# Analytic Number Theory

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Problem Sheet 1

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**Problem 1. (The asymptotic formula in the prime number theorem)**

(a) Let $\psi(x) = \sum_{n \le x} \Lambda(n)$ and $\mathrm{li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$. Show that for any increasing function $A(x)$ satisfying $A(x) \gg x^{1/2}$ the statements

$$\pi(x) = \mathrm{li}(x) + O(A(x))$$

and

$$\psi(x) = x + O(A(x) \log x)$$

are equivalent.

**OBS.** *It seems that the problem should have read*

$$\pi(x) = \mathrm{li}(x) + O(A(x)) \implies \psi(x) = x + O(A(x) \log x),$$

*but*

$$\psi(x) = x + O(A(x) \log x) \implies \pi(x) = \mathrm{li}(x) + O(A(x) \log \log x).$$

*I'm sorry for any confusion.*

(b) Show that for any $k \in \mathbb{N}$ one has

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^3} + \ldots + (k-1)!\frac{x}{(\log x)^k} + O\left(\frac{x}{(\log x)^{k+1}}\right).$$

**Solution.** (a) Let $\theta(x) = \sum_{p \le x} \log p$. Observe first that

$$\psi(x) - \theta(x) = \sum_{\substack{p^k \le x \\ k \ge 1}} \log p \ll \sum_{p^2 \ll x} \log x \ll \sqrt{x} \log x,$$

so it suffices to show the statement for $\theta$ in place of $\psi$. By partial summation we have

$$\begin{aligned}
\theta(x) = \sum_{p \le x} \log p &= \pi(x) \log x - \int_2^x \pi(t)/t \,\mathrm{d}t \\
&= \log x \,(\mathrm{li}(x) + O(A(x)) - \int_2^x \frac{\mathrm{li}(t)}{t} \,\mathrm{d}t + O\left(\int_2^x \frac{A(t)}{t} \,\mathrm{d}t\right) \\
&= \log x \,\mathrm{li}(x) - \int_2^x \frac{\mathrm{li}(t)}{t} \,\mathrm{d}t + O(A(x) \log x),
\end{aligned}$$

and partial integration shows that

$$\int_2^x \frac{\mathrm{li}(t)}{t} \,\mathrm{d}t = \mathrm{li}(x) \log x - x + C.$$

Conversely,

$$\pi(x) = \sum_{p\le x} \frac{\log p}{\log p} = \frac{\theta(x)}{\log x} - \int_2^x \left(-\frac{1}{t(\log t)^2}\right)\theta(t)\,\mathrm{d}t$$
$$= \frac{x}{\log x} + O(A(x)) + \int_2^x \frac{\mathrm{d}t}{(\log t)^2} + O\left(A(x)\int_2^x \frac{t}{(\log t)}\,\mathrm{d}t\right),$$

and the paenultimate term is

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = \int_2^x x\cdot\frac{1}{x(\log t)^2}\,\mathrm{d}t = -\frac{x}{\log x} + \mathrm{li}(x) + C.$$

Altogether we obtain

$$\pi(x) = \mathrm{li}(x) + O(A(x)) + O(A(x)\log\log x).$$

(b) For the second statement we show that for all $k \in \mathbb{N}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{(\log x)^k} = \frac{-k}{x(\log x)^{k+1}}$$

and hence

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^k} = \frac{x}{(\log x)^k} + k\int_2^x \frac{\mathrm{d}t}{(\log t)^{k+1}}.$$

Since also

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^{k+1}} = \int_2^x \frac{1}{t}\frac{t}{(\log t)^{k+1}}\,\mathrm{d}t \ll \frac{x}{(\log x)^{k+1}}\int_2^x \frac{1}{t}\,\mathrm{d}t \ll \frac{x}{(\log x)^k},$$

the statement follows by induction.

**Problem 2. (Multiplicative functions and Dirichlet series)**

(a) Let $\omega(n)$ the number of distinct prime divisors of $n$, and let $\Omega(n)$ be the total number of prime divisors with multiplicity. We define further

$$\nu(n) := 2^{\omega(n)}, \qquad \lambda(n) := (-1)^{\Omega(n)}, \qquad q(n) := \begin{cases} 1 & n \text{ is a square} \\ 0 & \text{else.} \end{cases}$$

Show that these functions are multiplicative.

(b) Show $\nu = \mathbb{1} * \mu^2$, $q = \mathbb{1} * \lambda$ and $\mu^2 * \lambda = \varepsilon$, and use these identities to determine $q * \nu$.

(c) Write the Dirichlet series $L(q,s)$, $L(\mu^2,s)$, $L(\nu,s)$ and $L(\lambda,s)$ in terms of $\zeta(s)$.
Hint: Start with $L(q,s)$, then use (b).

**Solution.** (a) Let $m = p_1^{e_1}\cdots p_i^{e_i}$ and $n = q_1^{f_1}\cdots q_j^{f_j}$. Since $(m,n) = 1$ we have $\{p_1,\dots,p_i\} \cap \{q_1,\dots,q_j\} = \emptyset$. Obviously, $mn$ is a square if and only if both $m$ and $n$ are squares. Furthermore, we have

$$\nu(mn) = 2^{i+j} = 2^i 2^j = \nu(m)\nu(n)$$
$$\lambda(mn) = (-1)^{e_1+\cdots+e_i+f_1+\cdots+f_j} = (-1)^{e_1+\cdots+e_i}(-1)^{f_1+\cdots+f_j} = \lambda(m)\lambda(n).$$

(b) It suffices to check prime powers. For the first identity we have

$$\mathbb{1} * \mu^2(p^k) = \sum_{i=0}^{k} \mu^2(p^i)\mathbb{1}(p^{k-i}) = 1\cdot 1 + 1\cdot 1 = 2 = \nu(p^k)$$

whenever $k \geq 1$, and $\nu(1) = 1$. In the second case we find $\lambda(p^k) = (-1)^k$ and thus

$$\mathbb{1} * \lambda(p^k) = \sum_{i=0}^{k} \lambda(p^i)\mathbb{1}(p^{k-i}) = \sum_{i=0}^{k}(-1)^k = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$$

This is just $q(p^k)$. The third identity follows from

$$\mu^2 * \lambda = \sum_{i=0}^{k} \mu(p^i)\lambda(p^{k-i}) = 1 \cdot 1 + 1 \cdot (-1) = 0$$

for $k \geq 1$, and $\mu^2 * \lambda(1) = 1 = \varepsilon(1)$.

Finally, we have $q * \nu = \mathbb{1} * \lambda * \mathbb{1} * \mu^2 = \mathbb{1} * \mathbb{1} = d$, where in the third step we used $\mu^2 * \lambda = \epsilon$.

(c) We have

$$L(q, s) = \sum_{n=m^2} \frac{1}{n^2} = \zeta(2s).$$

Furthermore, from $\mu^2 = \lambda^{-1}$ we have $q * \mu^2 = \mathbb{1}$, and hence $L(\mu^2, s) = \zeta(s)/L(q, s) = \zeta(s)/\zeta(2s)$. The last two are $L(\nu, s) = \zeta(s)L(\mu^2, s) = \zeta(s)^2/\zeta(2s)$ and $L(\lambda, s) = L(\mu^2, s)^{-1} = \zeta(2s)/\zeta(s)$.