An Introduction to things $\bar{\partial}$

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Introduction

These are the notes of a series of lectures given at the PCMI summer school 2008. They are intended to serve as an introduction to the weighted L^2-estimates for the $\bar\partial$-equation, by Kodaira, [**12**], Andreotti-Vesentini, [**2**] and, in the most complete form, Hörmander, [**10**].

Three lecture series on this topic was given at the school; this is the first and supposedly the most elementary of them. Therefore we start by a discussion of the one dimensional case, which I believe shows very clearly the main ideas. After that we discuss briefly the functional analytic set up which is needed in the case of higher dimensions. The third lecture presents the geometric notions needed to discuss the $\bar\partial$-equation for forms with values in a line bundle on a manifold, and then gives the fundamental existence theorems and L^2-estimates. The proofs differ a bit from the more common ones, that are based on use of the Kähler identities and the Kodaira-Nakano formula for twisted Laplace operators. Instead we use the so called $\partial\bar\partial$-Bochner-Kodaira method introduced by Siu, which in my opinion is the most elementary approach.

The remaining lectures deal with applications and generalizations of this basic material. Lecture 4 gives some basic facts about Bergman kernels associated to holomorphic line bundles and uses the $\bar\partial$-estimates to deduce a rudimentary asymptotic formula. Lecture 5 introduces singular metrics and generalizes the results of lecture 3 to this setting, leading up to the Demailly-Nadel and Kawamata-Viehveg vanishing theorems. The main topic of lecture 6 is the Ohsawa-Takegoshi extension theorem, which we prove by another application of the $\partial\bar\partial$-Bochner-Kodaira method. The final section discusses briefly one of the main recent applications of the L^2-theory, Siu's theorem on the 'invariance of plurigenera', which we prove following the method of Paun.

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LECTURE 1

The one-dimensional case.

In this chapter we state and prove the basic Hörmander L^2-estimate for the $\bar\partial$-equation in the case of one complex variable. This case contains the two main ideas in the subject, and almost no technical difficulties, so it is a good introduction to the discussion in later chapters.

1. The $\bar\partial$-equation in one variable.

Let Ω be any domain in $\mathbb{C}$, and ϕ any function satisfying

$$\frac{\partial^2\phi}{\partial z\partial\bar z} =: \Delta\phi > 0.$$

The one dimensional version of the theorem we will discuss says that we can solve any inhomogeneous $\bar\partial$-equation

$$\frac{\partial u}{\partial\bar z} = f$$

with a function u satisfying

$$\int |u|^2 e^{-\phi} \le \int \frac{|f|^2}{\Delta\phi} e^{-\phi}.$$

Even this one variable case is a very precise and useful result, and it is quite surprising that it was discovered in several variables first. Moreover, the proof when $n = 1$ is considerably more elementary than the general case.

We begin by giving the problem a dual formulation. Remember that, interpreted in the sense of distributions, the equation $\frac{\partial}{\partial\bar z}u = f$ means precisely that

$$-\int u\frac{\partial}{\partial\bar z}\alpha = \int f\alpha \tag{1.1}$$

for all $\alpha \in C^2_c(\Omega)$. To introduce the weighted L^2-norms of the theorem we substitute $\bar\alpha e^{-\phi}$ for α. The equality (1.1) then says

$$\int u\overline{\bar\partial^*_\phi\alpha}e^{-\phi} = \int f\bar\alpha e^{-\phi}, \tag{1.2}$$

where

$$\bar\partial^*_\phi\alpha =: -e^{\phi}\frac{\partial}{\partial z}(e^{-\phi}\alpha),$$

is the *formal adjoint* of the $\bar\partial$-operator with respect to our weighted scalar product

$$\langle f, g\rangle_\phi = \int f\bar g e^{-\phi}.$$

The following proposition is one of the key ideas in the subject. It reduces the proof of an existence statement to the proof of an inequality.

Proposition 1.1. *Given f there exists a solution, u, to $\frac{\partial}{\partial \bar{z}} u = f$ satisfying*

$$\int |u|^2 e^{-\phi} \leq C, \tag{1.3}$$

if and only if the estimate

$$|\int f\, \bar{\alpha} e^{-\phi}|^2 \leq C \int |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi} \tag{1.4}$$

holds for all $\alpha \in C_c^2(\Omega)$. On the other hand, for a given function $\mu > 0$, (1.4) holds for all f satisfying

$$\int \frac{|f|^2}{\mu} e^{-\phi} \leq C \tag{1.5}$$

if and only if

$$\int \mu |\alpha|^2 e^{-\phi} \leq \int |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi}, \tag{1.6}$$

holds for all $\alpha \in C_c^2(\Omega)$.

Proof: It is clear that if (1.2), and (1.3) hold, then (1.4) follows. Suppose conversely that the inequality (1.4) is true. Let

$$E = \{\bar{\partial}_\phi^* \alpha; \alpha \in C_c^2(\Omega)\},$$

and consider E as a subspace of

$$L^2(e^{-\phi}) = \{g \in L^2_{loc}; \int |g|^2 e^{-\phi} < \infty\}.$$

Define an antilinear functional on E by

$$L(\bar{\partial}_\phi^* \alpha) = \int f \bar{\alpha} e^{-\phi}.$$

The inequality (1.4) then says that L is (well defined and) of norm not exceeding C. By Hahn-Banach's extension theorem L can be extended to an antilinear form on all of $L^2(e^{-\phi})$, with the same norm. The Riesz representation theorem then implies that there is some element,u, in $L^2(e^{-\phi})$, with norm less than C, such that

$$L(g) = \int u \bar{g} e^{-\phi},$$

for all $g \in L^2(e^{-\phi})$. Choosing $g = \bar{\partial}_\phi^* \alpha$, we see that

$$\int u \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} = \int f \bar{\alpha} e^{-\phi},$$

so u solves $\frac{\partial}{\partial \bar{z}} u = f$.

The first part of the proposition is therefore proved. The second part is obvious if μ is identically equal to 1. The general case follows if we write

$$f \cdot \bar{\alpha} = (f/\sqrt{\mu}) \cdot \overline{(\sqrt{\mu}\alpha)}.$$

□

To complete the proof of Hörmander's theorem in the one-dimensional case it is therefore enough to prove an inequality of the form (1.4). This will be accomplished by the following integral identity. The general case of this basic identity (that we will see in chapter 3) is the other key idea in the L^2-theory.

Proposition 1.2. *Let Ω be a domain in $\mathbb{C}$ and let $\phi \in C^2(\Omega)$. Let $\alpha \in C^2_c(\Omega)$. Then*

$$\int \Delta\phi|\alpha|^2 e^{-\phi} + \int |\frac{\partial}{\partial\bar{z}}\alpha|^2 e^{-\phi} = \int |\bar\partial^*_\phi \alpha|^2 e^{-\phi} \tag{1.7}$$

Proof: Since α has compact support we can integrate by parts and get

$$\int |\bar\partial^*_\phi \alpha|^2 e^{-\phi} = \int \bar\partial\bar\partial^*_\phi \alpha \cdot \bar\alpha e^{-\phi}.$$

Next note that

$$\bar\partial^*_\phi \alpha = -\frac{\partial}{\partial z}\alpha + \phi_z \alpha,$$

so that

$$\bar\partial\bar\partial^*_\phi \alpha = -\Delta\alpha + \phi_z \frac{\partial}{\partial\bar{z}}\alpha + \Delta\phi\alpha = \bar\partial^*_\phi \frac{\partial}{\partial\bar{z}}\alpha + \Delta\phi\alpha.$$

Hence

$$\int |\bar\partial^*_\phi \alpha|^2 e^{-\phi} = \int \Delta\phi|\alpha|^2 e^{-\phi} + \int |\frac{\partial}{\partial\bar{z}}\alpha|^2 e^{-\phi}$$

and the proof is complete. □

Combining the last two propositions we now immediately conclude

Theorem 1.3. *Let Ω be a domain in $\mathbb{C}$ and suppose $\phi \in C^2(\Omega)$ satisfies $\Delta\phi \geq 0$. Then, for any f in $L^2_{loc}\Omega$ there is a solution u to $\frac{\partial}{\partial\bar{z}}u = f$ satisfying*

$$\int |u|^2 e^{-\phi} \leq \int \frac{|f|^2}{\Delta\phi} e^{-\phi}.$$

Note that the theorem says two things: the $\bar\partial$-equation can be solved, and there is a good estimate for the solution. If we disregard the first aspect we get as a corollary the following Poincaré type inequality for the $\bar\partial$-operator.

Corollary 1.4. *Suppose $\phi \in C^2(\Omega)$ satisfies $\Delta\phi \geq 0$. Let u be a C^1 function in a domain Ω such that*

$$\int_\Omega u\bar{h}e^{-\phi} = 0$$

for any holomorphic function h in $L^2(\Omega, e^{-\phi})$. Then

$$\int_\Omega |u|^2 e^{-\phi} \leq \int_\Omega \frac{|\bar\partial u|^2}{\Delta\phi} e^{-\phi}. \tag{1.8}$$

Proof: The previous theorem says that the equation

$$\bar\partial v = \bar\partial u$$

has *some* solution satisfying (0.8). But, the condition that u is orthogonal to all holomorphic functions means that u is the minimal solution to this equation. Hence u satisfies the estimate as well which is what the corollary claims. □

2. An alternative proof of the basic identity.

We have proved the basic identity using integration by parts. We will here give an alternative proof, akin to the classical *Bochner method* from differential geometry, which gives a more general statement. This method can be generalized in a rather surprising way to the case of higher dimensions and complex manifolds, and we will get back to it in Chapter 3.

The idea is to calculate the Laplacian of the weighted norm of a test function α. Again we use the complex Laplacian

$$\Delta = \frac{\partial^2}{\partial z \partial \bar{z}}.$$

To compute $\Delta|\alpha|^2 e^{-\phi}$ we will use the product rule in the form

$$\frac{\partial}{\partial z}(u \cdot \bar{v} e^{-\phi}) = \delta u \cdot \bar{v} e^{-\phi} + u \cdot \overline{\frac{\partial}{\partial \bar{z}} v} e^{-\phi},$$

where

$$\delta := e^{\phi} \frac{\partial}{\partial z} e^{-\phi} = \frac{\partial}{\partial z} - \phi_z = -\bar{\partial}^*_{\phi},$$

and a similar computation for $\frac{\partial}{\partial \bar{z}}$. Applying this twice we obtain

$$\Delta|\alpha|^2 e^{-\phi} = \alpha \cdot \overline{\frac{\partial}{\partial \bar{z}} \delta \alpha} e^{-\phi} + \delta \frac{\partial}{\partial \bar{z}} \alpha \cdot \bar{\alpha} e^{-\phi} + |\frac{\partial}{\partial \bar{z}} \alpha|^2 e^{-\phi} + |\delta \alpha|^2 e^{-\phi}.$$

Then we apply the commutation rule

$$\delta \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \delta + \Delta \phi$$

and obtain

$$\Delta|\alpha|^2 e^{-\phi} = 2\Re \frac{\partial}{\partial \bar{z}} \delta \alpha \cdot \bar{\alpha} e^{-\phi} + |\frac{\partial}{\partial \bar{z}} \alpha|^2 e^{-\phi} + |\delta \alpha|^2 e^{-\phi} + \Delta \phi |\alpha|^2 e^{-\phi}. \quad (1.9)$$

Integrating this over a domain containing the support of α, and recalling that $\delta = -\bar{\partial}^*_{\phi}$ we obtain the basic identity.

3. *An application: Inequalities of Brunn-Minkowski type

In this section we will give an application of the corollary, or rather of an even simpler, real variable variant of it.

Proposition 3.1. *Let ϕ be a convex function on $\mathbb{R}$ of class C^2 with strictly positive second derivative. Let u be a function of class C^1 such that*

$$\int_{\mathbb{R}} u e^{-\phi} = 0.$$

Then

$$\int_{\mathbb{R}} |u|^2 e^{-\phi} \leq \int_{\mathbb{R}} \frac{|u'|^2}{\phi''} e^{-\phi}.$$

Note that this is formally similar to the the statement of Corollary 1.1.4. We have replaced a subharmonic ϕ by a convex ϕ, and require u to be orthogonal to constants (elements of the kernel of d) instead of orthogonal to holomorphic functions (elements of the kernel of $\bar{\partial}$).

Exercise: Prove the proposition imitating the proof of Corollary 1.0.4. □

We shall now use this proposition to prove a generalization of the Brunn Minkowski theorem.

Theorem 3.2. *Let $\phi(t,x)$ be a convex function on $\mathbb{R}^m_t \times \mathbb{R}^n_x$. Define the function $\tilde{\phi}(t)$ by*

$$e^{-\tilde{\phi}(t)} = \int_{\mathbb{R}^n} e^{-\phi(t,x)} dx.$$

Then $\tilde{\phi}$ is convex.

Proof: We start by a few reductions. By Fubini's theorem, we may assume that $n = 1$. Since convexity means convexity on any line, we may also assume that $m = 1$. Now

$$\tilde{\phi}(t) = -\log \int e^{-\phi(t,x)} dx.$$

Differentiating once with respect to t we get

$$\tilde{\phi}' = \frac{\int \phi'_t e^{-\phi(t,x)} dx}{\int e^{-\phi(t,x)} dx},$$

and differentiating once more

$$\tilde{\phi}'' = \frac{\int (\phi''_{t,t} - (\phi'_t)^2) e^{-\phi} \int e^{-\phi} + (\int \phi'_t e^{-\phi})^2}{(\int e^{-\phi})^2}$$

Let a be the mean value

$$a := \frac{\int \phi'_t e^{-\phi}}{\int e^{-\phi}}.$$

Then the expression for the second derivative simplifies to

$$\tilde{\phi}'' = \frac{\int (\phi''_{t,t} - (\phi'_t - a)^2) e^{-\phi}}{\int e^{-\phi}}.$$

It is now time to use the inequality in the proposition. Since $u := \phi'_t - a$ by construction has integral 0 against the weight $e^{-\phi}$, the proposition shows that

$$\tilde{\phi}'' \geq \frac{\int (\phi''_{t,t} - (\phi''_{t,x})^2/\phi''_{x,x}) e^{-\phi}}{\int e^{-\phi}}.$$

Since ϕ is convex

$$\phi''_{t,t} - (\phi''_{t,x})^2/\phi''_{x,x} = (\phi''_{t,t}\phi''_{x,x} - (\phi''_{t,x})^2)/\phi''_{x,x} \geq 0.$$

This completes the proof. □

Theorem 1.2.2 was first given by Prekopa, [**18**], but the proof we have given is essentially due to Brascamp an Lieb, [**6**]. The theorem is a functional form of the Brunn Minkowski inequality, which can be stated as follows.

Theorem 3.3. *Let D be a convex open set in $\mathbb{R}^m_t \times \mathbb{R}^n_x$, and let D_t be the slices*

$$\{x; (t,x) \in D\}.$$

Let $|D_t|$ be the Lebesgue measure of D_t. Then

$$\log \frac{1}{|D_t|}$$

is convex.

Proof: Take ϕ to be the convex function that equals 0 in D and ∞ outside D in the previous result. Admittedly, this is not a classical convex function, but it can be written as an increasing limit of smooth convex functions, to which the proof above applies. □

There is another, perhaps more common, way of stating the Brunn Minkowski theorem. It says that if D_0 and D_1 are convex open sets in $\mathbb{R}^n$, then the volume of their Minkowski sum $tD_1 + (1-t)D_0$ satisfies

$$|tD_1 + (1-t)D_0|^{1/n} \geq t|D_1|^{1/n} + (1-t)|D_0|^{1/n},$$

if t lies between 0 and 1. This formulation can be obtained from the ("multiplicative") form we have given above in the following way.

(1) There is a convex open set D in $\mathbb{R}_t \times \mathbb{R}^n$ such that

$$tD_1 + (1-t)D_0 = D_t.$$

(2) Theorem 1.3.3 implies trivially that

$$|D_t| \geq \min(|D_0|, |D_1|).$$

(3) Hence, if A and B are convex open sets

$$|A+B|^{1/n} = |tA/t + (1-t)B/(1-t)|^{1/n} \geq \min(|A|^{1/n}/t, |B|^{1/n}/(1-t)).$$

Choosing

$$t = \frac{|A|^{1/n}}{|A|^{1/n} + |B|^{1/n}}$$

we then get

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n},$$

which is the usual ("additive") form of the inequality.

The multiplicative form has the advantage that it applies to many other measures besides Lebesgue measure. The additive statement – $-|D_t|^{1/n}$ is convex – is formally stronger, but in the presence of the natural homogenuity of Lebesgue measure the two statements are equivalent.

Note that the main point in the proof of Prekopa's theorem was the one-dimensional case – the general case then followed by a simple induction. This is clearly not the case for Brunn Minkowski; there the one-dimensional case does not say very much!

4. Regularity - a disclaimer.

In the previous discussion we have interpreted the $\bar\partial$-equation in the weak, or distributional sense. Thus,

$$\frac{\partial u}{\partial \bar z} = f$$

has been interpreted as saying that

$$-\int u \frac{\partial}{\partial \bar z}\alpha = \int f\alpha$$

for any smooth α of compact support. If f is smooth, this implies that u is also smooth and that the $\bar\partial$-equation is also satisfied in the classical pointwise sense. This is a consequence of the clasical *Weyl's lemma*, and holds also in several variables (at least if f is of bidegree $(0,1)$). We will not discuss these issues further in the notes; all the $\bar\partial$-equations we deal with are to be interpreted in the sense of distributions.

LECTURE 2

Functional analytic interlude.

In this chapter we begin to look at the $\bar\partial$-equation in higher dimension. For the moment we shall think of Ω as a domain in $\mathbb{C}^n$, and consider the $\bar\partial$-equation when the right hand side is a $(0,1)$-form, but this is for motivational purposes only –the formalism we develop will later apply also to the case of complex manifolds and general bidegrees.

