathcal{V}_f,$$
$$a(v_f, \mathcal{T}^*\phi_i) = -a(v_f, \phi_i) \quad \text{for all } v_f \in \mathcal{V}_f.$$

By linearity $\mathcal{T}u_c = \sum_i u_c^i \mathcal{T}\phi_i$ and $\mathcal{T}^* v_c = \sum_i v_c^i \mathcal{T}^*\phi_i$ and thus we are led to computing $\mathcal{T}\phi_i$ and $\mathcal{T}^*\phi_i$ for each coarse basis function ϕ_i.

We compte $\tilde{\mathcal{T}}\phi_i, \tilde{\mathcal{T}}^*\phi_i$ approximately by

- Restricting to a localized patches $\text{supp}(\phi_i) \subset \omega_i$
- Discretizing using a fine subgrid on ω_i

Refinement and Layers



We let H be coarse scale mesh size and h be fine scale mesh size. Further we let L denote the number of layers of coarse elements in the patch. Typically homogeneous Dirichlet boundary condition are used. (Compare over-sampling.)

Construction of U_f

Recall that $u_f \in \mathcal{V}_f$ solves

$$a(u_f, v_f) = l(v_f) \quad \text{for all } v_f \in \mathcal{V}_f$$

Using a partition of unity ψ_i we can split the right hand side as follows $l(v_f) = \sum_i l(\psi_i v_f)$ to get,

$$u_f = \sum_i u_f^i$$

$$a(u_f^i, v_f) = l(\psi_i v_f).$$

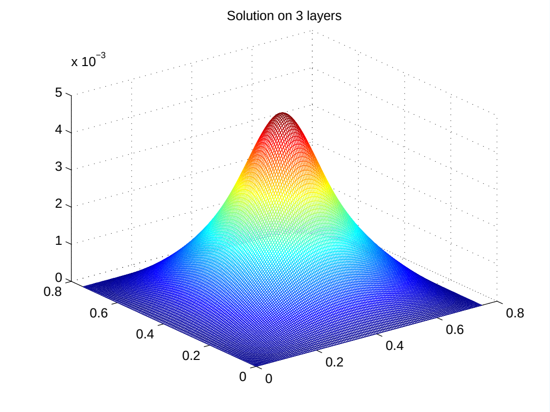
Again we find an approximation by restricting to patches and discretizing the subgrid. Typically $\psi_i = \phi_i$.

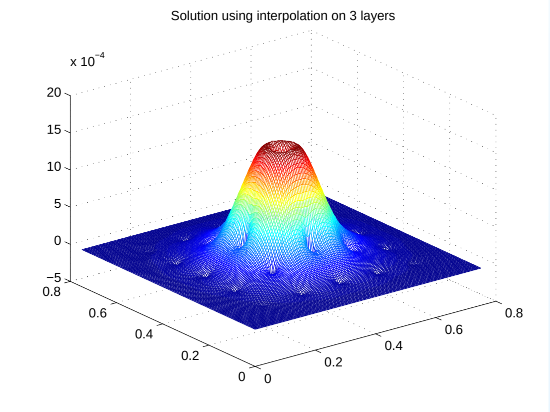
Simple Observation About Decay in $\mathcal{V}_f$ (Fourier)

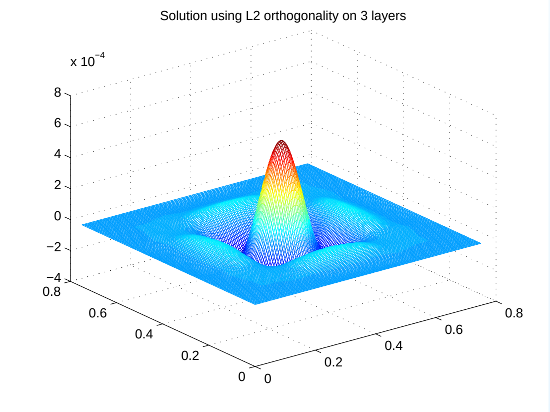
Consider,

$$-\Delta u = \varphi_i \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where φ_i has local support in Ω. The weak form reads: find $u \in \mathcal{W}$ s.t., $(\nabla u, \nabla v) = (\varphi_i, v)$ for all $v \in \mathcal{W}$.





