Analytic Number Theory

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Solutions to Problem Sheet 3

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Problem 1. Prove Corollary II.6.2 by partial summation.

Solution. Write

$$A(q) = \frac{q}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \text{ prim}}} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2,$$

then the left hand side is

$$\begin{aligned} \sum_{R<q\le Q} \frac{A(q)}{q} &= \sum_{1\le q\le Q} \frac{A(q)}{q} - \sum_{1\le q\le R} \frac{1}{q} A(q) \\ &= \frac{1}{Q} \sum_{1\le q\le Q} A(q) - \frac{1}{R} \sum_{1\le q\le R} A(q) + \int_R^Q \frac{1}{T^2} \sum_{1\le q\le T} A(q) \mathrm{d}T. \end{aligned}$$

Set

$$S(N) = \sum_{n=M+1}^{M+N} |a_n|^2.$$

Inserting the bound $\sum_{1\le q\le Q} A(q) \ll (N+Q^2)S(N)$ from Theorem II.6.1 yields

$$\begin{aligned} \sum_{R<q\le Q} \frac{1}{q} A(q) &\ll \frac{1}{Q}(N+Q^2)S(N) + \frac{1}{R}(N+R^2)S(N) + \int_R^Q \frac{1}{T^2}(N+T^2)S(N)\mathrm{d}T \\ &\ll \left(\frac{N}{Q} + Q + \frac{N}{R} + R \right) S(N) \ll \left(Q + \frac{N}{R} \right) S(N), \end{aligned}$$

since $Q > R$.

Problem 2. Modify the proof of the Barban–Davenport–Halberstam Theorem to show the Theorem of Bombieri–Friedlander–Iwaniec.

Solution. By the character relations of Lemma II.1.1 we have

$$\sum_{\substack{n\le x \\ n\equiv a \pmod q \\ (n,l)=1}} f(n) - \frac{\chi_0(n)}{\varphi(q)} \sum_{n\le x} f(n) = \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi\ne\chi_0}} \bar\chi(a) \sum_{n\le x} f(n)\chi(n),$$

and thus, by the same argument leading to (II.6.1),

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ q)}} f(n)-\frac{1}{\varphi(q)}\sum_{\substack{n\le x\\(n,q)=1}} f(n)\right|^2=\frac{1}{\varphi(q)}\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi\neq\chi_0}}\left|\sum_{n\le x} f(n)\chi(n)\right|^2.$$

As in the proof of the Barban–Davenport–Halberstam theorem, we have to get rid of non-primitive characters. If χ is induced by a character χ' (mod q') with $q=q'l$, we have

$$\sum_{n\le x} f(n)\chi(n)=\sum_{\substack{n\le x\\(n,l)=1}} f(n)\chi'(n),$$

so if we denote the left hand side of the Bombieri–Friedlander–Iwaniec theorem by S, we have

$$S\le\sum_{\substack{q'\ge 2,l\ge 1\\ q'l\le Q}}\frac{1}{\varphi(q')\varphi(l)}\sum_{\substack{\chi' \ (\mathrm{mod}\ q)'\\ \chi' \ \mathrm{prim}}}\left|\sum_{\substack{n\le x\\(n,l)=1}} f(n)\chi'(n)\right|^2.$$

As in the proof of B–D–H we first investigate the contribution of $q>R$ for some suitable R. From Corollary II.6.2 we have

$$\sum_{R<q'\le Q/l}\frac{1}{\varphi(q')}\sum_{\substack{\chi' \ (\mathrm{mod}\ q')\\ \chi' \ \mathrm{prim}}}\left|\sum_{\substack{n\le x\\(n,l)=1}} f(n)\chi'(n)\right|^2\ll\left(\frac{x}{R}+\frac{Q}{l}\right)\sum_{\substack{n\le x\\(n,l)=1}}|f(n)|^2.$$