1. Dual formulation of the $\bar\partial$-problem.

Denote by $D_{(0,1)}$ the class of $(0,1)$-forms whose coefficients are, say, of class C^2 with compact support in Ω. If f and α are $(0,1)$-forms we denote by $f\cdot\bar\alpha$ their pointwise scalar product, i e

$$f\cdot\bar\alpha=\sum f_j\bar\alpha_j.$$

The equation $\bar\partial u=f$, explicitly

$$\frac{\partial u}{\partial\bar z_j}=f_j,$$

taken in the sense of distributions, means that

$$\int f\cdot\alpha=-\int u\sum\frac{\partial\alpha_j}{\partial\bar z_j}, \tag{2.1}$$

for all $\alpha\in D_{(0,1)}$. Just like in the one-dimensional case we replace α by $\bar\alpha e^{-\phi}$ (where ϕ is a C^2-function which will later be chosen to be plurisubharmonic). The condition (2.1) is then equivalent to

$$\int f\cdot\bar\alpha e^{-\phi}=\int u\overline{\bar\partial^*_\phi\alpha}e^{-\phi} \tag{2.2}$$

for all $\alpha\in D_{(0,1)}$, where

$$\bar\partial^*_\phi\alpha=-e^{\phi}\sum\frac{\partial}{\partial z_j}(e^{-\phi}\alpha_j).$$

Assume now that we can find a solution, u, to $\bar\partial u=f$, satisfying

$$\int|u|^2e^{-\phi}\le C.$$

Then (2.2) implies

$$|\int f\cdot\bar\alpha e^{-\phi}|^2\le C\int|\bar\partial^*_\phi\alpha|^2e^{-\phi}.$$

The next proposition says that the converse of this also holds.

Proposition 1.1. *There is a solution, u, to the equation $\bar\partial u = f$ satisfying*

$$\int |u|^2 e^{-\phi} \leq C. \tag{2.3}$$

if and only if the inequality

$$|\int f \cdot \bar\alpha e^{-\phi}|^2 \leq C \int |\bar\partial_\phi^* \alpha|^2 e^{-\phi}. \tag{2.4}$$

holds for all $\alpha \in D_{(0,1)}$.

Proof: It only remains to prove that (2.4) implies that there is a solution to the $\bar\partial$-equation satisfying (2.3). This is done precisely as in the one-dimensional case (cf first part of Proposition 1.0.1). □

To prove inequality (2.4) one might, as in the one dimensional case, first try to prove an inequality of the form

$$\int |\alpha|^2 e^{-\phi} \leq C \int |\bar\partial_\phi^* \alpha|^2 e^{-\phi}.$$

The main problem in higher dimensions is that no such inequality can hold. Indeed, if it did, then we would be able to solve $\bar\partial u = f$, even when f does not satisfy the compatibility condition $\bar\partial f = 0$ – which is clearly not possible. Thus we must somehow feed this information, $\bar\partial f = 0$, into the method. This requires a little bit more of functional analysis.

Let us consider a linear operator T between two Hilbert spaces H_1 and H_2. Assume that T is *closed* and densely defined. The first condition means that the graph of T

$$\{(x, Tx); x \in \mathrm{Dom}(T)\}$$

is a closed linear subspace of the product $H_1 \times H_2$, or, in more concrete terms, that if x_n lie in the domain of T and converge to x, and if moreover Tx_n converge to y, then x lies in the domain of T and $Tx = y$. The main example to think of for the moment is $T = \bar\partial$,

$$H_1 = L^2(\Omega, e^{-\phi}),$$

the space of functions that are square integrable against the weight $e^{-\phi}$ and

$$H_2 = L^2_{(0,1)}(\Omega, e^{-\phi}),$$

the space of $(0, 1)$-forms that are square integrable against the weight $e^{-\phi}$. In this example, the domain of T is taken to be the space of all functions u in H_1 such that $\bar\partial u$ taken in the sense of distributions lies in H_2.

Exercise: Show that in this example T is closed and densely defined. □

The next proposition is the Hilbert space version of the theorem that a linear operator between finite dimensional vector spaces is surjective if and only if its adjoint is injective.

Proposition 1.2. *T is surjective if and only if there is a constant $c > 0$ such that for all y in the domain of T^**

$$c|y|^2 \leq |T^* y|^2. \tag{2.5}$$

More generally, let F be a closed subspace of H_2 containing the range of T. Then T is surjective onto F if and only if (2.5) holds for y in F intersected with the domain of T^*. In that case, for any y in F there is an x in H_1 such that

$$Tx = y$$

and

$$|x|^2 \leq \frac{1}{c}|y|^2.$$

PROOF. First assume 2.5 holds for all y in the domain of T^*. Take z in H_2 and define an antilinear functional on the range of T^* by

$$L(T^*y) = (z, y).$$

Our hypothesis implies that L is well defined and of norm at most $|z|\sqrt{c}$. By the Hahn-Banach theorem, L extends to a linear operator with the same norm on all of H_1, and by the Riesz representation theorem there is an element in H_1 of norm at most $|z|\sqrt{c}$ such that

$$(z, y) = L(T^*y) = (x, T^*y).$$

Since $T^{**} = T$ (this is a nontrivial statement!) it follows that $Tx = z$.

Suppose conversely that T is surjective. By the open mapping theorem there is then a constant so that for any y in H_2 we can solve $Tx = y$ with

$$|x|^2 \leq \frac{1}{c}|y|^2.$$

If y lies in the domain of T^*

$$|y|^2 = (y, Tx) = (T^*y, x) \leq \sqrt{\frac{1}{c}}|T^*y||y|,$$

so (2.5) holds.

Let now F be a closed subspace of H_2 containing the range of T. We leave as an exercise to prove that that the adjoint of T considered as a map from H_1 to F is the restriction of T^* to F. Hence the last part of the proposition follows from the first part. □

Notice that the first part of the proof is virtually identical to the proof of Proposition 1.1.1. In the application that we have in mind the space F will of course consist of the subspace of $\bar\partial$-closed forms – which is closed since differentiation is a continuous operation in distribution theory.

What we have gained with this proposition is that to prove solvability of the $\bar\partial$-equation we need only prove an inequality of type

$$\int |\alpha|^2 e^{-\phi} \leq C \int |\bar\partial^*_\phi \alpha|^2 e^{-\phi},$$

when α is a *closed* form. We will then automatically also get estimates for the solution. But, we have also lost something. In the one dimensional case we could work all the time with α smooth and of compact support; now we have to deal with forms in the domain of a rather abstract operator: the Hilbert space adjoint of $\bar\partial$. We will later overcome this complication by approximating a general element in the domain of T^* by smooth forms with compact support. This is a somewhat delicate business which we will, following Demailly [**8**], handle by introducing complete Kähler metrics. We illustrate the issue involved in the next subsection in a very simple model example.

1.1. The role of completeness in a simple model example.*

Let us consider the differential equation

$$\frac{du}{dx} = f$$

on an open interval I in $\mathbb{R}$. To imitate the weighted estimates for the $\bar\partial$-equation that we are dealing with we will study estimates for norms of the solution in the Hilbert space H with norm

$$\|u\|^2 = \int_I u^2 e^{-x^2/2}.$$

We consider the operator

$$u \mapsto \frac{du}{dx} =: Tu$$

as a closed densely defined operator and we let the domain of T consist of all functions u in $L^2(I, e^{-x^2/2})$ such that du/dx in the sense of distributions lie in $L^2(I, e^{-x^2/2})$.

The *formal adjoint* of the differential operator d/dx is the operator ϑ defined by

$$\int_I (du/dx) v e^{-x^2/2} = \int_I u \vartheta v e^{-x^2/2}$$

for all smooth u and v of compact support. Simple integration by parts shows that

$$\vartheta v = -dv/dx + xv.$$

On the other hand, the Hilbert space adjoint of T is the operator T^* satisfying

$$(Tu, v)_H = (u, T^* v)_H$$

for all u in the domain of T. This should be interpreted as saying that v lies in the domain of T^* if there is some w in H satisfying

$$(Tu, v)_H = (u, w)_H$$

and T^*v is then equal to w (which is uniquely determined since the domain of T is dense).

There are now two main cases: $I = \mathbb{R}$ (a complete manifold) or $I \neq \mathbb{R}$ (not complete). Assume first that $I = \mathbb{R}$. Then if v lies in the domain of T^* and we take u to be smooth with compact support, we see that

$$T^* v = \vartheta v$$

where the right hand side is taken in the sense of distributions. Hence ϑv must lie in H and must be equal to T^*v if v lies in the domain of T^*. Conversely if v and ϑv lies in H, and $\chi_k(x) = \chi(x/k)$ is a sequence of cut-off function tending to 1, we get

$$(u, \vartheta v)_H = \lim(\chi_k u, \vartheta v)_H = \lim(\chi_k' u, v)_H + (Tu, v)_H = (Tu, v)_H,$$

(since χ_k' goes to zero).

Hence we have a precise description of T^*: $T^*v = \vartheta v$ and its domain consist of functions for which ϑv is in H. Moreover, if v lies in the domain of T^* then $\chi_k v$ tends to v and $T^* \chi_k v$ tends to T^*v, so to prove the crucial inequality

$$\|v\|^2 \leq C \|T^* v\|^2$$

one may assume that v has compact support. In fact one may also assume that v is smooth, since we can achieve this by taking convolutions with a sequence of smooth functions tending to the Dirac measure.

Exercise: Assuming that v is smooth with compact suppport, prove that

$$\|v\|^2 \leq \|\vartheta v\|^2$$

□

Let us now take $I = (0, 1)$ instead. As before we see that if v lies in the domain of the adjoint, then $T^* v = \vartheta v$ and of course this expression must lie in H then. However, this is no longer sufficient to be in the domain of T^*, not even if v is smooth up to the boundary.

Exercise: Show that if v is smooth on $[0, 1]$ then v lies in the domain of T^* if and only if $v(0) = v(1) = 0$. □

Moreover it is in this case also less evident how to approximate an element in the domain with functions of compact support, and also how to regularize by taking convolutions. Any sequence of cut-off functions with compact support in I that tend to 1 must have unbounded derivatives, and there is also the problem to take convolutions near the boundary. These problems are certainly possible to overcome (and it *is* possible to solve the equation $du/dx = f$ on the unit interval), but the fact that they arise serve to illustrate the advantage of working with complete metrics. In the case of the $\bar\partial$-equation we shall see later that we may introduce complete (Kähler) metrics precisely in the domains where we expect to be able to solve the $\bar\partial$-equation.

LECTURE 3

The $\bar\partial$-equation on a complex manifold.

In this section we will discuss the $\bar\partial$-equation on an n-dimensional complex manifold X. We start by discussing the linear algebra of forms and metrics on complex manifolds, and then apply this formalism to derive the basic integral identity for $\bar\partial$ for forms with values in a line bundle (Proposition 3.4.1). In many texts this is done via the Kodaira-Nakano identity for associated Laplace operators, but we have chosen another route: Siu's $\partial\bar\partial$-Bochner-Kodaira method that generalizes the 'alternative proof' in the one dimensional case from section 1.1. This formalism is particularily efficient for forms of bidegree (n, q). It avoids the use of the so called 'Kähler identities' completely and also gives a stronger result that will be of use later when we discuss extension of holomorphic sections from subvarieties.

1. Metrics

The first thing we have to do is to find convenient expressions for norms of forms that will enable us to compute the adjoint of $\bar\partial$. For this we need a hermitian metric on X. This is by definition a hermitian scalar product on each complex tangent space

$$T_p^{1,0}(X).$$

If $z = (z_1, ...z_n)$ are local coordinates near p, then

$$\frac{\partial}{\partial z_1}, \cdots \frac{\partial}{\partial z_n}$$

at p form a basis for $T_p^{0,1}(X)$ with dual basis

$$dz_1, ...dz_n$$

for the complex cotangent space. With respect to this basis the metric is given by a hermitian metric $g = (g_{j,k})$, so that if

$$v = \sum v_j \frac{\partial}{\partial z_j},$$

then

$$|v|^2 = \sum g_{j,k} v_j \bar v_k.$$

We also get an induced metric on the cotangent space, such that if

$$\eta = \sum \eta_j dz_j,$$

then

$$|\eta|^2 = \sum g^{j,k} \eta_j \bar\eta_k.$$

The dual metric is defined so that

$$|\eta|^2 = \sup_{|v|\leq 1} |\eta(v)|^2.$$

Exercise: Show that $(g^{j,k})^* = (g_{j,k})^{-1}$. □

It is assumed that $g(p)$ varies smoothly with p. Formally,

$$g = \sum g_{j,k} dz_j \otimes d\bar z_k,$$

but it turns out to be more convenient to let the metric be represented by its *Kähler form*

$$\omega = \omega_g = i \sum g_{j,k} dz_j \wedge d\bar z_k.$$

One says that g is a Kähler metric if $d\omega = 0$. Obviously the euclidean metric with Kähler form

$$\beta = i \sum dz_j \wedge d\bar z_j$$

is Kähler. One can always choose local coordinates so that $\omega = \beta$ to order 0 at any given point. We say that the local coordinates z are normal at a point p if $\omega = \beta$ to first order at p. More precisely

$$g_{j,k} = \delta_{j,k}$$

and

$$dg_{j,k} = 0$$

at p. Clearly, if we can choose normal coordinates at any point, then $d\omega = 0$, so the metric must be Kähler.

Exercise: Verify that the converse to this also holds. (Hint: This is a bit tricky. Start from coordinates whose differentials are orthonormal at the point. Then look for new coordinates of the form

$$\zeta_j = z_j + Q_j(z)$$

where Q_j are quadratic in z.)

□

A metric determines a volume element on the manifold X. This is a differential form of maximal degree that can be written

$$dV_\omega = i^n \xi_1 \wedge \bar\xi_1 \wedge ...\xi_n \wedge \bar\xi_n$$

if ξ is an orthonormal basis for the holomorphic cotangent space. If we take another orthonormal basis it is related to the first one via a unitary linear transformation. Then the expression on the right hand side above gets multiplied by the modulus square of the determinant of the unitary tranformation, i e by 1. Hence the volume element is well defined. In terms of the Kähler form

$$\omega = i \sum g_{j,k} dz_j \wedge d\bar z_k = i \sum \xi_j \wedge \bar\xi_j,$$

$$dV_\omega = \omega^n/n!.$$

This formula is very convenient in computations. Note that there is nothing (at least nothing obvious) that corresponds to this for real manifolds. We will introduce the notation

$$\alpha_p := \alpha^p/p!$$

for expressions of this kind. If Y is a complex submanifold of dimension p, then the induced metric on Y has Kähler form ω restricted to Y, i e the pullback of ω to Y under the inclusion map. Hence the volume element on the submanifold is simply

$$\omega_p$$

restricted to Y.

Exercise: Show that this implies: If Y is a complex curve, i e a complex submanifold of dimension 1 in $\mathbb{C}^n$, then the area of Y equals the sum of the areas of the projections of Y onto the coordinate axes (counted with multiplicities).

How can this be geometrically possible? □

Exercise*: In general, the volume element on an oriented Riemannian manifold M, can be defined as

$$dV_M = \xi_1 \wedge ...\xi_n,$$

wher ξ_j form an oriented orthonormal basis for the space of real one-forms. Show that if Y is a submanifold of $\mathbb{C}^n$ of real dimension 2, then

$$dV_Y \geq \omega|Y$$

with equality (if and) only if Y is a complex curve. This is *Wirtinger's inequality.* Deduce that if Y and Y' are such submanifolds with the same boundary and Y' is complex, the Y' has area smaller than or equal to that of Y. (Hint: It is enough to prove Wirtinger's inequality for a linear subspace. Choose complex coordinates on $\mathbb{C}^n$ so that $\frac{\partial}{\partial x_1}$ and $a\frac{\partial}{\partial y_1}$ + remainder form an orthonormal basis for the tangent space.) □

2. Norms of forms

We have already said that if $\eta = \sum \eta_j dz_j$ is a $(1,0)$-form, and dz_j are orthonormal at a point, then

$$|\eta|^2 = \sum |\eta_j|^2.$$

We would now like to define the norm of a form of arbitrary bidegree by saying that if

$$\eta = \sum \eta_{I,J} dz_I \wedge d\bar{z}_J,$$

then

$$|\eta|^2 = \sum |\eta_{I,J}|^2, \tag{3.1}$$

if dz_i is orthonormal. It is however not crystal clear that this definition is independent of the choice of orthonormal basis. Fortunately there is a simple way out of this, at least for forms of bidegree $(p,0)$ or $(0,q)$.

Note first that if dz_j are orthonormal and we denote by

$$dV_J = \bigwedge_{j \in J} i dz_j \wedge d\bar{z}_j,$$

then

$$\omega_p = \sum dV_J,$$

the sum running over multiindices J of length p. Let η be a form of bidegree $(p,0)$. Then we can define the norm of η by

$$|\eta|^2 dV_\omega = c_p \eta \wedge \bar{\eta} \wedge \omega_{n-p},$$

where $c_p := i^{p^2}$ is a unimodular constant chosen to make the right hand side positive.

This constant, $c_p = i^{p^2}$ will appear again and again in the sequel, so it might be good to pause and see what it comes from. Say first that η is of the form

$$\eta = a_1 \wedge a_2 \wedge ...a_p,$$

where a_j are differentials dz_j. Then

$$(-1)^{p(p-1)/2}\eta\wedge\bar\eta = a_1\bar a_1\wedge ...a_p\wedge\bar a_p.$$

But $(-1)^{p(p-1)/2} = i^{p(p-1)} = i^{p^2}i^{-p}$, so

$$i^{p^2}\eta\wedge\bar\eta = ia_1\bar a_1\wedge ...ia_p\wedge\bar a_p,$$

which is a positive form (after wedging with ω_{n-p}). For general η we get a sum of such terms, so the same choice of c_p will do.

This definition is independent of choice of basis and coincides with (3.1), so (3.1) is indeed independent of choice of basis if η is of bidegree $(p,0)$. Of course, norms on forms of bidegree $(0,q)$ are defined similarily so that

$$|\eta|^2 = |\bar\eta|^2.$$

We shall next motivate why formula (3.1) is independent of choice of ON-basis for forms of general bidegree (p,q). Fix one choice of ON-basis dz_i and define the scalar product by (3.1). Then, if $\mu = \sum \mu_J dz_j$ is of bidegree $(p,0)$ and $\xi = \sum\xi_K d\bar z_K$ is of bidegree $(0,q)$ it follows that

$$\|\mu\wedge\xi\|^2 = \sum |\mu_J|^2|\xi_K|^2 = \|\mu\|^2\|\xi\|^2,$$

and polarizing that

$$(\mu\wedge\xi,\mu'\wedge\xi') = (\mu,\mu')(\xi,\xi').$$

From this it follows that if $\eta = \sum\eta_{J,K}dw_J\wedge d\bar w_K$

$$\|\eta\|^2 = \sum\eta_{JK}\bar\eta_{LM}(dw_J,dw_L)(d\bar w_K,d\bar w_M).$$

But our previous discussion for $(p,0)$-forms shows that

$$(dw_J,dw_K) = \delta_{JK}$$

so (3.1) holds for any ON-basis.

We will later have use for a special formula for forms of bidegree (n,q).