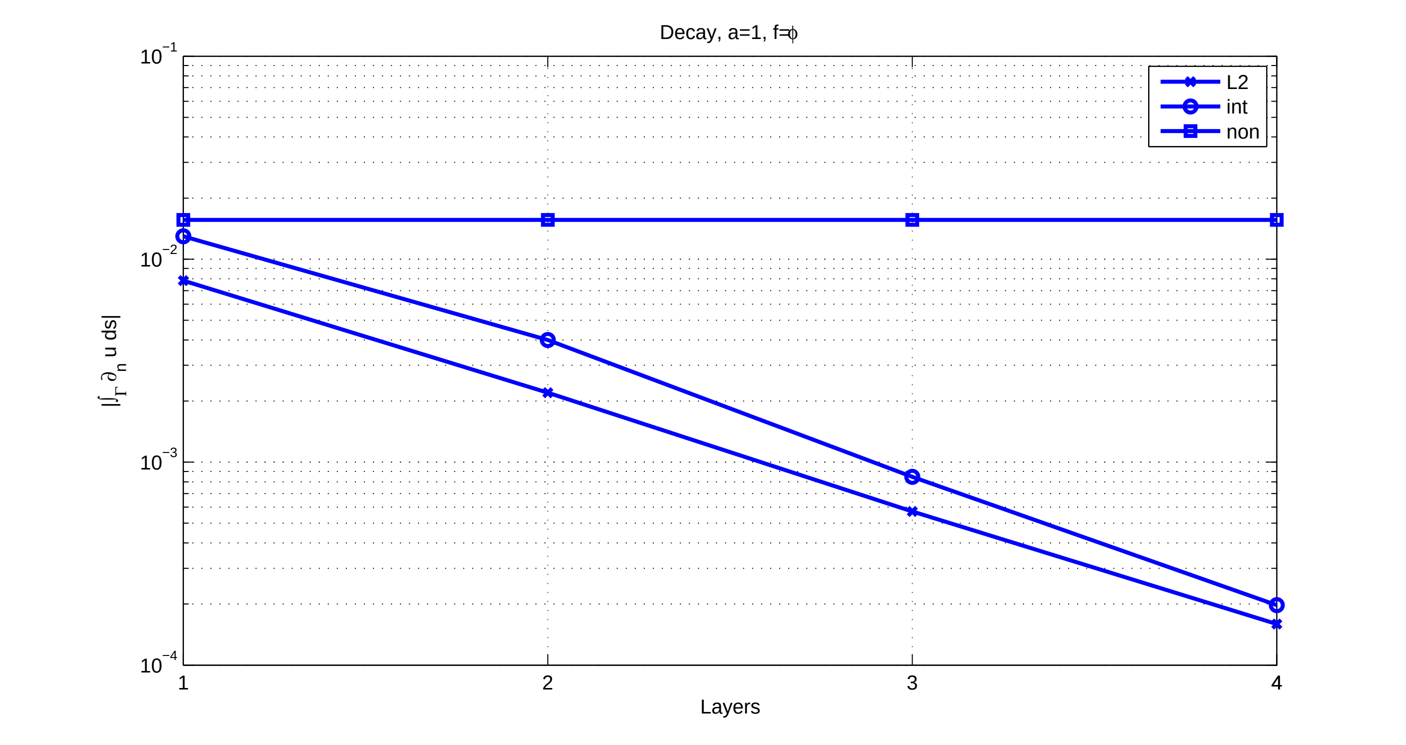


To the left $\mathcal{W} = \mathcal{V}_c \oplus \mathcal{V}_f$, middle $\mathcal{W} = \mathcal{V}_f$ using hierarchical split, and right $\mathcal{W} = \mathcal{V}_f$ using L^2-orthogonal split.

Constraints are realized using Lagrangian multipliers.

Simple Observation About Decay in $\mathcal{V}_f$

Decay of flux integrated over the boundary.



We see exponential decay with respect distance measured in number of coarse elements. This effect gives rapid convergence as the patch size increases.

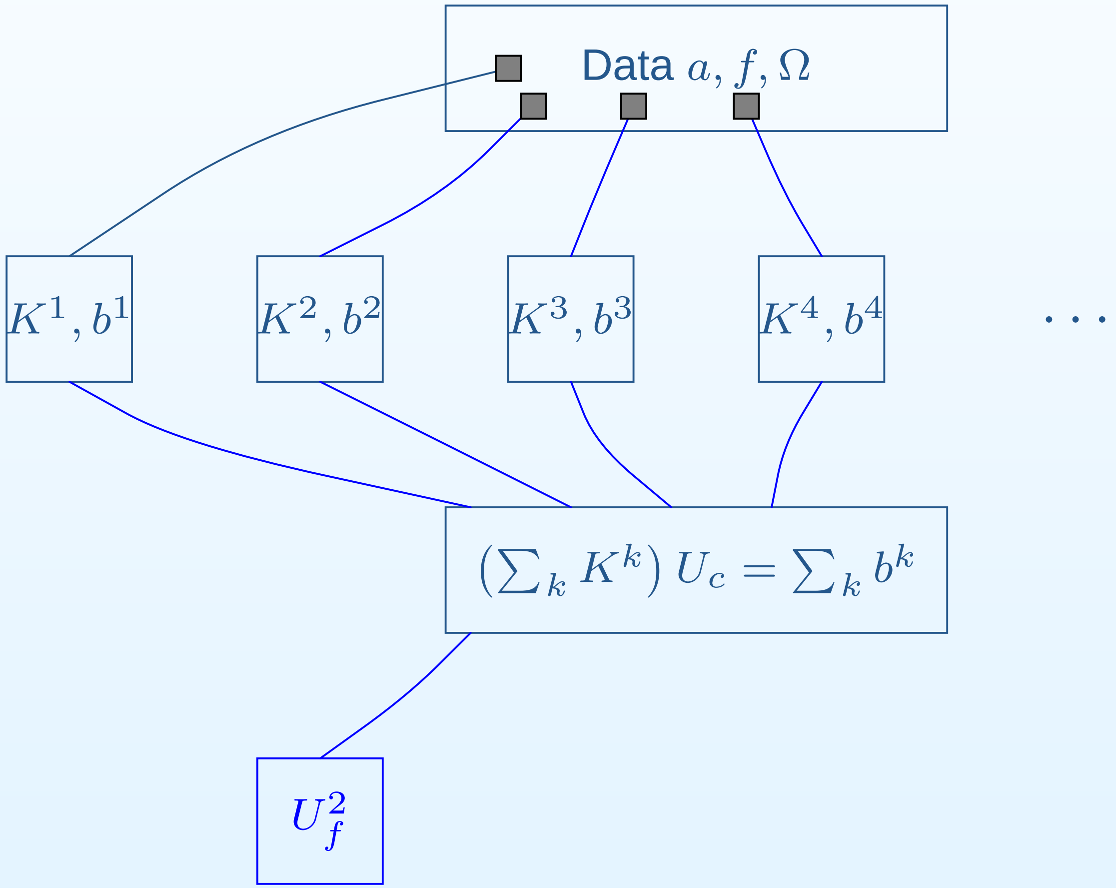
Summary of the Method so Far

- The fine scales in the problem are isolated by introducing splitting operators $I + \mathcal{T}$ and $I + \mathcal{T}^*$ that are orthogonal to $\mathcal{V}_f$ with respect to $a(\cdot, \cdot)$
- The fine scale equations are decoupled $\mathcal{T}\phi_i$, $\mathcal{T}^*\phi_i$, u_f^i
- We note rapid decay which allows us to restrict local solutions to patches ω_i
- We use local computations to modify the coarse scale equation
- Fine scale features can be reconstructed given the coarse scale solution

We will show a posteriori error estimates and adaptive strategies later in the talk.

Parallel Structure

One local problem for each coarse dof, minimal communication.



