ALMOST EQUAL SUMMANDS IN WARING'S PROBLEM WITH SHIFTS

KIRSTI D. BIGGS

Abstract. A result of Wright from 1937 shows that there are arbitrarily large natural numbers which cannot be represented as sums of s kth powers of natural numbers which are constrained to lie within a narrow region. We show that the analogue of this result holds in the shifted version of Waring's problem.

Waring's problem with shifts asks whether, given $k, s \in \mathbb{N}$ and $\eta \in (0, 1]$, along with shifts $\theta_1, \ldots, \theta_s \in (0, 1)$ with $\theta_1 \notin \mathbb{Q}$, we can find solutions in natural numbers x_i to the following inequality, for all sufficiently large $\tau \in \mathbb{R}$:

$$\left|(x_1 - \theta_1)^k + \ldots + (x_s - \theta_s)^k - \tau\right| < \eta. \tag{1}$$

This problem was originally studied by Chow in [3]. In [1], the author showed that an asymptotic formula for the number of solutions to (1) can be obtained whenever $k \geq 4$ and $s \geq k^2 + (3k - 1)/4$. The corresponding result for $k = 3$ and $s \geq 11$ is due to Chow in [2].

An interesting variant is to consider solutions of (1) subject to the additional condition

$$\left|x_i - (\tau/s)^{1/k}\right| < y(\tau), \quad (1 \leq i \leq s),$$

for some function $y(\tau)$. In other words, we are confining our variables to be within a small distance of the "average" value.

In 1937, Wright studied this question in the setting of the classical version of Waring's problem, and proved in [6] that there exist arbitrarily large natural numbers n which cannot be represented as sums of s kth powers of natural numbers x_i satisfying the condition $\left|x_i^k - n/s\right| < n^{1-1/2k}\phi(n)$ for $1 \leq i \leq s$, no matter how large s is taken. Here, $\phi(n)$ is a function satisfying $\phi(n) \to 0$ as $n \to \infty$.

In [4] and [5], Daemen showed that if we widen the permitted region slightly, we can once again guarantee solutions in the classical case. Specifically, he obtains a lower bound on the number of solutions under the condition

$$\left|x_i - (n/s)^{1/k}\right| < cn^{1/2k}, \quad (1 \leq i \leq s),$$

for a suitably large constant c, and an asymptotic formula under the condition

$$\left|x_i - (n/s)^{1/k}\right| < n^{1/2k+\epsilon}, \quad (1 \leq i \leq s).$$

2010 *Mathematics Subject Classification.* 11D75, 11P05.
Key words and phrases. Waring's problem, Diophantine inequalities.
The author is supported by EPSRC Doctoral Training Partnership EP/M507994/1.

In this note, we show that (a slight strengthening of) Wright's result remains true in the shifted case. Specifically, we prove the following.

Theorem 1. *Let $s, k \geq 2$ be natural numbers. Fix $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s) \in (0,1)^s$, and let $c, c' > 0$ be suitably small constants which may depend on s, k and $\boldsymbol{\theta}$. There exist arbitrarily large values of $\tau \in \mathbb{R}$ which cannot be approximated in the form (1), with $0 < \eta < c\tau^{1-2/k}$, subject to the additional condition that $\left|x_i - (\tau/s)^{1/k}\right| < c'\tau^{1/2k}$ for $1 \leq i \leq s$.*

Proof. This follows the structure of Wright's proof in [6], with minor adjustments to take into account the shifts present in our problem. As such, for $m \in \mathbb{N}$, we let $\tau_m = sm^k + km^{k-1}(s - \sum_{i=1}^s \theta_i)$, and we note that $\tau_m \to \infty$ as $m \to \infty$. Throughout the proof, we allow $c_1, c_2, \ldots$ to denote positive constants which do not depend on m, although they may depend on the fixed values of $s, k, \boldsymbol{\theta}, c$ and c'. We also note that $\eta < c\tau^{1-2/k}$ implies that $\eta \ll m^{k-2}$.