Lemma 2.1. *Let η be a form of bidegree (n,q). Then there is a unique form γ_η of bidegree $(n-q,0)$ such that*

$$\eta = \gamma_\eta\wedge\omega_q.$$

This follows immediately from a computation in ON-coordinates. Note that if in ON-coordinates

$$\eta = \sum\eta_J dz\wedge d\bar z_J,$$

γ_η equals

$$\sum\epsilon_J\eta_J dz_{J^c},$$

where ϵ_J are unimodular constants. Consequently

$$|\eta|^2 = |\gamma_\eta|^2.$$

In terms of Riemannian geometry, γ_η is the Hodge-* of η, again up to a unimodular constant. We have

$$|\eta|^2 dV_\omega = c_{n-q}\eta\wedge\bar\gamma_\eta.$$

Polarizing we get

$$(\eta,\xi)dV_\omega = c_{n-q}\eta\wedge\bar\gamma_\xi, \tag{3.2}$$

a formula that will be very helpful later to compute adjoints.

3. Line bundles

Formally, a line bundle L over a complex manifold X is a complex manifold fibered over X
$$p : L \to X$$
in such a way that all fibers $L_x = p^{-1}(x)$ are complex lines. One also requires that the fibration be *locally trivial*, so that each point in X has a neighbourhood U with $p^{-1}(U)$ isomorphic to $U \times \mathbb{C}$ via a fiber preserving holomorphic map
$$g_U : p^{-1}(U) \to U \times \mathbb{C},$$
which is linear on each fiber. If V is another trivializing open set, we then get a map
$$g_U g_V^{-1} : U \cap V \times \mathbb{C} \to U \cap V \times \mathbb{C}.$$
Since this map is fiber preserving and linear on fibers, it must be given as
$$g_U g_V^{-1}(z, \xi) = (z, g_{UV}(z)\xi)$$
where g_{UV} is a zero free holomorphic function of z. The maps g_{UV} arising in this way are known as *transition functions* of the line bundle. Notice that they satisfy the *cocycle conditions*
$$g_{UV} g_{VU} = 1$$
and
$$g_{UV} g_{VW} g_{WU} = 1.$$
Conversely, given an open covering U_j of the base manifold X, and a corresponding collection of functions $g_{ij} = g_{U_i U_j}$ satisfying the cocycle conditions we can define a line bundle over X having these as transition functions. For this, one starts with the disjoint union
$$\bigcup_i U_i \times \mathbb{C}$$
and identifies (z, ξ) in $U_j \times \mathbb{C}$ with $(z, g_{ij}(z)\xi$ in $U_i \times \mathbb{C}$. A *section* of a line bundle is a map s from (a subset of) X to L satisfying
$$p \circ s(z) = z.$$
In terms of the local trivializations, s is given by a collection of scalar functions
$$s_i = g_{U_i} s,$$
satisfying
$$s_i = g_{ij} s_j$$
A main reason to be interested in line bundles comes from the following example:

Example: Let $\mathbb{P}^n$ be n-dimensional projective space, i e the set of equivalence classes
$$[z_0, z_1 ... z_n] = [z]$$
where z lies in $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence $[z] = [\lambda z]$ for λ in $\mathbb{C} \setminus \{0\}$. Denote by π the natural projection map from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{P}^n$, $\pi(z) = [z]$. Let
$$U_i = \{[z]; z_i \neq 0\}$$
Then each U_i is isomorphic to $\mathbb{C}^n$, e g $U_0 = \{[1, \zeta]; \zeta \in \mathbb{C}^n\}$. We now define, for each integer k, the line bundles $\mathcal{O}(k)$, using the transition functions
$$g_{ij} = (z_j/z_i)^k.$$

A section of $\mathcal{O}(k)$ is then a collection of functions s_i on U_i such that

$$z_j^k s_j = z_i^k s_i$$

on overlaps. Then $h := z_i^k s_i \circ \pi$ is a globally defined k-homogeneous function on $\mathbb{C}^{n+1} \setminus \{0\}$, and working backwards we conversely see that any k-homogeneous function gives rise to a section of $\mathcal{O}(k)$. For k nonnegative it follows that the global holomorphic sections extend across the origin (they are bounded near 0) and therefore correspond to homogeneous polynomials of degree k (their Taylor expansion near 0 is homogeneous) . Such a polynomial is of course uniquely determined by its restriction to the dense set U_0, where it can be thought of as a polynomial on $\mathbb{C}^n$ of degree k (put $z_0 = 1$). For k negative there are no global holomorphic sections. □

This example has much wider scope than is visible at first sight. Any line bundle over a complex manifold restricts to a line bundle over submanifolds. Hence $\mathcal{O}(k)$ restricts to a line bundle over any submanifold of $\mathbb{P}^n$, and the bundles we obtain in this way have at least the sections that come from restricting homogeneous polynomials.

Example: The *canonical bundle* K_X of a complex manifold X is a line bundle whose sections are forms of bidegree $(n, 0)$. Locally such section can be written

$$s = s_i dz_1^i \wedge ...dz_n^i =: s_i dz^i,$$

where $(z_1^i, ...z_n^i)$ are local coordinates. If

$$g_{ij} dz^i = dz^j,$$

then g_{ij} are transition functions for the canonical bundle. □

Exercise: Prove that on $\mathbb{P}^n$ the canonical bundle is equal to $\mathcal{O}(-(n+1))$. □

A (classical) metric on a line bundle is a smoothly varying norm on each fiber, which thus enables us to define the norm $\|s\|$ of any section of the bundle. Under the representation of s as a collection of local functions s_i, the metric becomes represented by a collection of smooth real valued functions ϕ_i so that

$$\|s\|^2 = |s_i|^2 e^{-\phi_i}.$$

Since $s_i = g_{ij} s_j$ on overlaps the local representatives for the metric must be related by

$$\log |g_{ij}|^2 = \phi_i - \phi_j$$

on overlaps. We will in the sequel denote metrics by e g ϕ, with the understanding that this means a collection of local functions ϕ_j related in this manner.

Note that if ϕ is a metric on L and χ is a function, then

$$\phi + \chi$$

is also a metric.

Exercise Show that if ϕ is some metric on L, all other metrics on L can be written $\phi + \chi$ where χ is a function. □

Since $\log |g_{ij}|^2$ is pluriharmonic, we see that the form

$$c(\phi) = \partial\bar\partial\phi_i$$

is globally defined (even though the ϕ_i:s are not global). By definition, $c(\phi)$ is the *curvature form* of the metric.

Example: Let $\rho_i\sqrt{-1}dz^i \wedge d\bar{z}^i$ be the local Kähler forms of a metric on a complex manifold X of dimension 1. Then

$$-\Delta \log \rho_i$$

is the classical Gaussian curvature of the corresponding Riemannian metric $\rho(dx^2 + dy^2)$, and $\partial\bar{\partial} - \log \rho$ is the curvature form. If $s = (s_i)$ is a section of K_X, then

$$|s_i|^2 dz^i \wedge d\bar{z}^i$$

is globally well defined, so $|s_i|^2/\rho_i$ is also global. Therefore

$$\rho_i = e^{\phi_i}$$

where ϕ is a metric on K_X and the curvature of this metric is the negative of the Gaussian curvature of the underlying manifold. (The minus sign comes from ϕ being a metric on K_X which is the dual of the tangent space.) In higher dimensions a Kähler metric on X also induces a metric on K_X. The curvature of this induced metric on K_X is the negative of the *Ricci curvature* of the Kähler metric on X. □

Definition: We say that a line bundle L equipped with a metric ϕ is *positive* if $ic(\phi)$ is a positive form (i e if all the local representatives ϕ_i are strictly plurisubharmonic). L itself is called positive (or equivalently *ample* in the algebraic geometry terminology) if it has some metric with positive curvature. □

Example/Exercise: We have seen that for the line bundles $\mathcal{O}(k)$ on $\mathbb{P}^n$ their global holomorphic sections are in one-one correspondence with functions on $\mathbb{C}^{n+1}$ that are homogeneous of degree k. Under this correspondence:

$$s \to z_i^k s_i \circ \pi := h$$

a metric on $\mathcal{O}(k)$ can be defined by

$$\|s\|^2 = |h|^2/|z|^{2k}.$$

(This defines $\|s\|$ as a function on $\mathbb{C}^{n+1}$ that is homogeneous of degree 0, i e a function on $\mathbb{P}^n$.) Show that if we represent this metric by local functions ϕ_i then

$$\phi_0(1, \zeta) = k \log(1 + |\zeta|^2).$$

Hence $\mathcal{O}(k)$ is positive (ample) if $k > 0$. □

Example: Let D be a *divisor* in X. Intuitively, this is a hypersurface, every branch of which is endowed with certain multiplicities; positive or negative. D is called *effective* if all multiplicities are nonnegative. Locally, in an open set U, a divisor can be defined by a meromorphic function s_U in U which vanishes to the given multiplicity on every branch of D that intersects U. Another local holomorphic function s_V defines the same divisor if

$$g_{UV} : s_U/s_V$$

is holomorphic and zerofree on $U \cap V$. Then, if U_i is a covering of X by open sets and $g_{ij} := g_{U_iU_j}$, g_{ij} the g_{ij} satisfy the cocycle condition so they form transition functions of a certain line bundle which is usually denoted (D). Any line bundle arising in this way from a divisor has a meromorphic section, and any line bundle arising from an effective divisor has a holomorphic section. □

If L and F are line bundles over X, we can form a new line bundle $L \otimes F$ by taking tensor products of the fibers. If g_{ij} and h_{ij} are the transition functions of L and F respectively (by passing to a common refinement we may assume they are defined w r t the same covering with open sets), then $g_{ij}h_{ij}$ are the transition functions of $L\otimes F$. If $\phi=(\phi_i)$ is a metric on L and $\psi=(\psi_i)$ is a metric on F, then $\phi+\psi$ is a metric on $L\otimes F$. For this reason one sometimes uses additive notation for the tensor product of vector bundles so that

$$L\otimes F := L+F.$$

We will use this convention (often) in the sequel and write in particular

$$L^{\otimes k}=kL$$

for the product of L with itself k times.

3.2. Forms with values in a line bundle.

Recall that a section of a line bundle L can be thought of as a collection of functions (s_i), each defined in an open set U_i, where all the U_i together make up an open covering of the base manifold X. Moreover, the s_i are related by

$$s_i=g_{ij}s_j$$

where g_{ij} are the transition functions of the line bundle. What this means is that in each U_i we have a local basis element $e_i:=g_{U_i}^{-1}1$ for the fibers of L, and the section can be recovered from the local functions as

$$s=s_i\otimes e_i.$$

The norm of s with respect to a metric ϕ on L is then defined by

$$\|s\|^2=|s_i|^2\|e_i\|^2=|s_i|^2e^{-\phi_i},$$

so the local representatives ϕ_i of the metric satisfy

$$e^{-\phi_i}=\|e_i\|^2.$$

In the same way a differential form η with values in L is given by a collection of local forms η_i in U_i, such that

$$\eta_i=g_{ij}\eta_j,$$

and we can recover η from the local representatives as

$$\eta=\eta_i\otimes e_i.$$

Given a metric ϕ on L we can now define the norm of an L valued form by

$$\|\eta\|^2=|\eta_i|^2e^{-\phi_i},$$

where the norm of η_i is defined with respect to some metric ω on the underlying manifold. The tranformation rules for η_i and ϕ_i under change of local trivialization then show that the norm is well defined. Polarizing we also get a scalar product on L-valued forms. Notice that expressions like

$$\eta_i\wedge\bar\xi_ie^{-\phi_i}=:\eta\wedge\xi e^{-\phi}$$

also become well defined for the same reason. We will write $\langle,\rangle$ for the scalar product on forms, so that for L-valued η and ξ

$$\langle\eta,\xi\rangle=(\eta_i,\xi_i)e^{-\phi_i}.$$

Note now that the $\bar\partial$-operator is well defined on sections of L and on forms with values L:

$$\bar\partial\eta=\bar\partial\eta_i\otimes e_i.$$

Since the transition function are holomorphic this definition is independent of which local representative η_i that we choose. However we can not define the d or ∂ operator on L-valued forms in this way unless the transition functions are locally constant. Instead we define a differential operator of degree $(1,0)$ on L valued forms by

$$\delta\eta = (e^{\phi_i}\partial e^{-\phi_i}\eta_i)\otimes e_i.$$

Then the operator δ satisfies

$$\bar\partial(\eta\wedge\bar\xi e^{-\phi}) = \bar\partial\eta\wedge\bar\xi e^{-\phi} + (-1)^{\deg\eta}\eta\wedge\overline{\delta\xi}e^{-\phi}, \tag{3.3}$$

so, since the $\bar\partial$-operator is well defined, the δ-operator must be well defined as well.

Exercise: Show by hand that δ is independent of choice of local trivialization. □

Together δ and $\bar\partial$ make up the analog of the d-operator for L-valued forms

$$D = \delta + \bar\partial,$$

which is the *Chern connection* operator defined by the metric on L. Then

$$D^2 = \delta\bar\partial + \bar\partial\delta = -\bar\partial\partial\phi = c(\phi),$$

the curvature of the connection. It is instructive to compare this formalism to the calculations in section 1.1.

4. Calculation of the adjoint and the basic identity.

We now set out to determine the formal adjoint of the $\bar\partial$ operator on L-valued forms.We will be principally interested in the case of forms of bidegree (n,q). Recall that (section 3.2) we have defined a map

$$\eta\to\gamma_\eta$$

mapping forms of bidegree (n,q) to forms of bidegree $(n-q,0)$, and that the scalar product on forms satisfies

$$(\eta,\xi)dV_\omega = c_{n-q}\eta\wedge\bar\gamma_\xi. \tag{3.4}$$

The formal adjoint of the $\bar\partial$ operator, $\bar\partial^*_\phi$ must, for any η of bidegree $(n,q-1)$, satisfy

$$\int\langle\bar\partial\eta,\xi\rangle dV_\omega = \int\langle\eta,\bar\partial^*_\phi\xi\rangle dV_\omega.$$

By (3.4), the left hand side equals

$$c_{n-q}\int\bar\partial\eta\wedge\overline{\gamma_\xi}e^{-\phi},$$

which by Stokes' theorem and formula (3.3) equals

$$c_{n-q}(-1)^{n-q}\int\eta\wedge\overline{\delta\gamma_\xi}e^{-\phi}.$$

The right hand side is

$$c_{n-q+1}\int\eta\wedge\overline{\gamma_{\bar\partial^*_\phi\xi}}e^{-\phi}.$$

Since $c_{n-q+1} = i(-1)^{n-q}c_{n-q}$ we see that $\bar\partial^*_\phi$ satisfies

$$\gamma_{\bar\partial^*_\phi\xi)} = i\delta\xi.$$

We will now use this description of the adjoint to derive a basic integral identity generalizing what we found in the one dimensional case. Let α be an L-valued form of bidegree (n,q). We will follow the method outlined in section 1.1, but instead of considering the norm of the form α we will use α to define a differential form of bidegree $(n-1, n-1)$. (This method was introduced by Siu in [**19**].) In the one dimensional case an $(n-1, n-1)$-form is a function, and in that case our construction will repeat what we did in section 1.1.

DEFINITION : Let α be a differential form of bidegree (n, q). Then

$$T_\alpha := c_{n-q}\gamma_\alpha \wedge \bar\gamma_\alpha \wedge \omega_{q-1} e^{-\phi},$$

where the unimodular constant $c_{n-q} = i^{(n-q)^2}$ is chosen so that $T_\alpha \geq 0$. □

Proposition 4.1. *Let α be a smooth (n, q)-form on X. Then, if ω is Kähler,*

$$i\partial\bar\partial T_\alpha = (-2\Re\langle\bar\partial\bar\partial^*_\phi\alpha, \alpha\rangle + \|\bar\partial\gamma_\alpha\|^2 + \|\bar\partial^*_\phi\alpha\|^2 - \|\bar\partial\alpha\|^2)dV_\omega + ic(\phi) \wedge T_\alpha, \tag{3.5}$$

and, if α has compact support,

$$\int ic(\phi) \wedge T_\alpha + \int \|\bar\partial\gamma_\alpha\|^2 dV_\omega = \int \|\bar\partial\alpha\|^2 + \|\bar\partial^*_\phi\alpha\|^2 \tag{3.6}$$

PROOF. Note first that the second formula follows immediately from the first one, since the integral of the left hand side of (3.5) vanishes by Stokes' theorem and

$$\int \langle\bar\partial\bar\partial^*_\phi\alpha, \alpha\rangle dV_\omega = \int \|\bar\partial^*_\phi\alpha\|^2.$$

To prove (3.5) we shall basically use (3.3) twice, keeping in mind that $d\omega = 0$. (This is one place where we use the Kähler assumption. Notice however that for the perhaps most important case, $q = 1$, we don't need the Kähler assumption – here.)We then get

$$i\partial\bar\partial T_\alpha =$$

$$ic_{n-q}(\delta\bar\partial\gamma_\alpha \wedge \bar\gamma_\alpha \wedge \omega_{q-1} + \gamma_\alpha \wedge \overline{\bar\partial\delta\gamma_\alpha} \wedge \omega_{q-1} +$$

$$+(-1)^{n-q}\delta\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1} + (-1)^{n-q+1}\bar\partial\gamma_\alpha \wedge \overline{\bar\partial\gamma_\alpha} \wedge \omega_{q-1})e^{-\phi}.$$

Now we use the commutation rule

$$\delta\bar\partial + \bar\partial\delta = c(\phi)$$

in the first term. The first two terms then combine to give the first and last two terms in (3.5). Moreover, by our formula for $\bar\partial^*_\phi$

$$ic_{n-q}(-1)^{n-q}\delta\gamma_\alpha \wedge \overline{\delta\gamma_\alpha} \wedge \omega_{q-1} = \|\bar\partial^*_\phi\alpha\|^2 dV_\omega.$$

(Again, it is useful to note that $c_{n-q+1} = i(-1)^{n-q}c_{n-q}$.) It only remains to analyse the term

$$\bar\partial\gamma_\alpha \wedge \overline{\bar\partial\gamma_\alpha} \wedge \omega_{q-1},$$

which even though it looks simple enough is actually the trickiest term. By the lemma below it equals

$$(\|\bar\partial\gamma_\alpha\|^2 - \|\bar\partial\alpha\|^2)dV_\omega,$$

since $\gamma_\alpha \wedge \omega_q = \alpha$ implies $\bar\partial\gamma_\alpha \wedge \omega_q = \bar\partial\alpha$ by the Kähler assumption (this is the only place where we use the Kähler assumption when $q = 1$). This finishes the proof. □

Lemma 4.2. *Let ξ be a form of bidegree $(n-q, 1)$. Then*

$$ic_{n-q}(-1)^{n-q-1}\xi \wedge \bar\xi \wedge \omega_{q-1} = (\|\xi\|^2 - \|\xi \wedge \omega_q\|^2)dV_\omega.$$

Proof. We give the proof for $q = 1$, leaving the general case as an exercise for the reader. Since we are dealing with a pointwise formula we may choose a ON basis dz_j for the differentials. Write

$$\xi = \sum \xi_{jk} d\hat{z}_j \wedge d\bar{z}_k.$$

Here $d\hat{z}_j$ denotes the product of all differentials dz_i except dz_j, ordered so that

$$dz_j \wedge d\hat{z}_j = dz := dz_1 \wedge ...dz_n.$$

Then one checks that

$$ic_{n-q}(-1)^{n-q-1}\xi \wedge \bar{\xi} \wedge \omega_{q-1} = \sum \xi_{jk}\bar{\xi}_{kj} dV_\omega.$$

On the other hand

$$\|\xi \wedge \omega\|^2 = \sum_{j<k} |\xi_{jk} - \xi_{kj}|^2 = \sum |\xi_{jk}|^2 - \sum \xi_{jk}\bar{\xi}_{kj},$$

so

$$\sum \xi_{jk}\bar{\xi}_{kj} = \|\xi\|^2 - \|\xi \wedge \omega\|^2.$$

□

Corollary 4.3. *Assume the curvature form of the metric ϕ is strictly positive so that*

$$ic(\phi) \geq c\omega$$

for some positive c. Then

$$qc \int \|\alpha\|^2 dV_\omega + \int \|\bar{\partial}\gamma_\alpha\|^2 dV_\omega \leq \int \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*_\phi \alpha\|^2 \tag{3.7}$$

Proof. This follows from (3.6) since $ic(\phi) \geq c\omega$ implies

$$ic(\phi) \wedge T_\alpha \geq cT_\alpha \wedge \omega = qc\gamma_\alpha \wedge \bar{\gamma}_\alpha \wedge \omega_q = qc\|\alpha\|^2 dV_\omega$$

□

5. The main existence theorem and L^2-estimate for compact manifolds.

We are now ready to state and prove the main existence theorem for the $\bar{\partial}$-equation for positive line bundles over compact manifolds. Apart from the estimate for the solution obatined, this is the celebrated *Kodaira vanishing theorem*, [**12**].