Data transfer

Local solves

Global solve

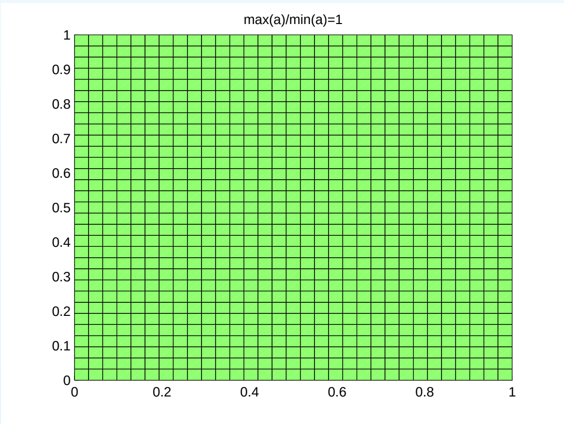
Postprocessing

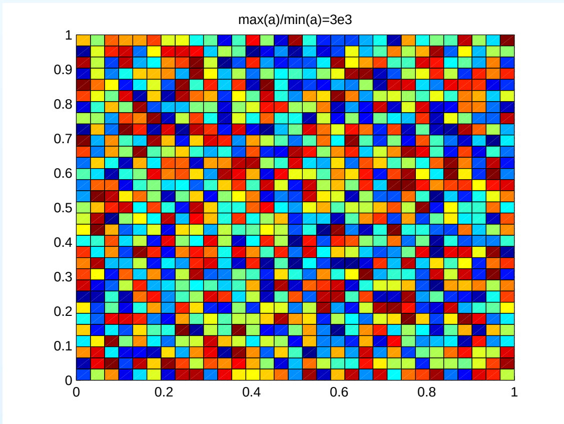
Application to a Mixed Problem

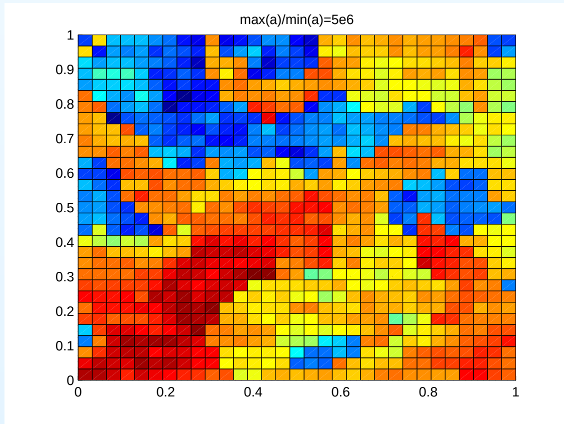
Poisson equation on mixed form:

$$\begin{cases} \frac{1}{a}\sigma - \nabla u = 0 & \text{in } \Omega \\ -\nabla \cdot \sigma = f & \text{in } \Omega \\ n \cdot \sigma = 0 & \text{on } \Gamma \end{cases}$$

where the permeability a is constant, random, or taken from the SPE data set (upperness in log-scale),







Splitting Based on RT-elements

We use lowest order RT basis functions together with piecewise constants.

- Let Π_c be the RT-interpolant onto the space of lowest order RT functions $\mathcal{V}_c$ and P_c be the L^2-projection onto the space of piecewise constants W_c
- We define $\mathcal{W}_f = (I - P_c)\mathcal{W}$, $\mathcal{W} = L^2(\Omega)$
- We define $\mathcal{V}_f = (I - \Pi_c)\mathcal{V}$, $\mathcal{V} = H(\text{div}; \Omega)$
- This means $\sigma_c = \pi_c \sigma \in \mathcal{V}_c \quad u_c = P_c u \in \mathcal{W}_c$.
- Thus we are using an L^2-orthogonal splitting in the scalar variable.

Hierarchical split for lagrangian elements leads to nodal exactness in the coarse solution $U_c \approx \pi_c u$ while here we get exactness of average values on coarse elements $U_c \approx P_c u$.

Fine Scale Equations for $\mathcal{T}^* = \mathcal{T}$

We note that $\mathcal{T} = \{\mathcal{T}_u, \mathcal{T}_\sigma\}$ has two components, that solves: find $\mathcal{T}_u(w_c, v_c) \in \mathcal{W}_f$ and $\mathcal{T}_\sigma(w_c, v_c) \in \mathcal{V}_f$ such that,

$$\left\{ \begin{aligned} (\tfrac{1}{a}\mathcal{T}_\sigma(w_c, v_c), v_f) + (\mathcal{T}_u(w_c, v_c), \nabla \cdot v_f) &= -(\tfrac{1}{a} v_c, v_f) - (w_c, \nabla \cdot v_f) \\ -(\nabla \cdot \mathcal{T}_\sigma(w_c, v_c), w_f) &= (\nabla \cdot v_c, w_f) \end{aligned} \right.$$

for all $w_f \in \mathcal{W}_f$ and $v_f \in \mathcal{V}_f$. We note that red terms vanish due to the construction of the spaces, since for coarse elements K

$$(w_f, \nabla \cdot v_c) = \sum_K \nabla \cdot v_c \int_K w_f \, dx = 0,$$

$$(w_c, \nabla \cdot v_f) = \sum_K w_c \int_K \nabla \cdot v_f \, dx = \sum_K w_c \int_{\partial K} n \cdot v_f \, ds = 0.$$

This means that $\mathcal{T}_u(w_c, v_c) = \mathcal{T}_u v_c$ and $\mathcal{T}_\sigma(w_c, v_c) = \mathcal{T}_\sigma v_c$.

Coarse Scale Equation

Find $\sigma_c \in \mathcal{V}_c$, $\sigma_f \in \mathcal{V}_f$, $u_c \in \mathcal{W}_c$, and $u_f \in \mathcal{W}_f$ such that,

$$\begin{cases} (\frac{1}{a}(\sigma_c + \mathcal{T}_\sigma \sigma_c), v_c + \mathcal{T}_\sigma v_c) + (u_c, \nabla \cdot v_c) & = -(\frac{1}{a}\sigma_f, v_c + \mathcal{T}_\sigma v_c) \\ -(\nabla \cdot \sigma_c, w_c) & = (f, w_c) \\ (\frac{1}{a}\sigma_f, v_f) + (u_f, \nabla \cdot v_f) & = 0 \\ -(\nabla \cdot \sigma_f, w_f) & = (f, w_f) \end{cases}$$

for all $v_c \in \mathcal{V}_c$, $v_f \in \mathcal{V}_f$, $w_c \in \mathcal{W}_c$, and $w_f \in \mathcal{W}_f$.