Suppose τ_m satisfies (1) with $0 < \eta < c\tau_m^{1-2/k}$ and $\left|x_i - (\tau_m/s)^{1/k}\right| < c'\tau_m^{1/2k}$ for $1 \leq i \leq s$. We write $x_i = m + a_i$, and observe that

$$\begin{aligned} m^{k-1}\left|a_i\right| &= m^{k-1}\left|x_i - m\right| \\ &\leq m^{k-1}\Big(\left|x_i - (\tau_m/s)^{1/k}\right| + \left|(\tau_m/s)^{1/k} - m\right|\Big) \\ &\leq c'm^{k-1}\tau_m^{1/2k} + \left|\tau_m/s - m^k\right|. \end{aligned}$$

Using the definition of τ_m, we obtain

$$m^{k-1}\left|a_i\right| \leq c_1 m^{k-1}m^{1/2} + km^{k-1}(1 - s^{-1}\sum_{i=1}^s \theta_i),$$

and therefore $|a_i| \leq c_2 m^{1/2}$ for $1 \leq i \leq s$. Expanding (1), we see that

$$\begin{aligned} \eta &> \left|\sum_{i=1}^s (x_i - \theta_i)^k - \tau_m\right| \\ &= \left|\sum_{i=1}^s (m + a_i - \theta_i)^k - \Big(sm^k + km^{k-1}(s - \sum_{i=1}^s \theta_i)\Big)\right| \qquad (2) \\ &\geq km^{k-1}\left|s - \sum_{i=1}^s a_i\right| - \left|\sum_{j=2}^k \binom{k}{j} m^{k-j}\sum_{i=1}^s (a_i - \theta_i)^j\right|. \end{aligned}$$

Rearranging, this gives

$$\begin{aligned} \left|s - \sum_{i=1}^s a_i\right| &< \eta k^{-1}m^{1-k} + \left|\sum_{j=2}^k \binom{k}{j} k^{-1}m^{1-j}\sum_{i=1}^s (a_i - \theta_i)^j\right| \\ &\leq \eta k^{-1}m^{1-k} + \sum_{j=2}^k \binom{k}{j} k^{-1}m^{1-j}s(c_3 m^{1/2})^j \\ &\leq c_4. \end{aligned}$$

By choosing our original c, c' to be sufficiently small, we may conclude that $c_4 \leq 1$, which implies that $\sum_{i=1}^{s} a_i = s$. Substituting this back into (2), when $k = 2$ we obtain

$$\eta > \binom{k}{2} m^{k-2} \sum_{i=1}^{s} (a_i - \theta_i)^2,$$

and consequently

$$\sum_{i=1}^{s} (a_i - \theta_i)^2 < c_5,$$

which is a contradiction if we choose c, c' sufficiently small, since we have $\sum_{i=1}^{s} (a_i - \theta_i)^2 \gg 1$.

When $k \geq 3$, we obtain

$$\begin{aligned} \eta &> \left| \sum_{j=2}^{k} \binom{k}{j} m^{k-j} \sum_{i=1}^{s} (a_i - \theta_i)^j \right| \\ &\geq \binom{k}{2} m^{k-2} \sum_{i=1}^{s} (a_i - \theta_i)^2 - \left| \sum_{j=3}^{k} \binom{k}{j} m^{k-j} \sum_{i=1}^{s} (a_i - \theta_i)^j \right|. \end{aligned}$$

Consequently,

$$\begin{aligned} \binom{k}{2} m^{k-2} \sum_{i=1}^{s} (a_i - \theta_i)^2 &< \eta + \sum_{j=3}^{k} \binom{k}{j} m^{k-j} \sum_{i=1}^{s} |a_i - \theta_i|^j \\ &\leq \eta + \sum_{j=3}^{k} \binom{k}{j} m^{k-j} (c_3 m^{1/2})^{j-2} \sum_{i=1}^{s} (a_i - \theta_i)^2 \\ &\leq \eta + c_6 m^{k-5/2} \sum_{i=1}^{s} (a_i - \theta_i)^2, \end{aligned}$$

and so

$$\sum_{i=1}^{s} (a_i - \theta_i)^2 < c_7 + c_8 m^{-1/2} \sum_{i=1}^{s} (a_i - \theta_i)^2,$$

which is again a contradiction when m is large.