Theorem 5.1. *Let L be a holomorphic line bundle endowed with a metric ϕ over a compact complex manifold X. Assume the metric ϕ has (strictly) positive curvature and that*

$$ic(\phi) \geq c\omega$$

where $c(\phi) = \partial\bar{\partial}\phi$ is the curvature of ϕ and ω is some Kähler form on X.

Let f be a $\bar{\partial}$-closed (n, q)-form ($q > 0$) with values in L. Then there is an $(n, q-1)$-form u with values in L such that

$$\bar{\partial}u = f$$

and

$$\|u\|^2 \leq \frac{1}{cq}\|f\|^2.$$

For the proof we use the strategy from section 2 and consider the two Hilbert spaces

$$H_1 = L^2_{(n,q-1)}(X, L, \phi)$$

and

$$H_2 = L^2_{(n,q)}(X, L, \phi)$$

of L-valued forms in L^2 with respect to the metric ϕ, and the closed and densely defined operator $T = \bar\partial$ from H_1 to H_2. By the discussion in section 2, the theorem follows if we can establish the dual inequality

$$cq\|\alpha\|^2 \le \|T^*\alpha\|^2 \tag{3.8}$$

for any $\bar\partial$-closed α in the domain of the Hilbert space adjoint T^* of the operator T. Such an estimate will follow from Corollary 3.4.3 together with the following important approximation lemma.

Lemma 5.2. *If X is compact any smooth (n, q) form α lies in the domain of T^* and*

$$T^*\alpha = \bar\partial^*_\phi\alpha.$$

If α lies in the domain of T^* and in addition $\bar\partial\alpha$ (in the sense of distributions) lies in $H_3 := L^2_{(n,q+1)}(X, L, \phi)$ then there is a sequence α_k of smooth L-valued (n, q) forms such that

$$\|\alpha_k - \alpha\|^2,$$

$$\|\bar\partial^*_\phi\alpha_k - T^*\alpha\|^2,$$

and

$$\|\bar\partial\alpha_k - \bar\partial\alpha\|^2$$

all tend to zero.

Proof. The first statement means that if u lies in the domain of T (i e if $\bar\partial u$ taken in the sense of distributions lies in H_2) and α is smooth, then

$$\int \langle \bar\partial u, \alpha\rangle dV_\omega = \int \langle u, \bar\partial^*_\phi \alpha\rangle dV_\omega.$$

But this is exactly the definition of $\bar\partial u$ in the sense of distributions. The main part of the lemma is the possibility to approximate a general form by smooth forms in the way described.

We first claim that if α lies in Dom(T^*) and if χ is a realvalued smooth function on X, then $\chi\alpha$ also lies in Dom(T^*). For this we need to prove that

$$|(\bar\partial u, \chi\alpha)_{H_2}| \le C\|u\|_{H_1}.$$

But

$$(\bar\partial u, \chi\alpha)_{H_2} = (\chi\bar\partial u, \alpha)_{H_2} = (\bar\partial\chi u, \alpha)_{H_2} - (\bar\partial\chi \wedge u, \alpha)_{H_2}.$$

The first term on the right hand side satisfies the desired estimate since α lies in Dom(T^*), and the second term is trivially OK too.

We now choose χ to belong to an appropriate partition of unity with small supports, and decompose

$$\alpha = \sum \chi_j\alpha.$$

It is enough to approximate each term, so we see that we may assume that α has support in a coordinate neighbourhood which moreover trivializes L.

We then simply approximate α by taking the convolution with a sequence of functions

$$\chi_k(z) : k^{2n}\chi(kz)$$

where χ is smooth with compact support and has integral equal to 1. It is then a standard property of such convolutions that

$$\alpha_k := \chi_k * \alpha$$

converges to α in L^2. Since $\bar\partial\alpha_k = \chi_k * \bar\partial\alpha$ it follows that $\bar\partial\alpha_k$ converges to $\bar\partial\alpha$ as well. The remaining property, that $T^*\alpha_k$ converges to T^* is somewhat more delicate, but follows from a classical result of Friedrich, that we omit. □

Given the lemma, we can apply Corollary 3.4.3 to each α_k and conclude that

$$cq\|\alpha_k\|^2 \leq \|T^*\alpha_k\|^2 + \|\bar\partial\alpha_k\|^2.$$

If α lies in the domain of T^* and is $\bar\partial$-closed we get 3.8 by taking limits. This completes the proof of Theorem 3.5.1.

6. Complete Kähler manifolds

In this section we shall prove that Theorem 3.5.1 also holds for certain noncompact manifolds, namely those that carry some complete Kähler metric. We stress that we do not need to assume that the Kähler metric appearing in the final estimates is complete, only that the manifold has *some* complete metric. First we need to recall some definitions.

Definition : A complex manifold X with an hermitian metric (or a Riemannian manifold) is *complete* if there is some function

$$\chi : X \to [0, \infty)$$

which is proper and satifies

$$\|d\chi\| \leq C.$$

□

Admittedly, this definition is not overly intuitive. The more intuitive definition is that the hermitian (or Riemannian) metric induces a structure on X as a metric space, the distance between points being the infimum of the length of paths connecting the two points. The Riemannian manifold is then complete if this metric space is complete, i e if any Cauchy sequence has a limit. One can prove however that this definition is equivalent to the one we have given, but we have chosen the definition above since it is more convenient in applications. (It is both easier to verify and to apply.) Notice that if we have two Hermitian metrics with Kähler forms ω_1 and ω_2 respectively, if $\omega_1 \leq C\omega_2$ for some constant C and ω_1 is complete, then ω_2 is also complete (the same function χ will do).

Recall that a domain Ω in $\mathbb{C}^n$ is pseudoconvex if it has a strictly plurisubharmonic exhaustion function. This means that there exists some function

$$\psi : X \to [0, \infty)$$

which is proper and satisfies $i\partial\bar\partial\psi > 0$. More generally, a complex manifold is *Stein* if it has some strictly plurisubharmonic exhaustion.

Proposition 6.1. *Any Stein manifold has a complete Kähler metric.*

PROOF. Let $\Psi = e^\psi$ where ψ is strictly plurisubharmonic and exhaustive, and let

$$\omega = i\partial\bar\partial\Psi = (i\partial\psi\wedge\bar\partial\psi + i\partial\bar\partial\psi)e^\psi.$$

This is then a Kähler metric (since ψ is strictly plurisubharmonic) and we claim that it is complete. For this we take

$$\chi = e^{\psi/2}.$$

Then

$$\partial\chi\wedge\bar\partial\chi = (1/4)i\partial\psi\wedge\bar\partial\psi e^\psi \leq \omega,$$

so $|\partial\chi|^2$ is bounded. □

Theorem 3.5.1 holds verbatim for manifolds having a complete Kähler metric, and hence in particular for Stein manifolds.

Theorem 6.2. *Let L be a holomorphic line bundle endowed with a metric ϕ over a complex manifold X which has some complete Kähler metric. Assume the metric ϕ on L has (strictly) positive curvature and that*

$$ic(\phi) \geq c\omega$$

where $c(\phi) = \partial\bar\partial\phi$ is the curvature of ϕ and ω is some Kähler form on X.

Let f be a $\bar\partial$-closed (n,q)-form ($q > 0$) with values in L. Then there is an $(n,q-1)$-form u with values in L such that

$$\bar\partial u = f$$

and

$$\|u\|^2 \leq \frac{1}{cq}\|f\|^2,$$

provided that the right hand side is finite.

To prove the theorem we first assume that the metric ω is itself complete. (Hopefully, it is useful here to refer back to section 2.0.1 for the same argument in a model example.) As in the previous section, we then need to establish an inequality

$$cq\|\alpha\|^2 \leq \|T^*\alpha\|^2 \tag{3.9}$$

for the adjoint of the operator $T = \bar\partial$, where α is $\bar\partial$-closed. (We use the same notation for the Hilbert spaces and operators between them as in the previous section.) An (n,q)- form α lies in the domain of T^* if for some η in H_1

$$(Tu,\alpha)_{H_2} = (u,\eta)_{H_1},$$

and then $T^*\alpha = \eta$. Testing this condition for u smooth with compact support it follows that $\eta = \bar\partial^*_\phi\alpha$, the formal adjoint of $\bar\partial$ acting on α in the sense of distributions.

Lemma 6.3. *Let α belong to the domain of T^* and assume that $\bar\partial\alpha$ lies in H_3. Then there is a sequence of forms with compact support α_k such that*

$$\|\alpha_k - \alpha\|^2,$$

$$\|\bar\partial^*_\phi\alpha_k - T^*\alpha\|^2,$$

and

$$\|\bar\partial\alpha_k - \bar\partial\alpha\|^2$$

all tend to zero.

PROOF. Let $g(t)$ be a smooth function on $\mathbb{R}$ which equals 1 for $t \leq 0$ and 0 for $t \geq 1$. Let

$$\chi_k = g(\chi - k),$$

where χ is a smooth exhaustion with bounded differential (which exists by the completeness). Then χ_k tends to 1 on X and has uniformly bounded differential. Let $\alpha_k = \chi_k \alpha$. The three limits are then easy to check using dominated convergence. □

The proof now follows the argument in the previous section. We approximate all the α_ks by forms with compact support that are moreover smooth. For these forms we apply Corollary 3.4.3, and then get 3.9 by taking limits (recall $\bar\partial\alpha = 0$). The theorem then follows from Proposition 2.0.5.

For the general case we now assume only that $ic(\phi) \geq c\omega$, but that there is on X some other metric with Kähler form ω' which is complete. To simplify the presentation, we shall assume that this metric is of the form

$$\omega' = i\partial\bar\partial\psi$$

where ψ is a strictly plurisubharmonic function. This is not necessary for the theorem, but makes the exposition simpler, and it is certainly satisfied in the Stein case. Let

$$\omega_{(k)} = \omega + \omega'/k.$$

All metrics $\omega_{(k)}$ are then complete, and if

$$\phi_{(k)} = \phi + c\psi/k$$

then $ic(\phi_{(k)}) \geq c\omega_{(k)}$. By the case we have just discussed, there is a solution u_k to $\bar\partial u_k = f$ with

$$cq\|u_k\|_k^2 \leq \|f\|_k^2,$$

where $\|\cdot\|_k$ are the norms with respect to the metrics $\omega_{(k)}$ on X and $\phi_{(k)}$ on L, provided the right hand side is finite.

Lemma 6.4. *Let $\omega_1 \leq \omega_2$ be two Kähler forms, and let $\|\cdot\|_{1,2}$ be the corresponding norms. Then, if f is an (n,q)-form*

$$\|f\|_2^2 dV_{\omega_2} \leq \|f\|_1^2 dV_{\omega_1}.$$

PROOF. Write $f = w \wedge g$ where w is $(n,0)$ and g is $(0,q)$. Then

$$\|f\|_i^2 = \|w\|_i^2 \|g\|_i^2$$

for both norms. But for $(n,0)$-forms

$$\|w\|^2 dV = c_n w \wedge \bar w$$

is independent of the metric. Hence we need only prove that

$$\|g\|_2^2 \leq \|g\|_1^2.$$

Choose a basis (at a point) such that

$$\omega_1 = i\sum dz_i \wedge d\bar z_i$$

and

$$\omega_2 = i\sum \lambda_i dz_i \wedge d\bar z_i,$$

with $\lambda_i \geq 1$. Then $\sqrt{\lambda_i} dz_i$ are orthonormal for ω_2. Hence, if $g = \sum g_J dz_j$

$$\|g\|_1^2 = \sum |g_J|^2$$

while

$$\|g\|_2^2 = \sum |g_J|^2/\lambda_J,$$

with λ_J the product of all λ_i for i in J. This proves the lemma. □

From the lemma we get that

$$\|f\|_k^2 \leq \|f\|^2,$$

so if the right hand side here is finite we have a uniform bound for all the norms with respect to $\omega_{(k)}$. We therefore get a uniform estimate for the norms of all u_k and the final theorem follows by taking limits of a suitably chosen subsequence, weakly convergent on any compact part of X.

□

Exercise Carry out the last part of the proof in detail! □

We stress once again that Theorem 3.6.2 applies to any Kähler metric, complete or not. The assumption that there exist some complete Kähler metric is a condition on the domain, but it is not visible in the final estimate. The assumption is satisfied by any pseudoconvex domain in $\mathbb{C}^n$ as we have seen, and the theorem therefore gives existence and estimates for $\bar\partial$ in e g the ball with respect to e g the Euclidean metric. This particular case is not much easier than the general case.

6.3. A basic fact of life on Stein manifolds

As an application of the results above for Stein manifolds, we shall now prove a Runge-like approximation theorem. Let X be a Stein manifold, so that we know that there is some smooth strictly plurisubharmonic exhaustion function on X. If Y is a compact subset of X we say that Y is *Runge* in X if for any open neighbourhood U of Y, there is some exhaustion as above which is negative on Y, and positive outside U. A main example of this situation is $X = \mathbb{C}^n$ and Y a polynomially convex subset, i e a compact that can be approximated arbitrarily well from outside by sets defined by polynomial inequalities, $\{|P_j| \leq 1, j = 1, 2...N\}$.

Exercise: Verify this! □

Theorem 6.5. *Let F be a holomorphic line bundle on X and let Y be a compact Runge subset of X. Then any holomorphic section of F defined in some open neighbourhood of Y can be approximated arbitrarily well in the supremum norm on Y by global sections of F. In particular, F has some nontrivial global holomorphic section.*

Proof. Let $L = F - K_X$ so that $F = K_X + L$. Then sections of F are $(n,0)$-forms with values in L, and F-valued $(0,1)$-forms are $(n,1)$-forms with values in L. Let h be a holomorphic section of F defined in a neighbourhood of Y. Choose some cut-off function χ which equals 1 in some open neighbourhood of Y and is supported in the open neighbourhood where h is defined. Then $v = \chi h$ and

$$f = \bar\partial v = \bar\partial \chi h$$

are global objects. We shall solve the $\bar\partial$-equation

$$\bar\partial u = f$$

in such a way that u is small on Y. Then $h' := v - u$ is a global holomorphic section of F that approximates h on Y.

By assumption there is some strictly plurisubharmonic plurisubharmonic exhaustion function on X, ψ, such that $\psi \leq 0$ on a neighbourhood V of Y and $\psi \geq \delta$ on the support of $\bar\partial\chi$. Let ϕ be some smooth metric on L such that $i\partial\bar\partial\phi \geq \omega$, where ω is some Kähler form on X. (Such a metric can always be constructed by choosing an arbitrary smooth metric on L and then adding $k \circ \psi$ where k is a sufficiently increasing convex function.) Let $\phi_m = \phi + m\psi$. By Theorem 3.6.2 we can solve

$$\bar\partial u = f$$

with u satisfying

$$\int |u|^2 e^{-\phi_m} \leq \int |f|^2 e^{-\phi_m}.$$

The right hand side here is bounded by $Ce^{-m\delta}$ so it follows that

$$\int_V |u|^2 \leq Ce^{-m\delta}.$$

But u is holomorphic near Y so it follows from the Cauchy estimates that u tends to 0 uniformly on Y. This completes the proof of the first part. To see that there are some compact Runge subsets on X, so that the theorem is not void, it suffices to take

$$Y = \{\psi \leq c\}.$$

If c is small enough, Y is a small neighbourhood of the minimum point of ψ, over which L is trivial, so we can find plenty of local holomorphic sections there. □

LECTURE 4

The Bergman kernel

In the first section of this chapter we will give the basic definitions and properties of Bergman kernels associated to Hilbert spaces of functions such that point evaluations are bounded linear functionals on the space. (The main example to keep in mind is an L^2 space of holomorphic functions on a domain in $\mathbb{C}^n$.) The next section discusses the analogous constructions for spaces of sections of a line bundle. This is very similar to the scalar valued case, but formally a bit different. In particular, the Bergman kernel is then no longer a function, but behaves like a metric on the line bundle. Here we will also give the simplest asymptotic estimates for Bergman kernels associated to high powers of the line bundle.