Again we have used

- $(w_f, \nabla \cdot v_c) = (w_c, \nabla \cdot w_f) = 0$
- $(\nabla \cdot \mathcal{T}_\sigma v_c, w_f) = 0$
- $(f, \mathcal{T}_u v_c) = -(\frac{1}{a}\sigma_f, v_c + \mathcal{T}_\sigma v_c)$

Approximate Fine Scales

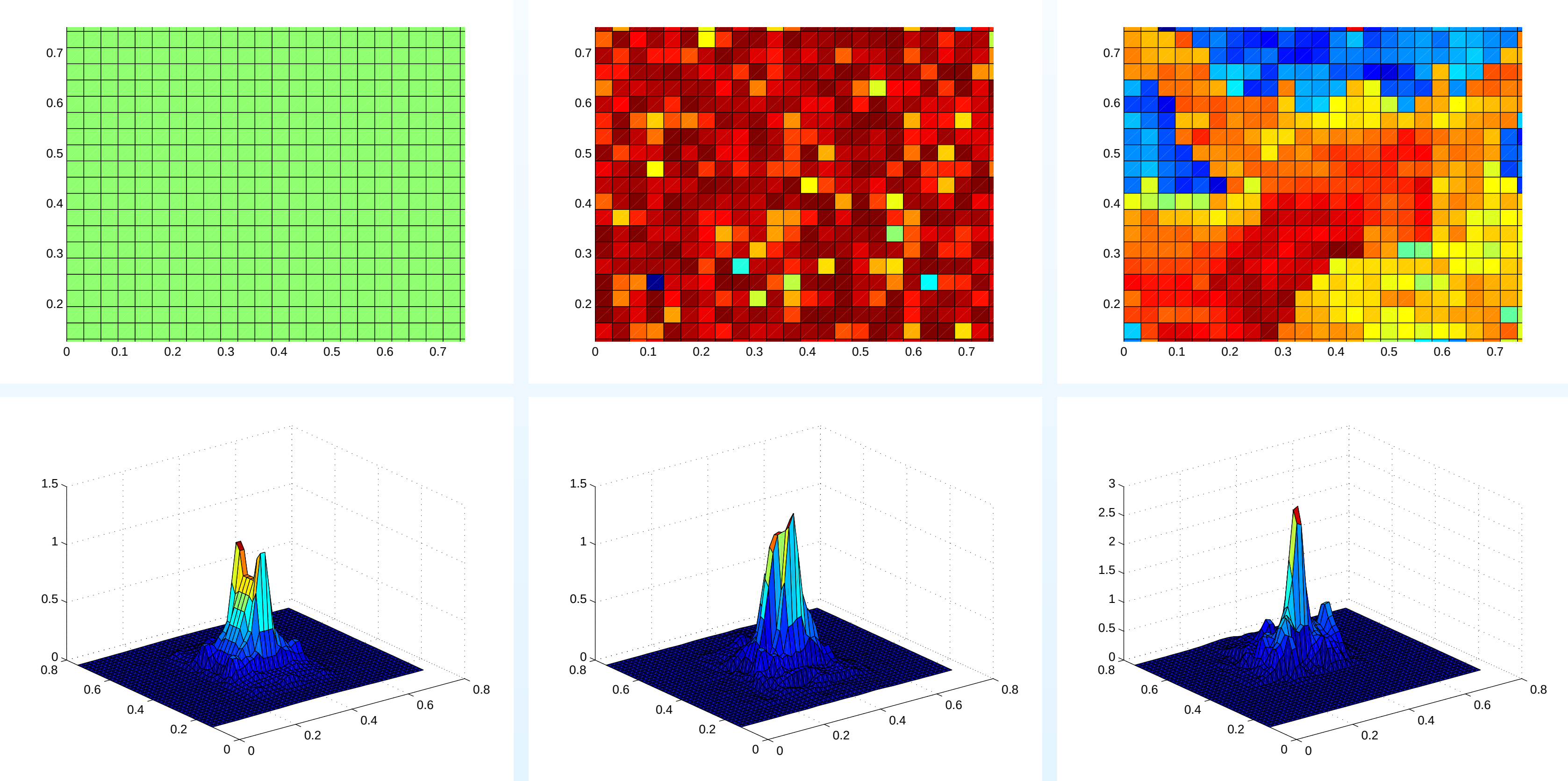
$$\begin{cases} (\frac{1}{a}\mathcal{T}_\sigma v_c, v_f) + (\mathcal{T}_u v_c, \nabla \cdot v_f) = -(\frac{1}{a} v_c, v_f) \\ \qquad\qquad\qquad\qquad\qquad -(\nabla \cdot \mathcal{T}_\sigma v_c, w_f) = 0 \end{cases}$$

- We apply the same idea as in the abstract framework
- Note that $\mathcal{T}_\sigma v_c = \sum_i v_c^i \mathcal{T}_\sigma \phi_i$, here ϕ_i are the Raviart-Thomas basis functions
- Solve the local problem driven by the basis functions (one problem for each basis function)
- Localize by restricting the problems to patches, apply homogeneous Neumann conditions
- Discretize using a suitable subgrid

The same technique is used for $\{u_f, \sigma_f\}$.

Example of Local Solutions $\tilde{\mathcal{T}}_\sigma \phi_i$

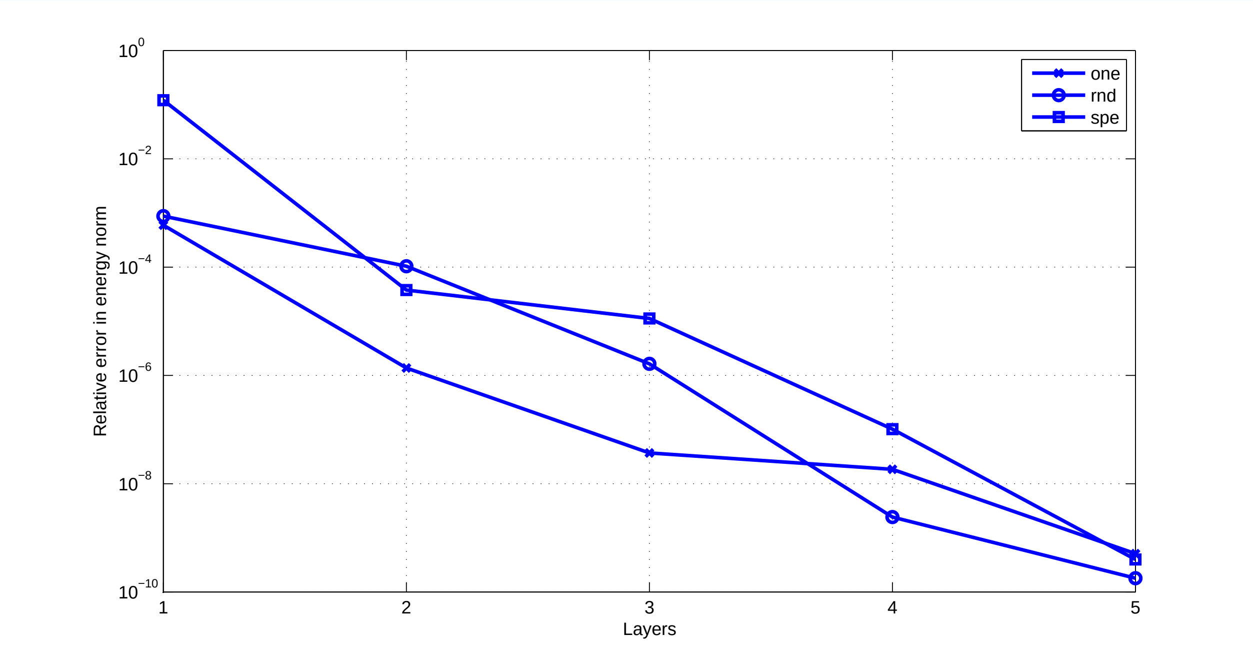
$$\left\{ \begin{aligned} (\tfrac{1}{a}\tilde{\mathcal{T}}_\sigma \phi_i, v_f) + (\tilde{\mathcal{T}}_u \phi_i, \nabla \cdot v_f) &= -(\tfrac{1}{a}\phi_i, v_f) \\ -(\nabla \cdot \tilde{\mathcal{T}}_\sigma \phi_i, w_f) &= 0. \end{aligned} \right.$$



