Analytic Number Theory

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Problem Sheet 1

Due date: Tuesday, 09.02.2016

Problem 1. (The asymptotic formula in the prime number theorem)

(a) Let $\psi(x) = \sum_{n \le x} \Lambda(n)$ and $\mathrm{li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$. Show that for any increasing function $A(x)$ satisfying $A(x) \gg x^{1/2}$ the statements

$$\pi(x) = \mathrm{li}(x) + O(A(x))$$

and

$$\psi(x) = x + O(A(x) \log x)$$

are equivalent.

(b) Show that for any $k \in \mathbb{N}$ one has

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^3} + \ldots + (k-1)!\frac{x}{(\log x)^k} + O\left(\frac{x}{(\log x)^{k+1}}\right).$$

Problem 2. (Multiplicative functions and Dirichlet series)

(a) Let $\omega(n)$ the number of distinct prime divisors of n, and let $\Omega(n)$ be the total number of prime divisors with multiplicity. We define further

$$\nu(n) := 2^{\omega(n)}, \qquad \lambda(n) := (-1)^{\Omega(n)}, \qquad q(n) := \begin{cases} 1 & n \text{ is a square} \\ 0 & \text{else.} \end{cases}$$

Show that these functions are multiplicative.

(b) Show $\nu = \mathbb{1} * \mu^2$, $q = \mathbb{1} * \lambda$ and $\mu^2 * \lambda = \varepsilon$, and use these identities to determine $q * \nu$.

(c) Write the Dirichlet series $L(q, s)$, $L(\mu^2, s)$, $L(\nu, s)$ and $L(\lambda, s)$ in terms of $\zeta(s)$.
Hint: Start with $L(q, s)$, then use (b).