1. Generalities

Let (X, μ) be a measure space, and let $\mathcal{H}$ be a closed subspace of $L^2(X, \mu)$. Assume that $\mathcal{H}$ is such that each function in $\mathcal{H}$ has a pointwise value at every point, and that the map

$$h \mapsto h(x)$$

is a bounded linear functional on $\mathcal{H}$. A basic example of this situation is that X is a domain in $\mathbb{C}^n$, $d\mu = d\lambda$ is Lebesgue measure, or $d\mu = e^{-\phi} d\lambda$ where ϕ is a weight function locally bounded from above. Then, by the Riesz representation theorem, there is for any x in X a unique element k_x of $\mathcal{H}$ such that

$$h(x) = (h, k_x).$$

Definition: The function k_x is the *Bergman kernel* for the point x. The function $K(x) = k_x(x)$ is the Bergman kernel on the diagonal. □

Another way to obtain the Bergman kernel is to start from an orthonormal basis of $\mathcal{H}$, h_j. Then

$$k_x(y) = \sum h_j(y)\overline{h_j(x)}$$

and

$$K(x) = \sum |h_j(x)|^2.$$

This requires some justification:

Proposition 1.1. *For any $N < \infty$*

$$\sum^N |h_j(x)|^2 \leq K(x).$$

Proof. Let $h = \sum^N a_j h_j$ with $\sum |a_j|^2 \leq 1$. Then $\|h\| \leq 1$ so

$$|h(x)|^2 = |(h, k_x)|^2 \leq (k_x, k_x) = K(x).$$

Since the coefficients a_j are here arbitrary, this implies that

$$\sum^N |h_j(x)|^2 \leq K(x).$$

□

Hence we see that

$$\sum^\infty |h_j(x)|^2$$

and therefore

$$\sum^\infty h_j(y)\overline{h_j(x)}$$

converges pointwise. Moreover

$$\sum h_j(\cdot)\overline{h_j(x)}$$

converges in L^2 to some function h_x. Then, for any l,

$$(h_l, h_x) = h_l(x)$$

so h_x must be equal to k_x.

From this expression for the Bergman kernel, we see that if we take the scalar product

$$(g, k_x) := \hat{g}(x)$$

for a function g which is not necessarily in $\mathcal{H}$, then

$$\hat{g} = \sum h_j(g, h_j)$$

is the orthogonal projection of g on $\mathcal{H}$. This gives us yet another way of looking at the Bergman kernel: it is the kernel of the orthogonal projection of a function on $\mathcal{H}$.

Another important property is the extremal characterization of the Bergman kernel on the diagonal:

Proposition 1.2. *The Bergman kernel on the diagonal $K(x)$ is the extremum*

$$s_x := \sup |h(x)|^2$$

over all elements h in $\mathcal{H}$ of norm at most 1. In other words, $K(x)$ is the norm of the point evalution at x.

PROOF.

$$s_x = \sup |(h, k_x)|^2 = \|k_x\|^2 = (k_x, k_x) = K(x).$$

□

Much of the strength of the Bergman kernel comes from the interplay between those characterizations of k_x. Here is one striking example.

Proposition 1.3. *Let $\mathcal{H}$ be a subspace of $L^2(X, \mu)$ consisting of continuous functions, and assume that for any h in $\mathcal{H}$*

$$|h(x)|^2 \leq C(x)\|h\|^2.$$

Then

$$dim\mathcal{H} \leq \int_X C(x)d\mu.$$

PROOF. The first condition implies that $K(x) \leq C(x)$. Hence if h_j is an ON-basis for $\mathcal{H}$

$$\dim\mathcal{H} = \int_X \sum |h_j|^2 = \int_X K(x) \leq \int_X C(x) d\mu.$$

□

We end this section with an asymptotic estimate for the Bergman kernel associated to weighted L^2-spaces of holomorphic functions. Let ϕ be a function of class C^2 in an open set Ω in $\mathbb{C}^n$. Consider the space $L^2_k := L^2(\Omega, e^{-k\phi})$ and its subspace $\mathcal{H}_k$ of holomorphic functions. Let K_k be the Bergman kernel on the diagonal for $\mathcal{H}_k$.

Proposition 1.4.

$$\limsup_{k\mapsto\infty} \frac{1}{k^n} K_k(z) e^{-k\phi(z)} d\lambda \leq \pi^{-n}\chi_0(z)(i\partial\bar{\partial}\phi)_n,$$

where χ_0 is the characteristic function of the open set X_0 where $i\partial\bar{\partial}\phi$ is positive.

PROOF. We will use the extremal characterization of $K_k(z)$, so let h be a holomorphic function in Ω such that

$$\int_\Omega |h|^2 e^{-k\phi} \leq 1.$$

Say $z = 0$. We need to estimate $h(0)$. Assume first that ϕ has the Taylor expansion

$$\phi(\zeta) = \phi(0) + \sum \lambda_j |\zeta_j|^2 + o(|\zeta|^2)$$

with $\lambda_j \geq 0$ near the origin. Then for any finite A

$$1 \geq \int_{|\zeta|^2 \leq A/k} |h|^2 \, e^{-k\phi} \geq (1-\epsilon_k) e^{-k\phi(0)} \int_{|\zeta|^2 \leq A/k} |h|^2 \, e^{-k\sum \lambda_j |\zeta_j|^2},$$

where ϵ_k tends to zero.

By the mean value property of holomorphic functions the last integral dominates

$$|h(0|^2 \int_{|\zeta|^2 \leq A/k} e^{-k\sum \lambda_j |\zeta_j|^2} = |h(0)|^2 (\lambda_1 ... \lambda_n k^n)^{-1} \pi^n (1-\delta_A)),$$

where δ_A tends to 0 as A goes to infinity. Hence

$$h(0) e^{-k\phi(0)}/k^n \leq \pi^{-n}(\lambda_1 ... \lambda_n)(1+2\delta_A)(1+2\epsilon_k)$$

for A and k large. Taking the supremum over all h of norm at most 1 we get that

$$K(0) e^{-k\phi(0)}/k^n \leq \pi^{-n}(\lambda_1 ... \lambda_n)(1+2\delta_A)(1+2\epsilon_k)$$

and the claimed estimate follows by letting first k and then A tend to infinity, since

$$\lambda_1 ... \lambda_n d\lambda = (i\partial\bar{\partial}\phi)_n.$$

If on the other hand ϕ has the same type of Taylor expansion with one of the eigenvalues λ_j negative or 0 it follows from a similar argument that $K_k e^{-k\phi}(0)$ tends to 0.

The same argument applies if ϕ has the Taylor expansion

$$\phi(\zeta) = \phi(0) + q(z, \bar{z}) + o(|\zeta|^2)$$

at the origin, where $q(z,\bar z)$ is an hermitian form, since we can then diagonalize q by a unitary change of variables. A general ϕ has the Taylor expansion

$$\phi(\zeta) = \phi(0) + q(z,\bar z) + 2\Re p(\zeta) + o(|\zeta|^2)$$

where p is a holomorphic polynomial of degree 2 and no constant term. The substitution

$$h \mapsto h e^{-p}$$

reduces this case to the one we have treated. □

2. Bergman kernels associated to complex line bundles

When generalizing the notion of Bergman kernel to spaces of sections of a line bundle instead of scalar valued functions one runs into no serious problem – but several minor complications. The first is that point evaluations are not well defined. The value of a section at a point x is an element in the fiber over that point, so the Bergman kernel k_x will also be L-valued, or rather $\bar L$-valued, as a function of x. When restricting to the diagonal, $K(x) = k_x(x)$ will then take its values in $L \otimes \bar L$, which means that

$$\psi := \log K$$

will behave like a metric on L. To avoid going through this in detail we will instead focus directly on the Bergman kernel on the diagonal and define it using an orthonormal basis instead.

Definition: Let L be a holomorphic line bundle over a complex manifold X, and let ϕ be a metric on L. Let μ be a measure on X, and let u_j be an orthonormal basis for the space

$$H^0(X,L)$$

of global holomorphic sections of L, with respect to the scalar product

$$(u,v) = \int_X u\bar v e^{-\phi} d\mu.$$

Then

$$B := \sum |u_j|^2 e^{-\phi}$$

is the *Bergman function*, and

$$K(z) := B(z)e^{\phi}$$

is the *Bergman kernel* (on the diagonal) for L, ϕ, μ. □

It is clear that in case L is trivial so that ϕ is just a global function on X, this definition coincides with the definition we gave in the previous section. Notice that since

$$\log K = \log B + \phi$$

and $\log B$ is a function, $\log K$ is a metric on L, just like we expected. One checks that the sum defining B is convergent in much the same way as we did in the scalar valued case before. The extremal characterization of the Bergman kernel takes the following form.

Proposition 2.1.

$$B(z) = \sup |u(z)|^2 e^{-\phi(z)}$$

where the supremum is taken over all global holomorphic sections of L with L^2-norm at most 1.

PROOF. A global section of L^2-norm at most 1 can be written

$$u = \sum a_j u_j$$

where $\sum |a_j|^2 \leq 1$. Evaluating at z, the proposition follows immediately from this:

$$|u(z)|^2 e^{-\phi} \leq B(z)$$

and equality can occur. □

Consider now powers L^k of the line bundle L, and give them the metrics $k\phi$. For each k we then get a Bergman function, that we denote B_k. The next proposition generalizes 5.1.4.

Proposition 2.2. *Let μ be an arbitrary measure on X with smooth density. Then*

$$\limsup_{k \mapsto \infty} \frac{1}{k^n} B_k(z) d\mu \leq \pi^{-n} \chi_0(z) (i\partial\bar{\partial}\phi)_n,$$

where χ_0 is the characteristic function of the open set X_0 where $i\partial\bar{\partial}\phi$ is positive.

We omit the proof of this since it is virtually identical to the proof of 4.1.4 – the arguments there were entirely local, using only the submeanvalue property of holomorphic functions near z. It should however be noted explicitly that the measure μ plays a very small role in the asymptotic estimate. This may seem a bit surprising, but the reason is that the estimates are carried out in a shrinking neighbourhood of a given point. Different choices of μ (with smooth densities!) are then related by a multiplicative factor that is almost constant, and it is very easy to see that the expression

$$K d\mu$$

is unchanged if we multiply μ by a constant.

From this we can rather easily prove the next estimate on the dimension of the space of global sections to high powers of a line bundle.

Theorem 2.3. *Let X be compact and put*

$$h^0_k = dim\, H^0(X, L^k).$$

Then

$$\limsup \frac{h^0_k}{k^n} \leq \pi^{-n} \int_{X_0} (i\partial\bar{\partial}\phi)_n.$$

In particular, $h^0_k = O(k^n)$ for any line bundle over any compact manifold. Notice that the left hand side is independent of the choice of metrics.

The proof of Therorem 4.2.3 is, just like the proof of Proposition 4.1.3, based on the fact that

$$\text{dim} H^0(X, L^k) = \int B_k.$$

(Again, this holds since

$$B := \sum |u_j|^2 e^{-\phi}$$

and each term in this sum contributes a 1 to the integral). We now want to apply Fatou's lemma, and for this we claim first that

$$\frac{1}{k^n} B_k(z)$$

is uniformly bounded from above. To see this, we use the extremal characterization

$$B_k(z) = \sup |u(z)|^2 e^{-k\phi(z)}$$

where u is a section of L^k of unit L^2-norm. Fix the point $z = 0$ and choose a local trivialization of L near 0 in which ϕ gets the form

$$\phi(\zeta) = q(z, z) + o(|\zeta|^2)$$

(cf the end of the proof of Proposition 4.1.4). Then, if u has unit norm

$$1 \geq \int_{|\zeta|^2 \leq 1/k} |u|^2 \, e^{-k\phi} \geq C \int_{|\zeta|^2 \leq 1/k} |u|^2 \geq C' |u(0)|^2 k^n$$

by the submeanvalue property. Hence

$$|u(0)|^2 e^{-k\phi(0)} \leq A/k^n$$

, since $\phi(0) = 0$, so the uniform upper bound for B_k follows. Hence, by Fatou's lemma,

$$\limsup \int B_k/k^n \leq \int \limsup B_k/k^n \leq \pi^{-n} \int_{X_0} (i\partial\bar{\partial}\phi)_n$$

which proves the theorem. □

We finally note that in case the metric has positive curvature, equality actually holds in the asymptotic estimate for the bergman kernels.

Theorem 2.4. *Assume X is compact and $ic(\phi) > 0$. Then*

$$\lim \frac{h_k^0}{k^n} = \pi^{-n} \int_{X_0} (i\partial\bar{\partial}\phi)_n.$$

This follows just like before (this time using dominated convergence instead of Fatou's lemma) from the following precision of Proposition 4.2.2 for positive bundles.

Theorem 2.5. *Assume that X is compact and that $ic(\phi) > 0$. Define a Kähler metric on X by $\omega = ic(\phi)$ and let $dV_\omega = \omega_n$ be its volume form on X. Let B_k be the associated Bergman function. Then*

$$\lim B_k/k^n = \pi^{-n}.$$

To prove this last theorem we have to construct a global holomorphic section with a large value at a given point x. This is no longer a local estimate and we have to use $\bar{\partial}$-estimates to obtain it. For this we choose local coordinates z centered at x and choose a local trivialization so that our metric on L is represented by a function

$$\phi(z) = \sum \lambda_j |z_j|^2 + o(|z|^2)$$

near that point. For simplicity we can even change coordinates by a linear transformation so that

$$\phi(z) = |z|^2 + o(|z|^2).$$

Let χ be a smooth function in $\mathbb{C}^n$, supported in a ball with radius 2 centered at the origin and equal to 1 in the unit ball. Let δ_k be a sequence of numbers tending to zero (to be chosen later such that $\delta_k k^{1/2}$ goes to infinity) and put

$$\chi_k(z) = \chi(zk^{1/2}\delta_k).$$

We can consider χ_k to be a section of L supported in a shrinking neighbourhood of x. Simple estimates (using the change of variables $w = zk^{1/2}$) show that the L^2-norm of χ_k is asymptotic to $k^n\pi^n$, and the L^2 norm of $f_k := \bar\partial\chi_k$ is dominated by

$$k^{n+1}\delta_k^2 e^{-\delta_k^{-2}}.$$

We now want to apply Theorem 3.5.1 to solve the equation $\bar\partial u_k = f_k$. There is a very minor problem here that the bundle we are dealing with is L^k and does not contain the canonical bundle K_X as a summand. This is easily resolved by writing

$$L^k = K_X + F_k$$

with $F_k = L^k - K_X$. This bundle has a natural metric with curvature asymptotic to $k\omega$ so Theorem 3.5.1 does apply.

Hence we can solve $\bar\partial u_k = f_k$ with the L^2-norm of u_k controlled by $k^n\delta_k^2 e^{-\delta_k^{-2}}$. We can certainly choose δ_k so that this quatity goes to zero faster than polynomially. Since u_k is holomorphic near the origin, it then follows from the submeanvalue property that $u_k(0)$ also goes to zero.

Let $s_k = \chi_k - u_k$. Then s_k is a global holomorphic section of L^k with L^2-norm asymtotic to $k^n\phi^n$. Moreover, $|s_k(x)|^2 e^{-k\phi(x)}$ tends to 1. Hence

$$\limsup B_k(x)/k^n \geq \pi^{-n}$$

which together with Proposition 4.2.2 proves the theorem. □

This asymptotic formula for the Bergman function of a positive bundle was first obtained (in a sharper form) by Bouche, [**5**], and Tian, [**22**]. Much sharper asymptotic formulas, giving asymptotic developments for the Bergman function in powers of k, have been found later, see [**7**], [**24**] and also [**4**] .

2.4. The field of meromorphic functions on a compact complex manifold

In this section we will, as an application of the upper estimate on the dimension of the space of global sections, give a proof of a classical result of Siegel.

Theorem 2.6. *Let X be a compact complex manifold of dimension n, and let $g_1, ...g_{n+1}$ be meromorphic functions on X. Then there is a polynomial P such that*

$$P(g_1, ...g_{n+1}) = 0.$$

In other words, any $n+1$ meromorphic functions are algebraically dependent.

Proof. We first claim that there is a holomorphic line bundle L over X with a holomorphic section s, such that $s_j := sg_j$ are all holomorphic (sections of L). To see this, we argue as follows. A meromorphic function function g can be written locally, in a neighbourhood U of any given point,

$$g = f/h$$

where f and h are holomorphic, and by choosing them without common factors, this factorization is unique up to units, i e up to nonvanishing holomorphic factors. Cover X by open sets U_j where such a representation

$$g = f_j/h_j$$

holds. Then $h_j/h_i := h_{ij}$ are nonvanishing functions satisfying the cocycle condition, so they define a line bundle L of which $s = (h_i)$ is a section. (This is the line bundle associated to the *polar divisor* of g.) Clearly, gs is then holomorphic.

This proves the claim for one single meromorphic function; the case of an N-tuple (g_j) follows by taking the product of sections, one for each g_j.

With this in our hands, we consider sections to L^k of the form

$$s_0^{k_0}...s_{n+1}^{k_{n+1}},$$

where $k_0 + ...k_{n+1} = k$ and $s_0 = s$. The number of such sections is of the order k^{n+1}. (It is equal to the number of choices of $k_1, ...k_{n+1}$ with $k_1 + ...k_{n+1} \leq k$ which is the number of lattice points in k times the unit simplex in $\mathbb{R}^{n+1}$.) By our estimate for the dimension of the space of global sections of L^k $(= O(k^n))$ they must be linearly dependent for k large. This means that there is a homogenous polynomial in the s_j that vanishes identically. Clearing out the factor s we get a polynomial in g_j that vanishes identically. □

Exercise: Consider the meromorphic functions $1/z$ and $1/(z-1)$ on $\mathbb{P}^1$. Find a polynomial P satisfying the conclusion of the theorem. □

LECTURE 5

Singular metrics and the Kawamata-Viehweg vanishing theorem

In Chapter 3 we have defined a (smooth) metric on a line bundle L as a collection of functions ϕ_i defined on trivializing open sets U_i for L, satisfying the compatibility condition

$$\phi_i - \phi_j = \log |g_{ij}|^2$$

A *singular* metric on L is defined in the same way, but allowing ϕ_i to be non-smooth. To fix ideas one usually requires ϕ_i to be locally in L^1, but in practice ϕ_i is locally the difference between two plurisubharmonic functions and so lies in any L^p with $p < \infty$.

If L is *effective*, i e has a nontrivial global holomorphic section s, given by local holomorphic functions s_j, one can define a singular metric by $\phi = \log |s|^2$, i e $\phi_i = \log |s_i|^2$. More generally, if s^λ are global sections of L, then

$$\phi = \sum a_\lambda \log |s^\lambda|^2,$$

(for $\sum a_\lambda = 1$) and

$$\psi = \log \sum |s^\lambda|^2$$

are singular metrics on L. If the s^λs are holomorphic, then the singularities of the latter metric lies precisely on the common zero locus of the sections.