We use 3 layer patches and plot absolute value of the flux $|\tilde{\mathcal{T}}_\sigma \phi_i|$.

Example of Convergence

- Reference mesh has 32×32 elements
- The coarse mesh has 8×8 elements.
- We let $f = 1$ lower left corner and $f = -1$ in upper right, otherwise $f = 0$.



Error compared to reference solution.

Adaptive Multiscale Method

The adaptive version of the method (sometimes referred to as AVMS) builds on the following ingredients:

- Error estimation framework
- Adaptive strategy for tuning of critical discretization parameters

The method is designed so that:

$$\boxed{\text{error} \to 0 \text{ when } h \to 0 \text{ and } L \to \infty}$$

- A priori error estimates in progress.
- To circumvent difficulties with choosing discretization parameters h and L we use an adaptive algorithm based on a posteriori error estimates

A Posteriori Error Estimate (Conv.-Diff.-Reac.)

The following energy norm bound holds

$$\|u - U\|_a^2 = \|\sqrt{a}\nabla(u-U)\|^2 \leq C_a \sum_i \left(R_{\omega_i}^2 + R_{\partial\omega_i}^2\right)$$

where

$$R_{\omega_i}^2 = \|h(\phi_i f - U_c^i \mathcal{L}(\phi_i + \tilde{\mathcal{T}}_\sigma \phi_i) - \mathcal{L} U_f^i)\|_{\omega_i}^2$$
$$+ \sum_{K \in \omega_i} \|h^{1/2}[a\partial_n(U_c^i(\phi_i + \tilde{\mathcal{T}}\phi_i) + U_f^i)]\|_{\partial K \setminus \partial\omega_i}^2$$
$$R_{\partial\omega_i}^2 = \|h^{1/2} a\partial_n(U_{c,i}(\phi_i + \tilde{\mathcal{T}}\phi_i) + U_f^i)\|_{\partial\omega_i \setminus \Gamma}^2,$$

Similar linear functional estimates have also been derived using a dual problem. Note that h and the decay of $\tilde{\mathcal{T}}\phi_i$ and U_f^i determines the error.

A Posteriori Error Estimate (Poisson, mixed)

The following energy norm bound holds

$$\|\frac{1}{\sqrt{a}}(\sigma - \Sigma)\|^2 \le C_a \sum_i \left(R_{\omega_i}^2 + R_{\partial\omega_i}^2\right)$$

where

$$R_{\omega_i}^2 = \|\frac{1}{a}(\Sigma_c^i(\phi_i + \tilde{\mathcal{T}}_\sigma \phi_i) + \Sigma_f^i) - \nabla U_f^{i,*}\|_{\omega_i}^2$$
$$+ \|\frac{h}{a}(f\psi_i + \nabla\cdot(\Sigma_c^i(\phi_i + \tilde{\mathcal{T}}_\sigma\phi_i) + \Sigma_f^i))\|_{\omega_i}^2 + \sum_{K\in\omega_i} \|h^{-1/2}[U_f^{i,*}]\|_{\partial K}^2$$

$$R_{\partial\omega_i}^2 = \|h^{-1/2} U_f^{i,*}\|_{\partial\omega_i\setminus\Gamma}^2$$

U^* is a post processed version (Lovadina and Stenberg 06) of U, $C_a \sim \|\sqrt{a}\|_{L^\infty(\omega_i)}$.

Adaptive Strategy

We have the error bound

$$\|\sigma - \Sigma\|_a^2 \leq C_a \sum_i \left(R_{\omega_i}^2 + R_{\partial\omega_i}^2\right)$$

1. Let $h = H/2$ and $L = 1$ for all i.
2. Compute the solution $\{U, \Sigma\}$.
3. Calculate residuals for each coarse RT basis functions.
4. Mark large entries.
5. For marked entries $R_{\omega_i}^2$ let $h := h/2$.
6. For marked entries $R_{\partial\omega_i}^2$ let $L := L + 1$.
7. Return to 1 or stop if estimators are small enough.

Application in Oil Reservoir Simulation

We seek the water saturation s (oil is $1 - s$) that solves the system of a pressure and a transport equation,

$$\boxed{\begin{aligned} \frac{1}{a\lambda(s)}\sigma - \nabla u = 0 &\quad \text{in } \Omega, \\ -\nabla \cdot \sigma = q &\quad \text{in } \Omega, \\ n \cdot \sigma = 0 &\quad \text{on } \Gamma, \end{aligned}}$$

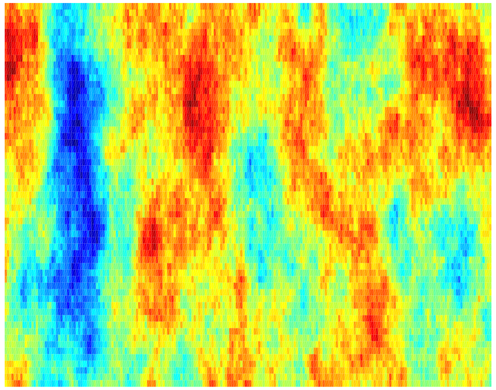
$$\dot{s} + \sigma \cdot \nabla f(s) = 0,$$

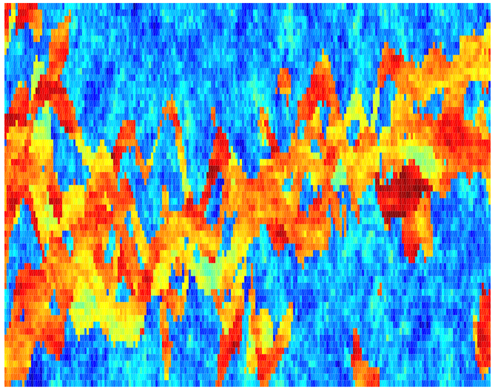
$f(s)$ is fractional flow function, $\lambda(s)$ is total mobility, and q is a source term.