Note that the right hand side only increases if we abandon the condition $(n,l)=1$ in the innermost sum. After summing over l we see

$$\begin{aligned}\sum_{l\le Q}\sum_{R<q'\le Q/l}\frac{1}{\varphi(q')\varphi(l)}\sum_{\substack{\chi' \ (\mathrm{mod}\ q')\\ \chi' \ \mathrm{prim}}}\left|\sum_{\substack{n\le x\\(n,l)=1}} f(n)\chi'(n)\right|^2 &\ll\sum_{l\le Q}\left(\frac{x}{R\varphi(l)}+\frac{Q}{l\varphi(l)}\right)\sum_{n\le x}|f(n)|^2\\ &\ll\left(\frac{x\log Q}{R}+Q\right)\sum_{n\le x}|f(n)|^2,\end{aligned}$$

where we used relation II.6.2.

For the terms with $q'\le R$ we sort into residue classes modulo q' and finde

$$\sum_{\substack{n\le x\\(n,l)=1}} f(n)\chi'(n)=\sum_{\substack{b=1\\(b,q')=1}}^{q'}\chi'(b)\sum_{\substack{n\le x\\ n\equiv b \ (\mathrm{mod}\ q')\\(n,l)=1}} f(n).$$

Instead of using the Siegel–Walfisz theorem, we now use the condition of B–F–I. This shows that the inner sum is

$$\sum_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ q)\\ (n,l)=1}} f(n) = \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,ql)=1}} f(n) + O\left(\frac{\sqrt{x}}{(\log x)^A}\left(\sum_{n\le x}|f(n)|^2\right)^{1/2}\right),$$

and therefore

$$\sum_{\substack{b=1\\ (b,q')=1}}^{q'} \chi'(b) \sum_{\substack{n\le x\\ n\equiv b \ (\mathrm{mod}\ q')\\ (n,l)=1}} f(n) = \frac{1}{\varphi(q)} \sum_{\substack{b=1\\ (b,q')=1}}^{q'} \chi'(b) \sum_{\substack{n\le x\\ (n,ql)=1}} f(n) + O\left(\frac{q'\sqrt{x}}{(\log x)^A}\left(\sum_{n\le x}|f(n)|^2\right)^{1/2}\right).$$

The first term of the right hand side vanishes by the character relations of Lemma II.1.1, and we find

$$\sum_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ q)\\ (n,l)=1}} f(n) \ll \frac{q'\sqrt{x}}{(\log x)^A}\left(\sum_{n\le x}|f(n)|^2\right)^{1/2}.$$

It follows that

$$\begin{aligned}\sum_{l\le Q}\sum_{q'\le R}\frac{1}{\varphi(q')\varphi(l)}\sum_{\substack{\chi' \ (\mathrm{mod}\ q')\\ \chi' \ \mathrm{prim}}}\left|\sum_{\substack{n\le x\\ (n,l)=1}} f(n)\chi'(n)\right|^2 &\ll \sum_{l\le Q}\sum_{q'\le R}\frac{1}{\varphi(l)}\frac{{q'}^2 x}{(\log x)^{2A}}\left(\sum_{n\le x}|f(n)|^2\right)\\ &\ll \frac{R^3 x\log Q}{(\log x)^{2A}}\left(\sum_{n\le x}|f(n)|^2\right).\end{aligned}$$

Altogether we have

$$\sum_{l\le Q}\sum_{1\le q'\le Q/l}\frac{1}{\varphi(q')\varphi(l)}\sum_{\substack{\chi' \ (\mathrm{mod}\ q')\\ \chi' \ \mathrm{prim}}}\left|\sum_{\substack{n\le x\\ (n,l)=1}} f(n)\chi'(n)\right|^2 \ll \left(\frac{x\log Q}{R} + Q + \frac{R^3 x\log Q}{(\log x)^{2A}}\right)\sum_{n\le x}|f(n)|^2,$$

and the first and last terms on the right hand side coincide when $R=(\log x)^A$.