1. The Demailly-Nadel vanishing theorem

Just as before we can define the curvature of a singular metric by

$$c(\phi) = \partial\bar\partial\phi = \partial\bar\partial\phi_i$$

on U_i. This is of course not necessarily a smooth form anymore, but merely a current. We say that L is *pseudoeffective* if L has a metric with nonnegative curvature current, meaning precisely that the local representatives can be chosen to be plurisubharmonic. The two metrics ϕ and ψ above constructed from sections have this property if s^λ are holomorphic and ($a_\lambda > 0$), so effective bundles are pseudoeffective. The principal aim of this chapter is to generalize the main existence theorems of chapter 3 to the setting of singular metrics. This is the content of the Demailly-Nadel vanishing theorem, see [**8**], [**15**] . Demailly actually proved this theorem in the more general setting of manifolds with complete Kähler metrics. This used his technique of regularisation of singular metrics. Here we will restrict attention to *projective* manifolds and Stein manifolds, which is more elementary. Since we have not defined projective manifolds yet in these notes, we adopt the preliminary definition that a compact manifold X is projective if there is some ample line bundle F over X (recall that F is ample if it has some smooth metric, ψ with strictly positive curvature). Later, we shall use the Deamilly-Nadel vanishing theorem to prove the Kodaira embedding theorem, which says that manifolds that carry ample line bundles are precisely those that can be embedded in projective space - hence the terminology. Projective manifolds are in particular Kähler, since $i\partial\bar{\partial}\psi$ is a Kähler form.

Theorem 1.1. *Let X be a complex manifold which is either projective or Stein. Let L be a holomorphic line bundle over X which has a possibly singular metric ϕ whose curvature satisfies*

$$i\partial\bar{\partial}\phi \geq \epsilon\omega,$$

where ω is a Kähler form. Let f be an L-valued $\bar{\partial}$-closed form of bidegree (n, q). Then there is a solution u to the equation $\bar{\partial}u = f$ satisfying

$$\|u\|^2_{\omega,\phi} \leq \frac{1}{\epsilon q}\|f\|^2_{\omega,\phi},$$

provided the right hand side is finite.

Notice the last proviso of the theorem. That the L^2-norm with respect to a singular metric is finite is not only a global condition, but also a local condition which imposes vanishing of f on the nonintegrability locus of the metric $e^{-\phi}$. In case $\phi = \log|s|^2$ it essentially means that f vanishes on the zero divisor of s.

The strategy of the proof is to reduce to the case of smooth metrics (we already know the theorem for smooth metrics) by regularizing the singular metric. This regularization is however a somewhat delicate business: on a compact (even projective) manifold it is in general not possible to approximate singular metrics with smooth metrics, keeping positivity of the curvature. Therefore we shall first prove the theorem in the Stein case – where regularization *is* possible – and then get the projective case from there.

Proposition 1.2. *Let X be a Stein manifold and let D be a relatively compact Stein subdomain of X. Let χ be a plurisubharmonic function on X. Then there is a sequence of smooth, strictly plurisubharmonic functions, χ_ν, defined in a neighbourhood of the closure of D, that decrease to χ on D. Moreover, if $i\partial\bar{\partial}\chi \geq \omega$ where ω is a Kähler form and $\delta > 0$ the $\chi_\nu s$ can be chosen so that $i\partial\bar{\partial}\chi_\nu \geq (1-\delta)\omega$.*

We will not give a complete proof of this result, but merely indicate some steps in the proof. First, since X is Stein, X can be (properly) embedded in $\mathbb{C}^N$, so we may assume from the start that X is a submanifold of $\mathbb{C}^N$. Let D' be a larger Stein open subset of X containing D in its interior. Then there is a neigbourhood U of D' in $\mathbb{C}^N$, such that D' is a *holomorphic retract* of U, i e there is a holomorphic map p from U to D' which is the identity on D' (see Forster and Ramspott...). Then $\chi' := \chi \circ p$ is plurisubharmonic in U, so by the well konown technique of convolution with an approximate identity χ' can be approximated by a decreasing sequence of strictly plurisubharmonic smooth functions on any relatively compact subset. The $\partial\bar{\partial}$ of the approximating functions will then be bounded from below of a quantity tending to $p^*\omega$ on compact subsets. Restricting to D the proposition follows.

Using the proposition we can prove Theorem 6.0.1. Assume first that L is the trivial line bundle, and that ϕ is a global plurisubharmonic function on X. Let D

be a relatively compact Stein open set in X, and choose approximating plurisubharmonic functions ϕ_ν as in the proposition. By the Hörmander L^2-estimates, we can solve the equation $\bar\partial u = f$ with $u = u_\nu$ satisfying

$$\|u_\nu\|^2_{\omega,\phi_\nu} \leq \frac{1}{\epsilon(1-\delta)q}\|f\|^2_{\omega,\phi_\nu} \leq \frac{1}{\epsilon(1-\delta)q}\|f\|^2_{\omega,\phi}.$$

If $\nu > \nu_0$, $\|\cdot\|_{\omega,\phi_\nu} \geq \|\cdot\|_{\omega,\phi_{\nu_0}}$ Therefore we may, using a diagonal argument, choose a subsequence of the u_ν converging weakly in $L^2(e^{-\phi_{\nu_0}})$, for any ν_0, to a limit, u_D . It is easily checked that u still solves the $\bar\partial$-equation and satisfies the required estimate in D. We then let D increase to X and take weak limits again.

If X is still Stein, but we no longer assume that L is trivial, we proceed as follows. Let s be some nontrivial holomorphic section of L. (Such a section exists by Theorem 3.6.5.) Let S be the zero divisor of s. Then $X \setminus S$ is still Stein (why ?) so we may apply the previous argument and solve our $\bar\partial$ equation on $X \setminus S$. The proof will then be completed by the following very important lemma.

Lemma 1.3. *Let M be a complex manifold and let S be a complex hypersurface in M. Let u and f be (possibly bundle valued) forms in L^2_{loc} of M satisfying $\bar\partial u = f$ outside of S. Then the same equation holds on all of M (in the sense of distributions).*

Proof. Since the statement is local we may assume that M is an open ball in $\mathbb{C}^n$, that u and f are scalar valued, and that $S = h^{-1}(0)$ where h is holomorphic in M. Let $\xi(w)$ be smooth and nonegative in $\mathbb{C}$, equal to 0 for $|w| \leq 1/2$ and equal to 1 outside of the unit disk. Put

$$\xi_\delta = \xi(h/\delta).$$

Then

$$\bar\partial(u\xi_\delta) = f\xi_\delta + \bar\partial\xi_\delta \wedge u.$$

We then let δ tend to 0, and it suffices to show that the last term on the right hand side goes to zero locally in L^1. For this it is by the Cauchy inequality enough to prove that

$$\int |\bar\partial\xi_\delta|^2 \leq \frac{C}{\delta^2}\int_{\delta/2\leq|h|\leq\delta} |\partial h|^2$$

is uniformly bounded. This is clear if h vanishes to order 1 on M. The general case is left as an exercise below. □

Exercise: Prove that if h is holomorphic in a ball of radius 2, then

$$\int_{\delta/2\leq|h|\leq\delta,|z|<1} |\partial h|^2 \leq C\delta^2.$$

(Hint: Prove first that if χ is a cutoff function equal to 1 on the ball with radius 1 and supported in the bigger ball, then

$$\int \chi\Delta\log(|h|^2 + \delta^2)$$

is bounded.) □

This completes the proof of the theorem in the Stein case: Our L^2 estimate for the solution u shows in particular that u is locally in (unweighted!) L^2 outside of S, so the lemma implies that u actually solves the $\bar\partial$-equation across S too.

A very similar argument is used to prove the projective case of Theorem 6.0.1. If X is projective there is some positive line bundle F over X. A sufficiently high power of F then has a nontrivial holomorhic section. The complement of the zerodivisor of that section is then Stein, so by the Stein case of the theorem we can solve our $\bar\partial$-equation there. But, then the lemma shows that we have actually solved the $\bar\partial$-equation in all of X.

2. The Kodaira embedding theorem

Let L be a line bundle over a compact complex manifold X, and let $s_0, ...s_N$ be a basis for the space, E, of global holomorphic sections of L (we assume that there are such sections so that the dimension of E is not zero). The *Kodaira map* associated to L is a holomorphic map from X to N-dimensional projective space, defined as follows :

$$\mathcal{K}(x) = [s_0(x), ...s_N(x)].$$

Some comments are in order. The s_js are sections to a line bundle so the values in the right hand side needs some interpretation. Choose a local trivialisation of L and let the right hand side mean the values of the sections s_j with respect to that trivialization. If we change to another trivialization, all s_j get multiplied by the same quantity, so the corresponding point in projective space is the same. Thus $\mathcal{K}$ is well defined - and of course holomorphic.

With the definition that we have just given, the Kodaira map is dependent of the choice of basis. Somewhat more elegantly we can consider the Kodaira map as a map from X to the projectivization of the dual space, E^*, of E: A point x in X is mapped to the element of E^* that is evaluation in that point with respect to some local trivialization near x, and then to the corresponding point in $\mathbb{P}(E^*)$. Again, when we take projectivization of E^* the final map will be independent of the choice of local trivialization. In terms of the basis above any section can be written

$$s = \sum a_j s_j.$$

Thus $a = (a_j)$ are the coordinates of s with respect to the basis and the evaluation map is given by

$$s(x) = \langle a, (s_j(x)) \rangle,$$

so our previous definition gives the coordinates of the evaluation map in the dual basis. The Kodaira embedding theorem says that if L is sufficiently positive, i e if the curvature of L is sufficiently large, then the Kodaira map is an embedding.

Theorem 2.1. *Let X be a compact complex manifold. Let L be a holomorphic line bundle over X, having a (smooth) metric ϕ of positive curvature. Then the Kodaira map for $K_X + L^k$ is an embedding if k is large enough.*

Proof. We will prove only that the Kodaira map is injective, leaving the injectivity of its differential as an exercise. We need to prove that if x_1 and x_2 are distinct points in X, then there is a section of kL that vanishes at x_2 but not at x_1. Take local coordinates near x_1, z, centered at x_1 and put

$$\xi_1 = \chi_1(z) n \log |z|^2$$

where χ_1 is a cut-off function with compact support in the coordinate neighbourhood that equals 1 near 0. Define ξ_2 in a similar way, and let $\xi = \xi_1 + \xi_2$ Then

$$i\partial\bar\partial\xi \geq -Ci\partial\bar\partial\phi,$$

where C can be taken independent of the choice of the x_is. Consider the bundle kL with k larger than $C+1$, and give kL the (singular) metric

$$\psi = k\phi + \xi,$$

so that $i\partial\bar\partial\psi \geq \omega = i\partial\bar\partial\phi$. We define a smooth section to $K_X + kL$ by

$$s' = \chi dz_1 \wedge ...dz_n,$$

where χ is again a cut-off function equal to 1 near 0, and we also arrange things so that χ vanishes near x_2 Put $f = \bar\partial s'$. We now apply the Demailly-Nadel vanishing theorem and find a solution u to $\bar\partial u = f$ such that

$$\|u\|^2_{\omega,\psi} \leq \|f\|^2.$$

Note that f vanishes near the points where $e^{-\psi}$ is not integrable (i e near x_1 and x_2), so the right hand side is finite and the theorem does apply. The left hand side is also finite so u must vanish at x_1 and x_2. Hence $s := s' - u$ is holomorphic, vanishes at x_2 but not at x_1. □

Exercise: Use a similar technique to find a holomorphic section of $K_X + kL$ with prescribed first order Taylor expansion at a given point x in X. Show that this means that the Kodaira map has nondegenerate differential if k is large enough. □

Thus we see that any compact X which carries a positive line bundle is actually biholomorphic to a smooth submanifold of some projective space. The converse to this statement is also true - since projective space carries the positive bundle $\mathcal{O}(1)$ we get a positive bundle on any submanifold by taking restrictions.

Finally a word on terminology. A line bundle is said to be *very ample* if its Kodaira map is an embedding. It is said to be *ample* if some power of it is very ample. Thus Kodaira's embedding theorem can be rephrased as saying that positive bundles are ample. Conversely, very ample bundles must be positive since they are isomorphic to the pullback of $\mathcal{O}(1)$ under the Kodaira embedding.

Exercise: Prove this!

Hence ample bundles are also positive, so positivity and ampleness are equivalent.

3. The Kawamata-Viehweg vanishing theorem

A line bundle L over a compact manifold is said to be *numerically effective* if for any $\epsilon > 0$ there is a smooth metric on L with

$$i\partial\bar\partial\phi > -\epsilon\omega,$$

where ω is any given Kähler form. This is not the original definition, which says that L is numerically effective if for some (and hence any) smooth metric ψ on L the integral

$$\int_C i\partial\bar\partial\psi \geq 0,$$

for any curve C in X. If X is projective, the *Kleiman criterion* says that this definition coincides with the one we have given. In a non-projective manifold the definition we have chosen (which originates with Demailly) is the only one possible, since there may not be any curves C to test the integral condition on. Moreover, because of its more analytic character it fits much better with the methods of these notes.

We already know from chapter 5 that if d_k is the dimension of the space of global sections of kL, then d_k grows with k at most at the rate

$$d_k \leq Ck^n.$$

We say that L is *big* if this maximal growth rate is actually attained, so that for k large

$$d_k \geq \epsilon k^n.$$

By the Bergman kernel asymptotics in the previous chapter any positive bundle is big. Bigness is however more general than positive. While, as we have seen in the previous section, "positive" means that the Kodaira map for kL is eventually an embedding, one can prove that bigness means that the image of the Kodaira map has maximal dimension ($= n$).

Using a method by Demailly ([**9**]) we shall now see that the Demailly-Nadel theorem implies a famous result of Kawamata [**11**] and Viehweg, [**23**].

Theorem 3.1. *Let X be a compact manifold with a holomorphic line bundle L that is both numerically effective and big. Then*

$$H^{n,q}(X, L) = 0,$$

for $q > 0$.

In order to prove the theorem it is enough to construct a singular metric, ϕ, on L, satisfying the hypotheses of the Demailly-Nadel vanishing theorem, which moreover is such that $e^{-\phi}$ is locally integrable. Then any smooth form has finite norm with respect to this metric, so theorem 5.1.1 implies that it is exact if it is closed. The first step is the following criterion of bigness, due to Kodaira.

Proposition 3.2. *A line bundle L on a projective manifold X is big (if and) only if some multiple of it can be written*

$$kL = A + E,$$

where A is ample and E is effective.

Proof. Since X is projective it carries some ample bundle A. Possibly after replacing A by some multiple we can assume that A has a nontrivial holomorphic section s. We can also arrange thing so that the zerodivisor, S of s is smooth. (This will be generically true if A is very ample, as follows from *Bertini's theorem*, a variant of Sard's lemma.) Now consider the short sequence of maps

$$H^0(X, kL - A) \mapsto H^0(X, kL) \mapsto H^0(X, kL|_S).$$

The first map here is taking products with s; the second is restricting to S. By the Bergman kernel asymptotics of Chapter 5, the dimension of the last space is at most Ck^{n-1}, and since L is big, the dimension of the middle space is of order k^n. Hence the kernel of the last map is nontrivial if k is large, and we let s_k be some element in the kernel. Then s_k can be written

$$s_k = st_k,$$

where t_k is some global holomorphic section of $E := kL - A$. Therefore E is effective, so we have proved the "only if" part. For the "if" part we note that if $kL = A + E$, then for any large integer m $H^0(X, mkL)$ has large dimension (since $H^0(X, mA)$ has large dimension and mE has at least some section). Hence

$$\liminf d_p/p^n > 0.$$

This actually implies that

$$\lim d_p/p^n > 0,$$

(since the limit exists), but we shall not prove it (nor use it). □

The next lemma i almost trivial.

Lemma 3.3. *If L is numerically effective and A is ample, then $L + A$ is ample.*

PROOF. Let ψ be a metric on A with strictly positive curvature. By assumption, L has a metric ϕ with

$$i\partial\bar{\partial}\phi > -i\partial\bar{\partial}\psi.$$

Hence $\phi + \psi$ is a metric on $L + A$ with strictly positive curvature. □

We can now conclude the proof of the Kawamata-Viehweg theorem. Write

$$kL = A + E,$$

with A ample and E effective. By the lemma

$$kL + L = A_1 + E$$

where A_1 is still ample. Iterating, we see that

$$(k + m)L = A_m + E,$$

where A_m is ample and E is the same effective bundle (it is important that E does not change with m). We equip $(k + m)L$ with the (singular) metric

$$\phi_m + \log |t|^2,$$

where ϕ_m has strictly positive curvature and t is one fixed section of E. Then

$$\phi_L := \frac{1}{k + m}(\phi_m + \log |t|^2)$$

is a metric on L with strictly positive (albeit small!) curvature current. Its only singularities come from the last term, so if m is large enough, $e^{-\phi_m}$ is integrable. This concludes the proof. □

Exercise: Justify the last part of the proof by showing that if h is a holomorphic function defined in a neighbourhood of a closed ball B, then

$$\int_B \frac{1}{|h|^\delta} < \infty$$

if δ is small enough. □

LECTURE 6

Adjunction and extension from divisors.

In this chapter we will discuss the Ohsawa-Takegoshi extension theorem, which deals with the extension from a divisor to the ambient space of holomorphic sections of a line bundle . The statement of the theorem is more natural if we consider *adjoint bundles* i e line bundles of the form

$$L + K_X,$$

where K_X is the canonical bundle of the manifold. This is of course only a matter of convenience, since any bundle can be put in this form, and we will afterwords also translate the theorem back to the non adjoint case. In the first section we discuss the relation between the canonical bundle of the divisor and the canonical bundle of the ambient space, which is described in the *adjunction formula*. There we also discuss some basic formulas describing the current of integration on a divisor that we will need in the sequel. In the second section we prove the extension theorem of Ohsawa-Takegoshi, see [**16**] for the first version of this theorem that has many variants. Apart from proving the possibility to extend sections under optimal conditions, this theorem gives sharp L^2-estimates for the extension. The hypotheses in the extension theorem involve an inequality between the curvatures of the line bundle L and the line bundle defined by the divisor. In the simple case of line bundles over the Riemann sphere the sections of these line bundles are polynomials, and the inequalities translate to an inequality between the number of points in the divisor, and the degree of an interpolating polynomial. In this way we see that the hypotheses are in fact sharp, already in the simplest case.

1. Adjunction and the currents defined by divisors.

Let S be a (smooth) hypersurface of X. Then S defines a divisor, that in turn defines an associated line bundle (S), having a global holomorphic section s which vanishes to degree 1 exactly on S. The adjunction formula expresses the canonical bundle of the hypersurface in terms of the canonical bundle of the ambient space and (S).