This is a simple model of two phase flow. Note the two way coupling, $\lambda(s)$ is one except at the water front.

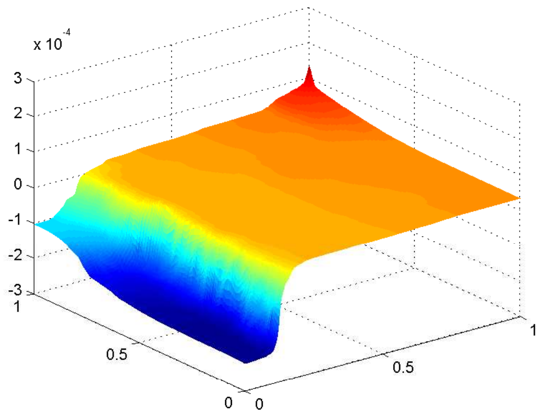
Application in Oil Reservoir Simulation

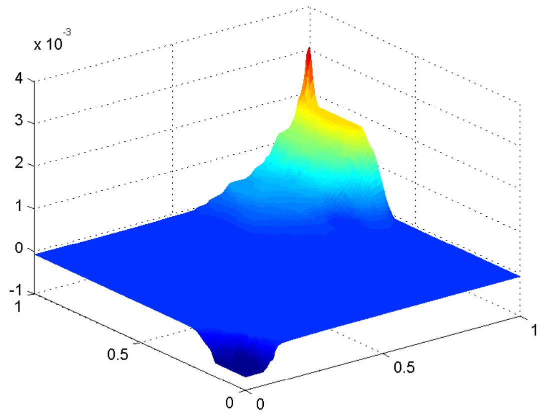
Layer 1 and 50 in the SPE comparative sol. proj. (log scale).





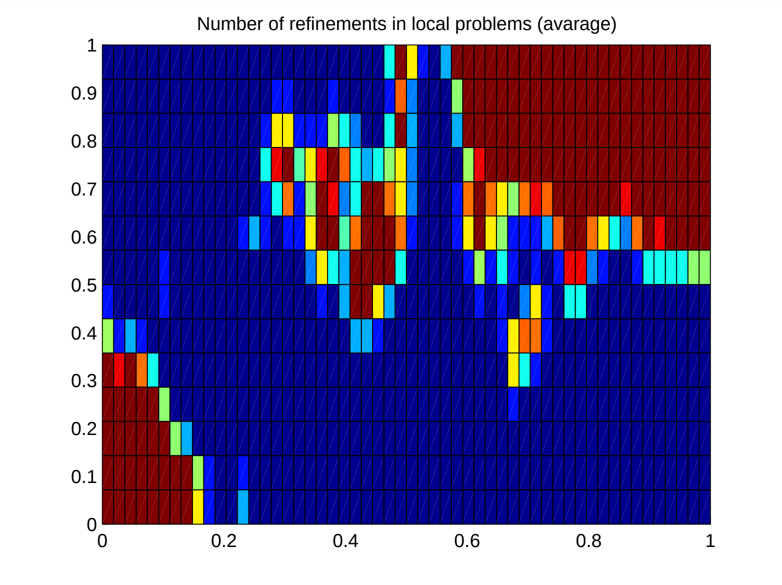
Plot of the sol. (pressure), $q = 1$ upper right $q = -1$ lower left.

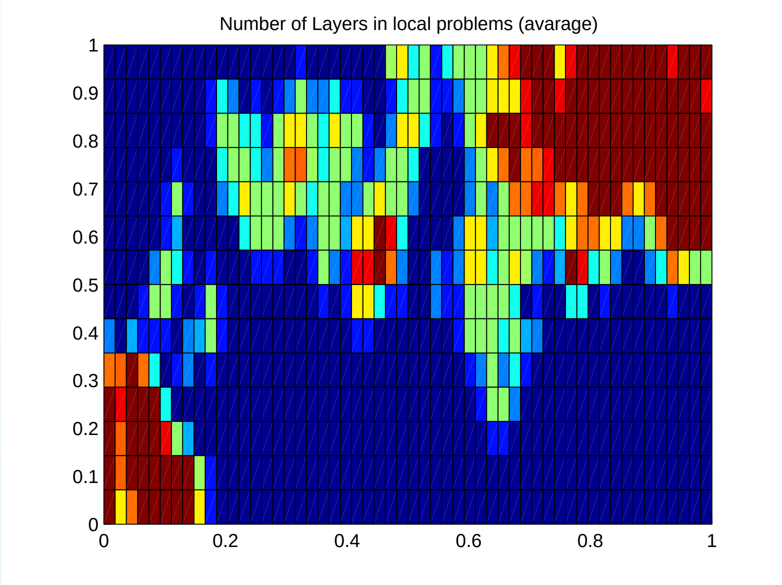




Refinements and layers SPE50

We use 55×15 coarse elements and a reference mesh with 440×120 elements.

