Analytic Number Theory

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Problem Sheet 2

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Problem 1. (The asymptotic formula in the prime number theorem)
We define Mertens' function as

$$M(x) = \sum_{n \le x} \mu(n).$$

Modify the proof of the prime number theorem to show that

$$M(x) \ll x \exp(-c(\log x)^{1/10})$$

for some constant $c > 0$.

Solution. Observe that $L(\mu, s) = 1/\zeta(s)$.

By Perron's formula (Theorem I.1.4) we have

$$M(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{\zeta(s)} \frac{\mathrm{d}s}{s} + O\left(\frac{x^c}{T\zeta(c)} + 1 + \frac{x \log x}{T}\right).$$

Unlike ζ, $1/\zeta$ does not have a pole at 1. It follows that we can find T and c' such that $1/\zeta$ has no poles in the rectangle $c \pm iT, c' \pm iT$, so by the residue theorem we find

$$M(x) \ll \int_{c+iT}^{c'+iT} \frac{x^s}{\zeta(s)} \frac{\mathrm{d}s}{s} + \int_{c'+iT}^{c'-iT} \frac{x^s}{\zeta(s)} \frac{\mathrm{d}s}{s} + \int_{c'-iT}^{c-iT} \frac{x^s}{\zeta(s)} \frac{\mathrm{d}s}{s} + \frac{x^c}{T\zeta(c)} + 1 + \frac{x \log x}{T}.$$

By Lemma I.4.2 we know that $\zeta(s) \gg (\log |t|)^{-7}$ in the region $|t| \ge 8$ and $1 - \delta(\log |t|)^{-9} < \sigma < 2$, so we set $\eta = \frac{\delta}{2}(\log |t|)^{-9}$ and $c = 1 + \eta$, $c' = 1 - \eta$. Since $|s| \ge |t|$ we have

$$\left| \int_{1+\eta+iT}^{1-\eta+iT} \frac{x^s}{\zeta(s)} \frac{\mathrm{d}s}{s} \right| \le \int_{-\eta}^{\eta} x^{1+\sigma} (\log T)^7 \frac{\mathrm{d}\sigma}{T} \ll \eta \frac{x^{1+\eta}(\log T)^7}{T} \ll \frac{x^{1+\eta}}{T(\log T)^2},$$

and the same bound holds for the third integral.

For the second integral we have $|s| > \frac{1}{2}(1 + |t|)$, and thus

$$\left| \int_{1-\eta+iT}^{1-\eta-iT} \frac{x^s}{\zeta(s)} \frac{\mathrm{d}s}{s} \right| \le \int_{-T}^{T} \frac{x^{1-\eta}}{\zeta(1-\eta+it)} \frac{\mathrm{d}t}{1+|t|}.$$

For $|t| > 8$ we can apply Lemma I.4.2 again, and for $|t| \le 8$ we use the bound $1/\zeta(s) \ll 1 - \sigma = \eta$ from Lemma I.4.2. Thus we have $1/\zeta(s) \ll (\log T)^7$ for all $|t| \le T$, and thus

$$\int_{-T}^{T} \frac{x^{1-\eta}}{|\zeta(1-\eta+it)|} \frac{\mathrm{d}t}{1+|t|} \ll x^{1-\eta}(\log T)^7 \int_{-T}^{T} \frac{\mathrm{d}t}{1+|t|} \ll x^{1-\eta}(\log T)^8.$$

Thus altogether we find

$$M(x) \ll \frac{x^{1+\eta}}{T(\log T)^2} + x^{1-\eta}(\log T)^8 + \frac{x^{1+\eta}(\log T)^7}{T} + 1 + \frac{x \log x}{T}.$$

We compare the third and the fourth term. Neglecting logarithms for the moment, these two terms are roughly equal if $T = x^{2\eta} = x^{\delta(\log T)^{-9}}$, or in other words $\log T = (\delta \log x)^{1/10}$. As in the lectures, inserting this bound yields the result.