Theorem 1.1. *Let S be a smooth hypersurface in a complex manifold X. Then the canonical bundle of S satisfies*

$$K_S = (K_X + (S))|_S.$$

Proof. Let u be any local section of K_S, i e a locally defined $n-1$-form in an open set in S. Then

$$ds \wedge u$$

is a form of bidegree $(n, 0)$ with values in (S) (ds is not globally defined on X but its restriction to S is a well defined $(1, 0)$-form with values in (S) (why?)). Thus the map that takes u to $ds \wedge u$ is well defined and it is easily checked that it is an isomorphism at any point. □

In the next section we will discuss 'extension' of sections u of $K_S + L|_S$ to sections U of $K_X + (S) + L$ over all of X. Extension is here to be taken in the sense of the adjunction theorem so that

$$U = ds \wedge u$$

on S. The method of proof we will use (see [**1**], [**3**]) reduces this extension problem to the problem of solving a $\bar\partial$-equation

$$\bar\partial v = f,$$

where

$$f = u \wedge [S],$$

$[S]$ being the *current of integration on* S. We therefore first discuss such currents of integration in an attempt to make the main lines of the proof understandable also to readers without much knowledge of the theory of currents.

By the *Lelong-Poincaré formula* the current of integration on a hypersurface S is given by

$$[S] = \frac{i}{2\pi}\partial\bar\partial \log |h|^2$$

if h is any (local) holomorphic function vanishing to degree one precisely on S. This means that if α is any compactly supported form of bidegree $(n-1, n-1)$ then

$$\int_S \alpha = \frac{1}{2\pi}\int_X \log |h|^2 i\partial\bar\partial\alpha.$$

Exercise: Prove this formula! (Hint: Since h vanishes to degree 1 and the formula is local, you may choose coordinates so that $h = z_1$. Use that, in one complex variable,

$$\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\frac{1}{2\pi}\log |\zeta|^2 = \delta_0,$$

a Dirac mass at $\zeta = 0$.) □

We will also have use for the closely related formula

$$[S] = \frac{i}{2\pi}\partial h \wedge \bar\partial\frac{1}{h}.$$

This can again be proved by choosing local coordinates so that $h = z_1$. It also follows from our previous expression for $[S]$ since

$$\partial\bar\partial \log |h|^2 = -\bar\partial\frac{\partial h}{h} = \partial h \wedge \bar\partial\frac{1}{h}.$$

The estimates that we will find for the extension U is in term of the L^2-norms

$$\int_X c_n U \wedge \bar U e^{-\phi-\psi},$$

(where ϕ and ψ are metrics on L and (S) respectively) and

$$\int_S c_{n-1} u \wedge \bar u e^{-\phi}.$$

These norms are well defined if U and u are $(n, 0)$ and $(n-1, 0)$ forms with values in $L + (S)$ and L respectively. They can also be expressed in terms of L^2

norms with respect to the volume elements on X and S. For this we need to use explicitly the Kähler form ω on X.

This form ω induces a norm on $(n, 0)$ forms on X by

$$c_n U \wedge \bar{U} = |U|^2_\omega \omega_n.$$

We can also write

$$|U|^2_\omega = |U|^2 e^{-\phi_\omega}$$

to emphazise that this way ω induces a metric ϕ_ω on the canonical bundle of X. Then

$$\int_X c_n U \wedge \bar{U} e^{-\phi-\psi} = \int_X |U|^2_\omega e^{-\phi-\psi} \omega_n = \int_X |U|^2 e^{-\phi-\psi-\phi_\omega} \omega_n$$

To express the integral over S in a similar way we need to use dS, the volume (or surface) measure on the hypersurface S. This is a measure on X (concentrated on S of course),and can hence be viewed as a current of bidegree (n, n). We define a corresponding current of bidegree $(0, 0)$ by

$$*dS\omega_n = dS.$$

We will use a representation of the current of integration on S that we state as a separate lemma.

Lemma 1.2.

$$[S] = \frac{ids \wedge d\bar{s}}{|ds|^2_\omega} * dS, \tag{6.1}$$

where the right hand side is defined by taking any local reprsentative for the section s.

PROOF. The formula means that if α is any form of degree $2n - 2$ on X then

$$\int_S \alpha$$

equals the integral of α against the right hand side. Since booth sides depend only on the restriction of α to S, and since on S any form of maximal degree is a multiple of the volume form, it is enough to verify this for $\alpha = gw_{n-1}$ where g is some function. But since $|ds|^2_\omega$ satisfies

$$|ds|^2_\omega \omega_n = ids \wedge d\bar{s} \wedge \omega_{n-1}$$

we get

$$\int g\omega_{n-1} \wedge \frac{ids \wedge d\bar{s}}{|ds|^2_\omega} * dS = \int g\omega_n * dS = \int gdS = \int_S g\omega_{n-1},$$

by the definition of the volume form on S as ω_{n-1}. This proves (6.1). □

Exercise*: This exercise outlines the corresponding formulas in the setting of real manifolds. It is not used in the sequel.

a. Let M be a smooth hypersurface in an n-dimensional Riemannian manifold X. Let M be defined locally by an equation $\rho = 0$, where $d\rho \neq 0$ on M, and let dV_M be the Riemannian volume element on M (so that dV_M is a form of degree $n - 1$ on M). Let γ be a (locally defined) form of degree $n - 1$ on X. Then the restriction of γ to M equals dV_M if and only if

$$\gamma \wedge d\rho/|d\rho| = dV_X,$$

the Riemannian volume element of X.

b. Let dM be the surface measure on M considered as a measure on X concentrated on M, and define $*dM$ by

$$*dMdV_X = dM.$$

Then the current of integration on M is given by

$$[M] = \frac{d\rho}{|d\rho|} * dM.$$

(In both (a) and (b) we assume that M is appropriately oriented.) □

2. The Ohsawa-Takegoshi extension theorem.

We can now state the "adjunction version" of the Ohsawa-Takegoshi extension theorem. We first deal with the case of a compact manifold and smooth metrics.

Theorem 2.1. *Let X be a compact Kähler manifold and let S be a smooth hypersurface in X, defined by a global holomorphic section s of the line bundle (S). Let L be a complex line bundle over all of X. Assume that L and (S) have smooth metrics, ϕ and ψ respectively, satisfying the curvature assumptions*

$$i\partial\bar{\partial}\phi \geq 0$$

and

$$i\partial\bar{\partial}\phi \geq \delta i\partial\bar{\partial}\psi,$$

with $\delta > 0$. Assume moreover that s is normalized so that

$$|s|^2 e^{-\psi} \leq e^{-1/\delta}.$$

Let finally u be a global holomorphic section of $K_S + L|_S$.

Then there is a global holomorphic section U of $K_X + (S) + L$ such that

$$U = ds \wedge u$$

on S and such that U satisfies the estimate

$$\int_X c_n U \wedge \bar{U} e^{-\phi-\psi} \leq C_\delta \int_S c_{n-1} u \wedge \bar{u} e^{-\phi}. \tag{6.2}$$

Here C_δ is a constant depending only on δ.

One very interesting feature of the statement is that the constant C_δ depends only on δ- this is one reason why the theorem is so useful. Even more remarkably, if $i\partial\bar{\partial}\psi = 0$ (or ≤ 0), then the only assumption on ϕ is that $i\partial\bar{\partial}\phi \geq 0$. Thus we get a uniform constant even if we don't have a strict lower bound for the curvature. Let now u be a (local or global) section of $K_S + L|_S$. Suppose U is some section of $K_X + (S) + L$ such that

$$U = ds \wedge u$$

on S. Put $v' = -(i/2\pi)U/s$ so that v is a section of $K_X + L$. Then

$$\bar{\partial}v' = -(i/2\pi)\bar{\partial}\frac{1}{s} \wedge U = (i/2\pi) ds \wedge \bar{\partial}\frac{1}{s} \wedge u = u \wedge [S],$$

since $\bar{\partial}(1/s)$ vanishes outside of S. Conversely, suppose we are able to solve the $\bar{\partial}$-equation

$$\bar{\partial}v = u \wedge [S]. \tag{6.3}$$

Then $v - v' = h$ is holomorphic and hence in particular smooth. Therefore sv is also holomorphic and satisfies

$$2\pi i s v = ds \wedge u$$

on S, so any such v gives us a solution to the extension problem. The extension problem is therefore completely equivalent to the problem of solving (6.3).

We should point out that if we assume that we have *strict* inequality in the curvature assumption for ϕ so that $i\partial\bar\partial\phi > 0$, and if we don't care about estimates for the solution, then the possibility of solving (6.3) follows from the Kodaira vanishing theorem. Indeed, the Kodaira theorem says that we can solve any equation $\bar\partial v = f$ if f is a smooth L-valued $(n,1)$-form and from this it follows that one can solve such an equation even if f is not smooth, but just a current. This is a well known fact - the cohomology defined with currents, and the cohomology defined with smooth forms are isomorphic - but we will not prove it.

To solve (6.3) we follow the method used in the proof of the Hörmander L^2-estimates, but a new twist is needed since the right hand side is now a form with measure coefficients instead of a form in L^2 . Put

$$f := u \wedge [S],$$

so that f is a form of bidegree $(n,1)$ with values in L. Indeed, u is a form of bidegree $(n-1,0)$ on S and we can extend it smoothly in an arbitrary way to a form of the same bidegree on X. The wedge product $u \wedge [S]$ is independent of the choice of extension since $[S]$ is supported on S and contains a factor ds.

As in the proof of Hörmander's theorem we need to estimate the "scalar product" between f and a smooth compactly supported form α of bidegree $(n,1)$. Write

$$\alpha = \gamma \wedge \omega,$$

where $\gamma = \gamma_\alpha$ is $(n-1,0)$ and ω is the Kähler form. As in chapter 3 the scalar product then becomes

$$\int_X f \wedge \bar\gamma e^{-\phi}$$

which equals

$$\int u \wedge \bar\gamma e^{-\phi} \wedge [S] = \int_S u \wedge \bar\gamma e^{-\phi}.$$

By Cauchy's inequality this can be estimated

$$|\int_S u \wedge \bar\gamma e^{-\phi}|^2 \leq \int c_{n-1}\gamma \wedge \bar\gamma e^{-\phi},$$

if we normalize things so that the L^2-norm of u over S equals 1.

To estimate this quantity we use again Proposition 3.4.1, but this time we multiply the formula for $i\partial\bar\partial T_\alpha$ by a certain function $w \geq 0$ before integrating. After applying Stokes' formula this introduces an extra term $i\partial\bar\partial w$ which is the key to the estimate. Take

$$w = -r \log |s|^2 e^{-\psi},$$

where $0 < r < 1$. We formulate the basic estimate as a separate lemma.

Lemma 2.2.

$$|\int_S c_{n-1}\gamma \wedge \bar\gamma e^{-\phi}|^2 \leq C_\delta(\int_X e^w c_n \partial_\phi\gamma \wedge \overline{\partial_\phi\gamma} e^{-\phi} + \int_X (w+1)|\bar\partial\alpha|^2 e^{-\phi}\omega_n)$$

PROOF. By our size estimate on s, $w \geq r/\delta$. Proposition 3.4.1 then implies after integration by parts that

$$c_{n-1}\left(\int wi\partial\bar\partial\phi\wedge\gamma\wedge\bar\gamma e^{-\phi}-\int i\partial\bar\partial w\wedge\gamma\wedge\bar\gamma e^{-\phi}\right)\leq$$

$$\leq 2c_{n-1}\int w\bar\partial\partial_\phi\gamma\wedge\bar\gamma e^{-\phi}+\int w|\bar\partial\alpha|^2e^{-\phi}\omega_n. \tag{6.4}$$

By the Lelong-Poincaré formula

$$i\partial\bar\partial w = ri\partial\bar\partial\psi - r[S]\leq (r/\delta)i\partial\bar\partial\phi - r[S].$$

The first term here gives a negative contribution to the second integral in the left hand side of (6.4). Since $w \geq r/\delta$ this term is however controlled by the first integral. Hence

$$r\int_S c_{n-1}\gamma\wedge\bar\gamma e^{-\phi}\leq 2c_{n-1}\int w\bar\partial\partial_\phi\gamma\wedge\bar\gamma e^{-\phi}+\int w|\bar\partial\alpha|^2e^{-\phi}\omega_n. \tag{6.5}$$

Apply Stokes' formula to the first term in the right hand side. We then get

$$\int w\bar\partial\partial_\phi\gamma\wedge\bar\gamma e^{-\phi}=\int w\partial_\phi\gamma\wedge\overline{\partial_\phi\gamma}e^{-\phi}+\int\bar\partial w\wedge\partial_\phi\gamma\wedge\bar\gamma e^{-\phi}.$$

The first term in the right hand side here is OK as it stands, but the second one needs some extra work. By the Cauchy inequality it is dominated by

$$|\int\bar\partial w\wedge\partial_\phi\gamma\wedge\bar\gamma e^{-\phi}|\leq\frac{1}{2}(c_n\int e^w\partial_\phi\gamma\wedge\overline{\partial_\phi\gamma}e^{-\phi}+e^{-w}\partial w\wedge\bar\partial w\wedge\gamma\wedge\bar\gamma e^{-\phi})$$

The first term on the right hand side is exactly what we want. The second term again contains γ, but it is less singular than the integral over S that we have just estimated, so it is at least intuitively clear that it should not cause any serious trouble.

To estimate it we use Proposition 3.4.1 once more. This time we multiply $i\partial\bar\partial T_\alpha$ by $W = 1 - e^{-w} \geq 1 - e^{-r/\delta}$ before integrating. Note that

$$i\partial\bar\partial W = r(i\partial\bar\partial\psi - i\partial w\wedge\bar\partial w)e^{-w}\leq\frac{r}{\delta}e^{-w}i\partial\bar\partial\phi - e^{-w}i\partial w\wedge\bar\partial w.$$

We then get

$$c_n\int(W-re^{-w}/\delta)i\partial\bar\partial\phi\wedge\gamma\wedge\bar\gamma e^{-\phi}+c_n\int e^{-w}\partial w\wedge\bar\partial w\wedge\gamma\wedge\bar\gamma e^{-\phi}\leq$$

$$c_n\int W\bar\partial\partial_\phi\gamma\wedge\bar\gamma e^{-\phi}+\int W|\bar\partial\alpha|^2e^{-\phi}\omega_n.$$

If δ is small enough $W - re^{-w}/\delta = 1-(1+r/\delta)e^{-w}\geq 0$ (since $w \geq r/\delta$). Hence we can neglect the first term on the left hand side. We then repeat the same procedure as above and apply Stokes' to the first integral in the right hand side. This again produces a good term plus an undesired term containing $\bar\partial W = \bar\partial we^{-w}$. Now however this term can be absorbed in the left hand side. The result is (since $e^{-w}\leq 1$ and $W \leq 1$) that

$$c_{n-1}\int e^{-w}i\partial w\wedge\bar\partial w\wedge\gamma\wedge\bar\gamma e^{-\phi}\leq C_\delta(\int c_n\partial_\phi\gamma\wedge\overline{\partial_\phi\gamma}e^{-\phi}+|\bar\partial\alpha|^2e^{-\phi}\omega_n).$$

Inserting this in our previous estimate we finally get

$$\int_S c_{n-1}\gamma\wedge\bar\gamma e^{-\phi}\leq C_\delta\Big(\int e^w c_n\partial_\phi\gamma\wedge\overline{\partial_\phi\gamma}e^{-\phi}+w|\bar\partial\alpha|^2\Big), \tag{6.6}$$

which proves the lemma. □

Let us now see how we get an existence theorem for $\bar\partial$ from this a priori estimate. The argument follows basically the reasoning in chapter 3 but it is complicated by two things. First, $f=u\wedge[S]$ is no longer a form in L^2. Therefore we can not use the same functional analysis set up as before. Second, we can not hope to get a solution in L^2 either. Remember that our solution v will basically be U/s where U solves the extension problem. Hence v will never be in L^2; the estimate we are looking for is an L^2 estimate for sv.

Normalize so that

$$c_{n-1}\int u\wedge\bar u e^{-\phi}=1.$$

The next lemma replaces the estimate for the scalar product $\langle f,\alpha\rangle$ from the standard L^2-theory.

Lemma 2.3.

$$|\int_S u\wedge\bar\gamma e^{-\phi}|^2\leq C_\delta\int e^w c_n\partial_\phi\gamma\wedge\overline{\partial_\phi\gamma}e^{-\phi}, \tag{6.7}$$

To understand the significance of the lemma, recall that $\partial_\phi\gamma=\bar\partial^*\alpha$. Hence we again estimate the 'scalar product' between f and α by $\bar\partial^*\alpha$. The additional weight factor e^w in the right hand side is the price we pay for not having f in L^2.

Proof. To prove (6.7) we decompose $\alpha=\alpha_1+\alpha_2$ where α_1 is $\bar\partial$-closed and α_2 is orthogonal to the space of $\bar\partial$-closed forms, and put $\alpha_j=\gamma_j\wedge\omega$. Since $f=u\wedge[S]$ is $\bar\partial$-closed we claim that

$$\int f\wedge\gamma e^{-\phi}=\langle f,\alpha\rangle=\langle f,\alpha_1\rangle=\int f\wedge\gamma_1 e^{-\phi}.$$

This would have been completely evident if f had been a form in L^2; as it is now it requires a small argument that we will only sketch. The main point is that due to elliptic regularity, α_1 and α_2 are still smooth. Moreover, when X is compact, the ranges of $\bar\partial$ and $\bar\partial^*$ are closed. Since α_2 is orthogonal to the kernel of $\bar\partial$, this implies that $\alpha_2=\bar\partial^*\chi$ for some smooth χ. Hence $\langle f,\alpha_2\rangle=\langle\bar\partial f,\chi\rangle=0$.

The claim (6.7) now follows since we may replace γ by γ_1 in the left hand side. This gives

$$|\int_S u\wedge\bar\gamma e^{-\phi}|^2=|\int_S u\wedge\bar\gamma_1 e^{-\phi}|^2\leq c_{n-1}\int_S\gamma_1\wedge\bar\gamma_1 e^{-\phi}$$

which by Lemma 6.2.3 is dominated by

$$C_\delta\int c_n e^w\partial_\phi\gamma_1\wedge\overline{\partial_\phi\gamma_1}e^{-\phi}$$

since $\bar\partial\alpha_1=0$. On the other hand $\partial_\phi\gamma_1=\partial_\phi\gamma$ since $\partial_\phi\gamma_2=\bar\partial^*\alpha_2=0$ since α_2 is orthogonal to closed forms. This proves the lemma. □

The theorem now follows in essentially the same way as we proved the standard L^2-estimate for $\bar\partial$. The Riesz representation theorem implies that there is some $(n,0)$-form η with

$$c_n \int \eta \wedge \bar\eta e^{w} e^{-\phi} \leq C_\delta$$

and

$$\int f \wedge \bar\gamma e^{-\phi} = \int_S u \wedge \bar\gamma e^{-\phi} = \int e^{w} \eta \wedge \overline{\partial_\phi \gamma} e^{-\phi},$$

for all smooth compactly supported $(n-1,0)$-forms γ. It might be appropriate to point out that this is the point where we use that we have chosen $w = -r \log |s|^2 e^{-\psi}$ with r smaller than 1. Then e^w is integrable so smooth forms lie in $L^2(e^w)$. Then $v = e^w \eta$ solves $\bar\partial v = f$ and

$$c_n \int v \wedge \bar v e^{-w} e^{-\phi} \leq C_\delta.$$

Concretely this means that

$$c_n \int v \wedge \bar v e^{-\phi} |s|^{2r} e^{-r\psi} \leq C_\delta.$$

Hence, since $U = sv$ and $|s|^2 e^{-\psi} \leq 1$,

$$\int_X c_n U \wedge \bar U e^{-\phi-\psi} \leq \int_X c_n v \wedge \bar v e^{-\phi} |s|^{2r} e^{-r\psi} \leq C_\delta,$$

which proves the theorem.