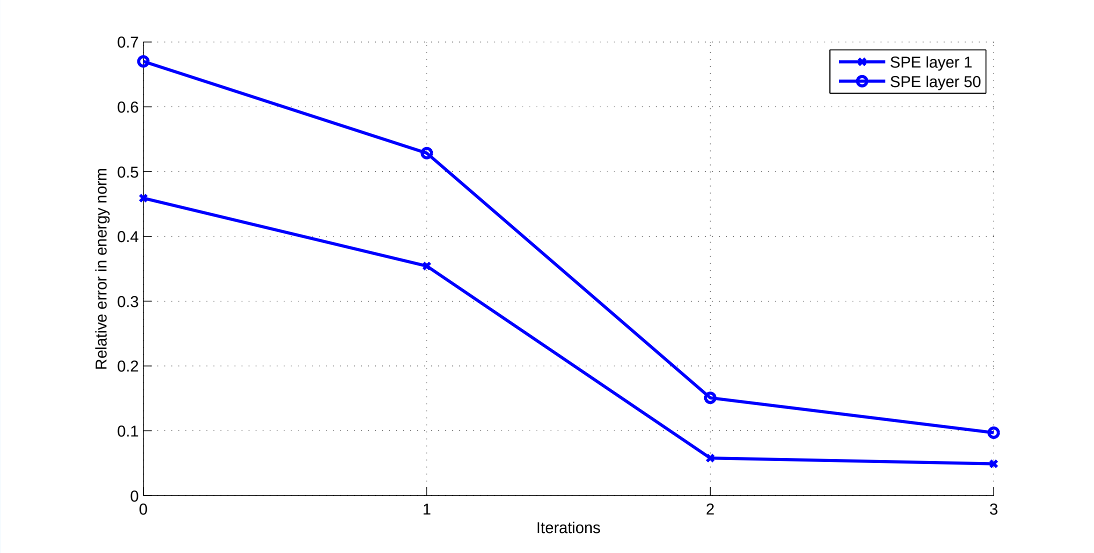




We start the adaptive algorithm with one refinement and one layer in all local problems. After three iterations in the algorithm marking 30%.

Convergence of Adaptive Algorithm

We compare error in energy norm with reference solution.

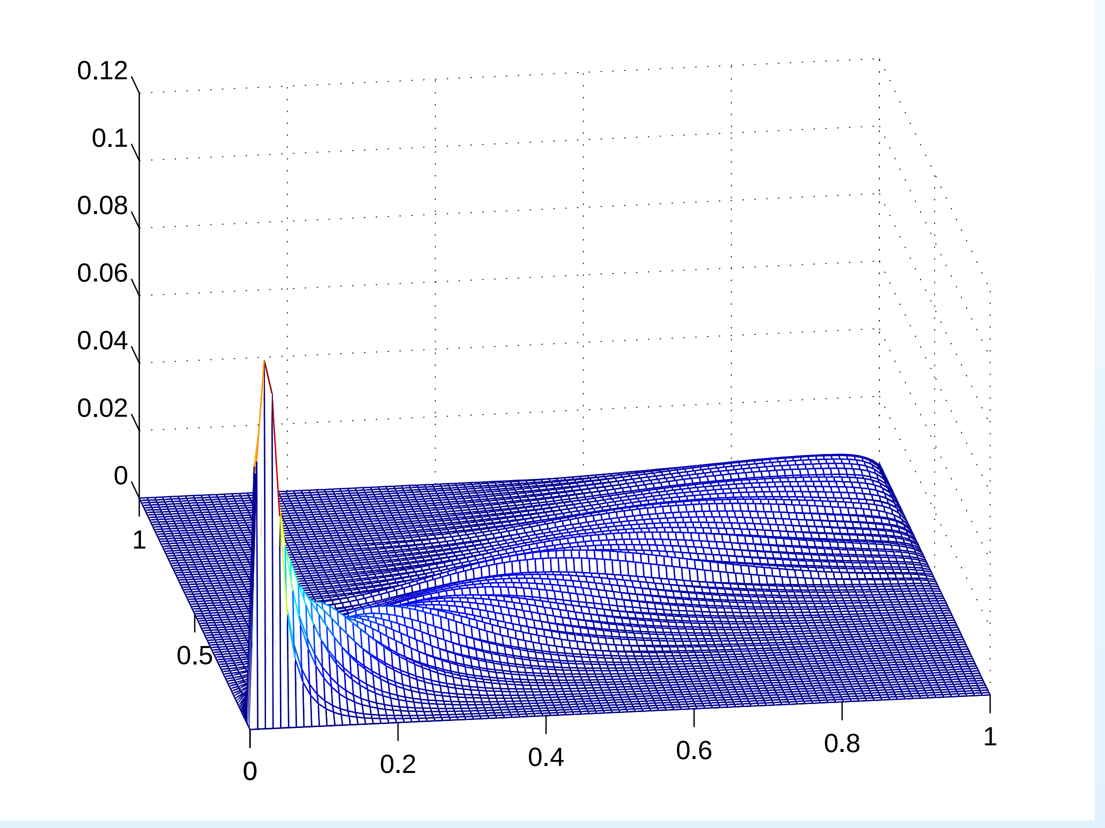


- Critical areas are found
- A majority of the patches uses one layer and one refinement.
- As the water front travels only local problems at the front need to be recomputed.

Convection Dominated Problem

$$\mathcal{L}u = -\epsilon\triangle u + \nabla \cdot (bu) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\epsilon = 0.01$, $f = 1$ lower left corner, $b = [b_x, b_x]$, b_x oscillates between 0.01 and 1.



Split Between Coarse and Fine Scales

- We let π_c be the interpolant and let $\mathcal{V}_f = (1 - \pi_c)\mathcal{V}$, i.e. an hierarchical split
- Let $a(v, w) = (\mathcal{L}v, w)$ for all $v, w \in \mathcal{V}$. Then we can define the fine scale equations as,