We conclude that for all sufficiently large m, it is impossible to approximate τ_m in the manner claimed. This completes the proof. □

Corollary 2. *For $s, k \geq 2$ natural numbers, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s) \in (0,1)^s$, and suitably small constants $C, C' > 0$, there exist arbitrarily wide gaps between real numbers τ for which the system*

$$\begin{aligned} \left|(x_1 - \theta_1)^k + \ldots + (x_s - \theta_s)^k - \tau\right| &< C\tau^{1-2/k} \\ \left|x_i - (\tau/s)^{1/k}\right| &< C'\tau^{1/2k}, \quad (1 \leq i \leq s) \end{aligned} \tag{3}$$

has a solution in natural numbers $x_1, \ldots, x_s$.

Proof. By Theorem 1, we fix $\tau_0 \in \mathbb{R}$ such that there is no solution in natural numbers $x_1, \ldots, x_s$ to $\left|(x_1-\theta_1)^k + \ldots + (x_s-\theta_s)^k - \tau_0\right| < c\tau_0^{1-2/k}$ with $\left|x_i - (\tau_0/s)^{1/k}\right| < c'\tau_0^{1/2k}$ for $1 \le i \le s$.

Let $0 < \delta \le C_0\tau_0^{1-2/k}$ for some $C_0 > 0$, and let $\tau \in [\tau_0-\delta, \tau_0+\delta]$. Let $C, C' > 0$ be suitably small constants depending on c, c' and C_0 to be chosen later, and suppose that $x_1 \ldots, x_s \in \mathbb{N}$ are such that (3) is satisfied.

We have

$$\begin{aligned}\left|(\tau/s)^{1/k} - (\tau_0/s)^{1/k}\right| &\le s^{-1/k}\left|(\tau_0-\delta)^{1/k} - \tau_0^{1/k}\right| \\ &\le C_1\delta\tau_0^{1/k-1},\end{aligned}$$

and consequently

$$\begin{aligned}\left|x_i - (\tau_0/s)^{1/k}\right| &\le \left|x_i - (\tau/s)^{1/k}\right| + \left|(\tau/s)^{1/k} - (\tau_0/s)^{1/k}\right| \\ &< C'\tau^{1/2k} + C_1\delta\tau_0^{1/k-1} \\ &\le C'(\tau_0+\delta)^{1/2k} + C_1C_0\tau_0^{-1/k} \\ &\le C_2\tau_0^{1/2k}.\end{aligned}$$

We also see that

$$\begin{aligned}\left|\sum_{i=1}^{s}(x_i-\theta_i)^k - \tau_0\right| &\le \left|\sum_{i=1}^{s}(x_i-\theta_i)^k - \tau\right| + |\tau-\tau_0| \\ &< C\tau^{1-2/k} + \delta \\ &\le C(\tau_0+\delta)^{1-2/k} + C_0\tau_0^{1-2/k} \\ &\le C_3\tau_0^{1-2/k}.\end{aligned}$$

Choosing C_0, C, C' small enough to ensure that $C_2 \le c'$ and $C_3 \le c$ gives a contradiction to our original choice of τ_0. Consequently, there is no solution to (3) in an interval of radius $\asymp \tau_0^{1-2/k}$ around τ_0. □

The author would like to thank Trevor Wooley for his supervision, and the anonymous referee for useful comments.

References

[1] K. D. Biggs. On the asymptotic formula in Waring's problem with shifts. *J. Number Theory* (2018).
[2] S. Chow. Sums of cubes with shifts. *J. Lond. Math. Soc. (2)* **91** (2015), no. 2, 343–366.
[3] S. Chow. Waring's problem with shifts. *Mathematika* **62** (2016), no. 1, 13–46.
[4] D. Daemen. The asymptotic formula for localized solutions in Waring's problem and approximations to Weyl sums. *Bull. Lond. Math. Soc.* **42** (2010), no. 1, 75–82.
[5] D. Daemen. Localized solutions in Waring's problem: the lower bound. *Acta Arith.* **142** (2010), no. 2, 129–143.
[6] E. M. Wright. The representation of a number as a sum of four 'almost equal' squares. *The Quarterly Journal of Mathematics* **8** (1937), 278–279.

School of Mathematics, University of Bristol, University Walk, Clifton, Bristol, BS8 1TW, United Kingdom

E-mail address: kirsti.biggs@bristol.ac.uk