Let us now translate this form of the Ohsawa-Takegoshi theorem to the non adjoint case. Let dS be the surface (or volume) measure on the hypersurface S induced by the Kähler metric ω.

Let $F := K_X + (S) + L$, and put $\phi_F = \phi_\omega + \phi + \psi$. Recall that

$$c_n U \wedge \bar U = |U|^2 e^{-\phi_\omega}.$$

Hence, by lemma 6.1.2 our estimate for U reads

$$\int_X |U|^2 e^{-\phi_F} \omega_n \leq C_\delta \int_S |U|^2 e^{-\phi_F} \frac{dS}{|ds|^2 e^{-\psi}}.$$

Hence we have the following form of Theorem 6.2.1.

Theorem 2.4. *Let F be a holomorphic line bundle over X and let S be a divisor in X. Assume F and (S) have metrics ϕ_F and ψ respectively satisfying*

$$i\partial\bar\partial\phi_F \geq (1+\delta) i\partial\bar\partial\psi + \phi_{K_X}$$

where ϕ_{K_X} is some smooth metric on K_X. Assume also that

$$i\partial\bar\partial\phi_F \geq i\partial\bar\partial\psi + \phi_{K_X}.$$

Then any holomorphic section U_0 of F over S extends holomorphically to a section of the same bundle over X satisfying

$$\int_X |U|^2 e^{-\phi_F} \omega_n \leq C_\delta \int_S |U_0|^2 e^{-\phi_F} \frac{dS}{|ds|^2 e^{-\psi}}.$$

Strictly speaking we have proved this theorem only in the case when $\phi_{K_X} = \phi_\omega$ is a metric on K_X coming from some metric ω on X. The general case however follows from this. An arbitrary metric on K_X differs from ϕ_ω by some smooth function χ, and we can write

$$\phi_{K_X} + \phi + \psi = \phi_\omega + (\chi + \phi) + \psi,$$

so changing ϕ_ω to ϕ_{K_X} is equivalent to changing ϕ to $\phi + \chi$.

Note that in this formulation the curvature assumption on the bundle F involves comparison with the canonical bundle. In the case when the canonical bundle is nonpositive (meaning that the Ricci curvature of X is nonnegative) this makes the hypotheses easier to fulfill, cf the exercise below.

Exercise: Let X be the Riemann sphere and let S be a divisor consisting of n points. Prove that there are sections of $\mathcal{O}(n-1)$ over X that attains arbitrary given values at the points, but that this in general is not true for $\mathcal{O}(n-2)$. (You need a polynomial of degree $n-1$ to interpolate at n points!). Check that this means that the curvature assumption in the Ohsawa-Takegoshi extension theorem is sharp – it is not possible to take $\delta = 0$. (Recall that $K_{\mathbb{P}^1} = \mathcal{O}(-2)$.) □

2.5. A more general version involving nonsmooth metrics and noncompact manifolds.

It is often important to be able to relax the assumptions in Theorem 6.2.1 as we have stated it. We will next give a version that allows for singular metrics and also noncompact manifolds. I do not know if the theorem holds as it stands for any complete Kähler manifold and line bundles with singular metrics. A reasonably general situation is the condition from [**14**] of a variety that becomes Stein after removal of some divisor. This certainly includes projective manifolds and of course Stein manifolds as well.

Theorem 2.5. *Let X be a complex manifold. Assume that X contains a divisor D such that $X \setminus D$ is Stein.*

Let S be a smooth hypersurface in X, defined by a global holomorphic section s of the line bundle (S), and let L be a complex line bundle over all of X. Assume that L and (S) have not necessarily smooth metrics, ϕ and ψ respectively, satisfying the curvature assumptions

$$i\partial\bar\partial\phi \geq 0$$

and

$$i\partial\bar\partial\phi \geq \delta i\partial\bar\partial\psi,$$

with $\delta > 0$. Assume moreover that s is normalized so that

$$|s|^2 e^{-\psi} \leq e^{-1/\delta}.$$

Let u be a global holomorphic section u of $K_S + L|_S$ such that

$$I := \int_S c_{n-1} u \wedge \bar u e^{-\phi} < \infty.$$

Then there is a global holomorphic section U of $K_X + (S) + L$ such that

$$U = ds \wedge u$$

on S and such that U satisfies the estimate

$$\int_X c_n U \wedge \bar{U} e^{-\phi-\psi} \leq C_\delta I. \tag{6.8}$$

Here C_δ is a constant depending only on δ.

Note that we have assumed that the L^2-norm of the section we wish to extend is finite. Just like in the Demailly-Nadel vanishing theorem this is a nontrivial condition even in the case of a compact manifold, since our line bundle metrics may have nonintegrable singularities. If it is not satisfied there is no guarantee that we can extend sections from S, with or without L^2-estimates!

We will not give a detailed proof of Theorem 6.2.5, but merely make a few remarks. First, it is enough to prove the theorem in the Stein case. The general case then follows from Lemma 5.1.3, which implies in particular that any holomorphic section on $X \setminus D$ extends holomorphically across D (given the L^2-condition). Hence, if we can extend u to $X \setminus D$, we automatically have an extension to all of X.It is then also enough to consider the case of smooth metrics, by the same arguments as in Chapter 5: Exhaust $X \setminus D$ by a sequence of relatively compact subdomains, on each of which we can approximate nonsmooth metrics with smooth ones. The basic estimate Lemma 6.2.2 is proved in exactly the same way on a noncompact manifold, provided we assume from the start that α and hence γ have compact support. The lemma then follows for not necessarily compactly supported forms if we first equip our Stein manifold with a complete Kähler metric. Then apply the lemma to $\chi_k \gamma$ where χ_k is an exhausting sequence of compactly supported cutoff functions having uniformly bounded gradients.

The only serious complication in the analysis is Lemma 6.2.3. In the proof of that lemma we decompose $\alpha = \alpha_1 + \alpha_2$ where α_1 is $\bar{\partial}$-closed and α_2 is orthogonal to the space of $\bar{\partial}$-closed forms. Then we write $\alpha_j = \gamma_j \wedge \omega$ and need to prove that

$$\int_S u \wedge \bar{\gamma}_2 e^{-\phi} = 0.$$

This is the "scalar product" between f and α_2, so it should be zero since f is $\bar{\partial}$-closed and α_2 is orthogonal to $\bar{\partial}$-closed forms, but again the crux is that f is not in L^2. So we need to approximate $f = u \wedge [S]$ in a suitable way by forms f_ϵ in L^2.

Again, we work on a relatively compact Stein subdomain of $X \setminus D$. Since this latter space is Stein, we can find a global holomorphic $(n-1, 0)$-form that restricts to u on S, in the sense that its pullback under the inclusion map equals u. We denote this global form by u too, and note that on our relatively compact subdomain u is bounded. We will approximate $f = u \wedge [S]$ by L^2-forms $f_\epsilon = u \wedge [S]_\epsilon$, where $[S]_\epsilon$ is a smooth approximation to $[S]$. For this we take

$$w_\epsilon = -\log(|s|^2 e^{-\psi} + \epsilon) = -\log(e^{-w} + \epsilon),$$

$$[S]_\epsilon = i\partial\bar{\partial} w_\epsilon - i\partial\bar{\partial}\psi,$$

and

$$f_\epsilon = u \wedge [S]_\epsilon.$$

Then f_ϵ is for positive ϵ a $\bar{\partial}$-closed form in L^2. We claim that

$$\int f \wedge \bar{\gamma}_2 e^{-\phi} = \lim \int f_\epsilon \wedge \bar{\gamma}_2 e^{-\phi}. \tag{6.9}$$

For this, note first that $\bar\partial\alpha_2 = \bar\partial\alpha$ and $\partial_\phi\gamma_2 = 0$. Hence it follows from Lemma 6.2.2 (or rather the noncompact version of that lemma) that

$$\int c_{n-1}\gamma_i \wedge \bar\gamma_i e^{-\phi} \wedge [S] < \infty.$$

A similar estimate is satisfied uniformly with $[S]$ replace by $[S]_\epsilon$ – it is proved in the same way, replacing w by w_ϵ To verify 6.9 we now decompose the integrals into two pieces, one close to the boundary and the remaining part. The first part is uniformly small if we are sufficiently close to the boundary since u is uniformly bounded. The other part converges as ϵ tends to zero since γ_2 is smooth. This completes the proof.

□

Exercise: Let ϕ be a plurisubharmonic function defined in a neighbourhood of the origin in $\mathbb{C}^n$. Use the Ohsawa-Takegoshi theorem to prove the following: If for some $\epsilon > 0$

$$\int_{|\zeta|<\epsilon} e^{-\phi(\zeta,0,\ldots 0)} dm(\zeta) < \infty,$$

then fore some other ϵ' the integral in n variables is also finite:

$$\int_{|z|<\epsilon'} e^{-\phi(z)} dm(z) < \infty.$$

Try to prove this without using the Ohsawa-Takegoshi extension theorem!

LECTURE 7

Deformational invariance of plurigenera.

In this chapter we will use the Ohsawa-Takegoshi theorem to prove a celebrated result of Siu, [**20**],[**21**], on the "invariance of plurigenera".

Recall that for a compact Riemann surface, the *genus* is the dimension of the space of holomorphic one-forms, i e the space of global holomorphic sections of the canonical bundle. In the same way the dimension of the space of global holomorphic sections of mK_X is called the m-genus, also for compact manifolds of any dimension. Collectively these are referred to as the *plurigenera* of the manifold. In the one dimensional case, the genus is a topological invariant, hence in particular invariant under deformations of the manifold. Siu's theorem is that in any dimension, all the plurigenera are invariant under deformations. The main point in Siu's proof is to show that the dimensions do not jump down when we perturb the manifold, and this is accomplished by an extension theorem. We will give here a simplification, due to Paun, [**17**], of Siu's original argument.

1. Extension of pluricanonical forms.

In this section we will prove Siu's theorem on the extension of sections of multiples of the canonical bundle from the central fiber of a projective family to the ambient space. First we need a few basic definitions.

Let X be a complex manifold together with a holomorphic map p from X to the unit disk. We will assume that p defines a smooth fibration with compact fibers. By this we mean that the differential of p is surjective everywhere and that the fibers $X_t = p^{-1}(t)$ are compact manifold. We can then think of the fibers as forming a family of complex manifolds and we say that this family is projective if there is a positive line bundle A over the total space X. The fiber X_0 is a smooth hypersurface defined by an equation $p = 0$.

In the notation of the previous chapter $S = X_0$, $s = p$ and (S) is now a *trivial* line bundle. The adjunction theorem says in this case that $K_X|_{X_0} = K_{X_0}$ the isomorphism being given by

$$u \mapsto U = dp \wedge u.$$

Abusing notation slightly we will identify u and $U|_{X_0}$. The main result of Siu, [**21**], is the following theorem.

Theorem 1.1. *Let u be a section of $mK_X|_{X_0}$. Then there is a holomorphic section U of mK_X over all of X that extends u.*

We give immediately the main corollary.

Corollary 1.2. *Let for any natural number m and any t in the disk $g_m(t)$ be the m-genus of the fiber X_t,*

$$g_m(t) = dim H^0(X_t, mK_{X_t}).$$

Then $g_m(t)$ is independent of t.

PROOF. The previous result shows that for any N the sets where $g_m(t)$ is at least N is open. A simple argument with normal families shows that it is also closed. The maximal value of N for which it is nonempty is the plurigenus. □

For the proof of Theorem 7.1.2 we will follow the method of Paun, [**17**], which simplified the original proof of Siu considerably. Notice first that the theorem follows immediately from Theorem 6.2.1 in case $m = 1$. In this case L and (S) are both trivial and ϕ and ψ are both zero, so by Theorem 6.2.1 u extends. For general m we write

$$(m-1)K_X =: L$$

so that $mK_X = K_x + L$. The crux of the matter is to find a metric ϕ on L over all of X with semipositive curvature current such that the section u that we want to extend satisfies

$$\int_{X_0} c_{n-1} u \wedge \bar{u}\, e^{-\phi} < \infty.$$

Then we can apply Theorem 6.2.1 again and get an extension of u. (We don't need any strict positivity of the curvature since the bundle (S) here is trivial!)

Over X_0 we can easily find such a metric. Since $\psi = \log |u|^2$ is a metric on mK_{X_0}, $\phi' = (1 - 1/m)\psi$ is a metric on $L = (m-1)K_X|_{X_0}$, and moreover

$$\int_{X_0} c_{n-1} u \wedge \bar{u}\, e^{-\phi'} = \int_{X_0} |u|^{2/m} < \infty.$$

The proof consists in finding an extension of ψ (and hence ϕ') as a metric with positive curvature current. This is simpler than extending u as a holomorphic section.

Let B be a line bundle over X that is sufficiently positive so that the following two conditions hold:

1. Any section of $pK_X + B$ over X_0 extends holomorphically to all of X if $p \leq m-1$.

2. Still for $p \leq m-1$, $pK_X + B$ is *base point free* over X_0, i e there is no point on X_0 where all sections to this bundle vanishes.

The first of these conditions is easy to achieve. Start with any smooth metric on K_X and some smooth metric of positive curvature on A. Let $B = lA$ for some l and take the induced metric on $pK_X + B$. This will have positive curvature if l is large enough and $p \leq m-1$. Hence, Theorem 6.2.1 implies again that any holomorphic section extends.

The second condition will also hold if l is large enough, since by the proof of the Kodaira embedding theorem we can then find sections of $pK_X + B$ with prescribed (i e nonzero) values at any point in the compact X_0.

Choose for $p \leq m-1$ a basis $(s_j^{(p)})$ for the space of global sections of $pK_X + B$ over the central fiber X_0.

Lemma 1.3. *For $k = 0, 1, \ldots$ and $p \leq m-1$ any section*

$$u^k s_j^{(p)}$$

of $(mk+p)K_X + B$ over X_0 extends holomorphically to all of X.

PROOF. We prove this by induction over $l = mk+p$, and we know by hypothesis that the statement holds for $l < m$, i e $k = 0$ and $p \leq m-1$. The first nontrivial step is therefore to extend $us_j^{(0)}$. Put

$$h_{m-1} = \sum_j |\widetilde{s_j^{(m-1)}}|^2$$

where $\tilde{s}$ means an extension of s. Then $h_{m-1} = e^{\phi_{m-1}}$ where ϕ_{m-1} is a metric on $(m-1)K_X + B$. Since this bundle is base point free this metric is actually smooth and

$$\int_{X_0} |us_j^{(0)}|^2 e^{-\phi_{m-1}}$$

is thus finite. By Theorem 6.2.1 we can find extensions of $us_j^{(0)}$ satisfying

$$\int_X |\widetilde{us_j^{(0)}}|^2 e^{-\phi_{m-1}} \leq C \int_{X_0} |us_j^{(0)}|^2 e^{-\phi_{m-1}}. \tag{7.1}$$

Put

$$h_m = \sum_j |\widetilde{us_j^{(0)}}|^2$$

and define ϕ_m so that $e^{\phi_m} = h_m$. The new metric ϕ_m is no longer smooth, but the only singularities come from u so we have that

$$\int_{X_0} |us_j^{(1)}|^2 e^{-\phi_m} < \infty$$

Then we can iterate the argument again and continuing this way the lemma follows. □

Notice also that during the proof of the lemma we obtain a sequence of metrics on $lK_X + B$,

$$h_l = \sum_j |\widetilde{u^k s_j^{(p)}}|^2$$

for $l = km + p$. These metrics satisfy good estimates namely

$$\int_X h_{l+1}/h_l \leq C \int_{X_0} h_{l+1}/h_l. \tag{7.2}$$

To verify this for $l = m-1$ we just sum over j in (7.1), and since all the metrics are constructed in a similar way all of the metrics satisfy (7.2). The integral in the right hand side here is

$$\int_{X_0} \sum |u^k s_j^{(p)}|^2 / \sum |u^k s_j^{(p-1)}|^2$$

if $p > 0$, and

$$\int_{X_0} \sum |u^k s_j^{(0)}|^2 / \sum |u^{k-1} s_j^{(m-1)}|^2$$

if $p = 0$. Hence they are bounded by a fixed constant, depending only on the choice of u and the choice of bases $s_j^{(p)}$, so we get

$$\int_X h_{l+1}/h_l \leq C.$$

By Hölder's inequality this implies that (define h_l to be some arbitrary smooth metric for $l < m-1$)

$$\int_X h_l^{1/l} = \int_X (h_l/h_{l-1})^{1/l}(h_{l-1}/h_{l-2})^{1/l}...h_1^{1/l} \leq$$

$$(\int_X h_l/h_{l-1})^{1/l}(\int_X h_{l-1}/h_{l-2})^{1/l}...(\int_X h_1)^{1/l} \leq C.$$

Take in particular $l = mk$ (the reason for this will be seen below). Thus by the submeanvalue property of plurisubharmonic functions

$$\phi_\infty := \limsup \frac{1}{mk}\phi_{mk}$$

is finite everywhere. Since ϕ_{mk} is a metric on $mkK_X + B$, ϕ_∞ is a metric on K_X (B disappears in the limit!).

After taking the upper semicontinuous regularization we get a metric ψ on K_X with semipositive curvature current which is greater than or equal to ϕ_∞ everywhere. On X_0

$$h_{km} = |u|^{2k}h$$

where h is a smooth positive function. Hence $e^\psi \geq e^{\phi_\infty} = |u|^{2/m}$ on X_0, so

$$\int_{X_0} |u|^2 e^{-(m-1)\psi} \leq \int_{X_0} |u|^{2/m} < \infty.$$

This completes the proof of the theorem. □

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