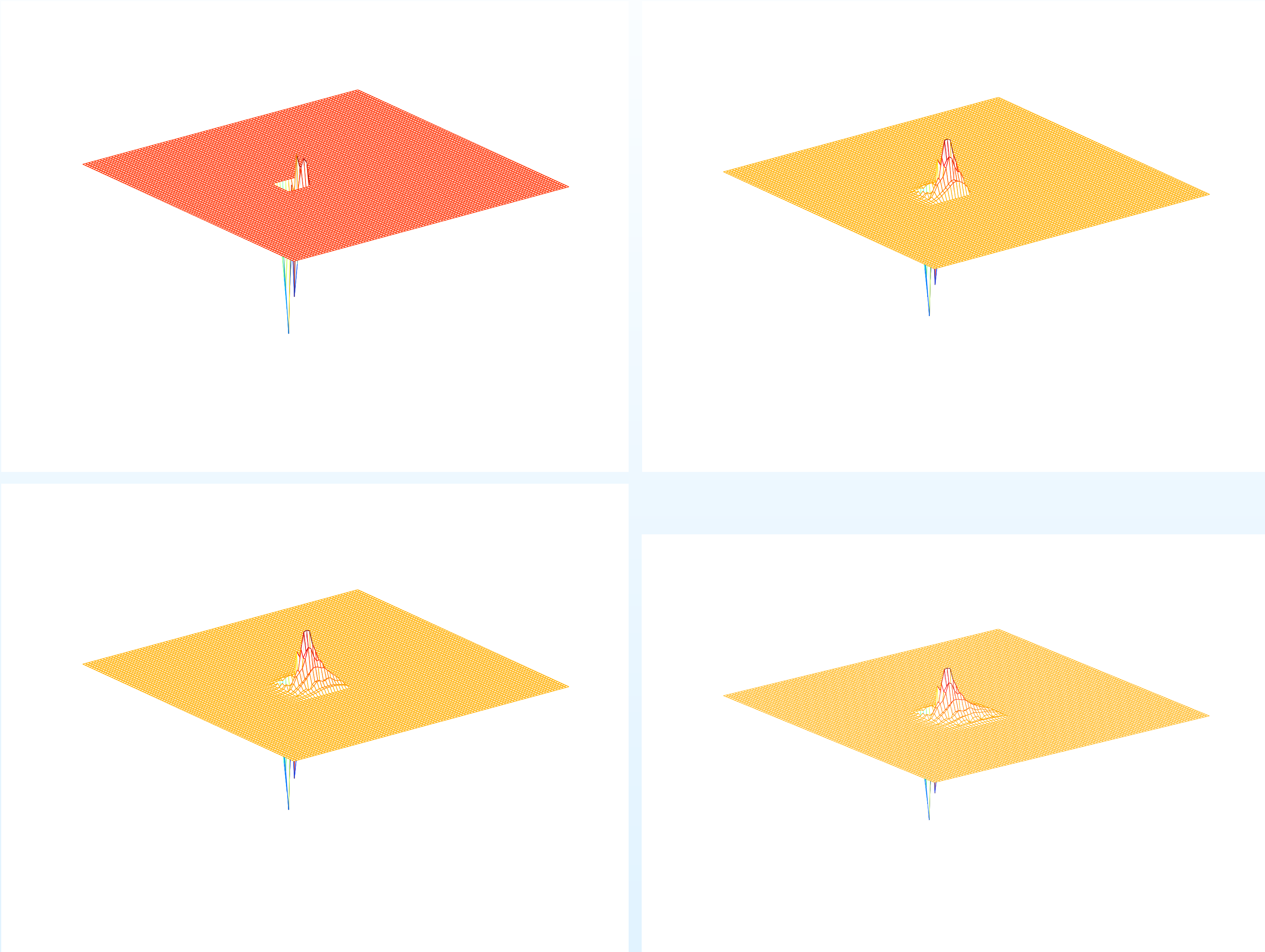
$$a(\mathcal{T}\phi_i, v_f) = -a(\phi_i, v_f),$$

$$a(v_f, \mathcal{T}^*\phi_i) = -a(v_f, \phi_i),$$

$$a(u_f^i, v_f) = l(\phi_i v_f),$$

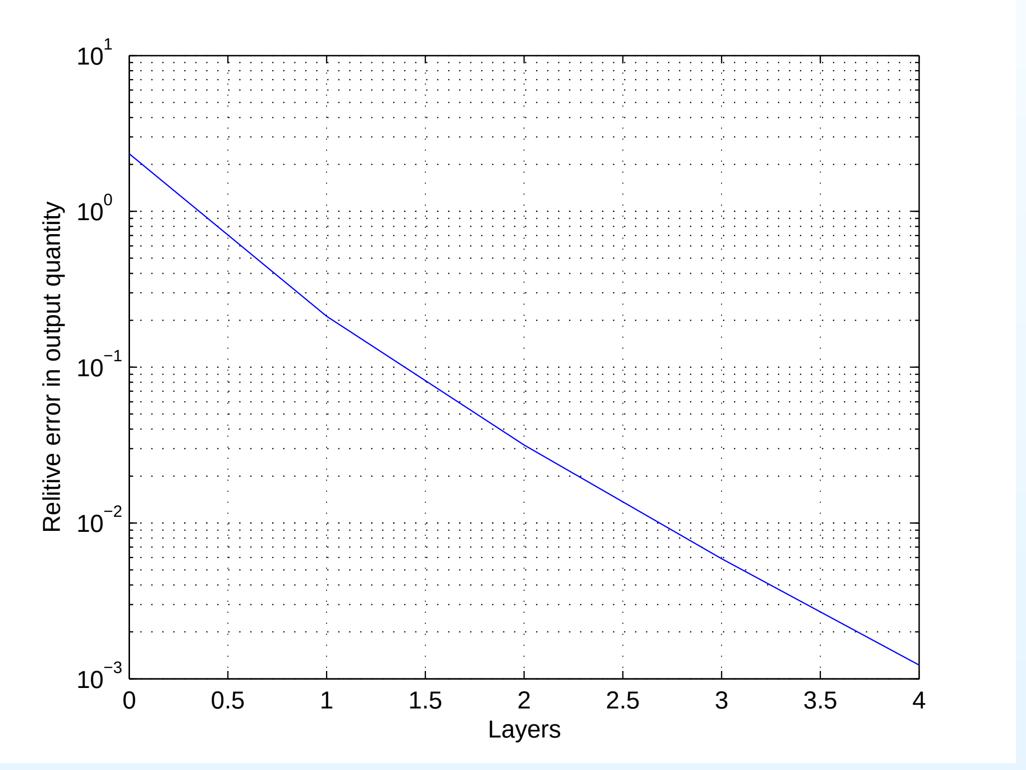
  for all $v_f \in \mathcal{V}_f$ and $i \in \mathcal{N}$ (coarse nodes).
- In numerics we so far only have tested $\mathcal{T}^* = 0$.

Solutions to Local Problems $U_c^i \tilde{\mathcal{T}} \phi_i + U_f^i$ (note angle)



Error in Multiscale Solution

Let $H = 1/24$, $h = H/4$ and study relative error $(U - U_{\text{ref}}, 1)/(U_{\text{ref}}, 1)$ compared to reference solution.



We observe exponential decay (slower). Note that the error using standard Galerkin on the coarse mesh is very high.

Summary

The adaptive variational multiscale method (AVMS) provides:

- Systematic technique for construction of a computable approximation of the fine scale part of the solution using decoupled localized subgrid problems.
- A posteriori error estimation framework (also for goal functionals)
- Adaptive algorithms for automatic tuning of critical discretization parameters
- Its applicable to a range of equations (only linear at this point)

The decay in $\mathcal{V}_f$ together with the adaptive strategy makes the method efficient.

Future Work

- Scale up numerics, parallel code, 3D.
- A priori error analysis, capture decay.
- More then two scales.
- Use Discontinuous Galerkin with L^2 orthogonal split between the scales.
- Multiscale approach to the coupled transport-pressure equation. (Time dependent problems Nordbotten 09)
- Tests on more realistic data, compare with other methods.