PERFECTOID SHIMURA VARIETIES AND THE CALEGARI–EMERTON CONJECTURES

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Abstract. We prove many new cases of a conjecture of Calegari-Emerton describing the qualitative properties of completed cohomology. The heart of our argument is a careful inductive analysis of completed cohomology on the Borel-Serre boundary. As a key input to this induction, we prove a new perfectoidness result for towers of minimally compactified Shimura varieties, generalizing previous work of Scholze.

Contents

1. Introduction

1.1. **Motivation for completed cohomology.** This paper is motivated by the notion of reciprocity in the Langlands program. Let $G/\mathbb{Q}$ be a connected reductive group. Roughly speaking, reciprocity is the expectation that there should be some precise relationship between

i) algebraic automorphic representations π of $G(\mathbb{A}_{\mathbb{Q}})$, and

ii) p-adic Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^{L}G(\overline{\mathbb{Q}_p})$ which are geometric in the sense of Fontaine-Mazur.

For a more precise conjectural formulation of this relationship, we refer the reader to [Clo90, BG14]. While there are many partial results, the general problem of reciprocity seems very difficult to attack, for (at least) two reasons:

1. Algebraic automorphic representations are inherently of an archimedean/real-analytic nature, while p-adic Galois representations are (of course) inherently p-adic.

2. Algebraic automorphic representations are rigid, while p-adic Galois representations naturally deform into positive-dimensional families.

These observations suggest that one should try to bridge the gap, by seeking a genuinely p-adic variant of the notion of automorphic representation, which is flexible enough to accommodate all p-adic Galois representations. At present, the only viable candidate for such a theory is the notion of *completed (co)homology*, introduced by Emerton [Eme06].

Let us recall the key definitions; we refer the reader to the body of the paper for any unexplained notation. Fix a connected reductive group $G/\mathbb{Q}$. Let $A \subset G$ be the maximal $\mathbb{Q}$-split central torus, and let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup. Let $X^G = G(\mathbb{R})/A(\mathbb{R})K_\infty$ be the (connected) symmetric space for G; we write X for X^G if G is clear. For any open compact subgroup $K \subset G(\mathbb{A}_f)$, we have the associated locally symmetric space $X_K = G(\mathbb{Q})^+\backslash(X \times G(\mathbb{A}_f))/K$.

Definition 1.1. *Let $K^p \subset G(\mathbb{A}_f^p)$ be any open compact subgroup. Then we define completed cohomology for G with tame level K^p as*

$$\tilde{H}^* = \varprojlim_n \varinjlim_{K_p \subset G(\mathbb{Q}_p)} H^*(X_{K^pK_p}, \mathbb{Z}/p^n\mathbb{Z}).$$

Similarly, we define completed homology for G with tame level K^p as

$$\tilde{H}_* = \varprojlim_{K_p \subset G(\mathbb{Q}_p)} H_*(X_{K^pK_p}, \mathbb{Z}_p).$$

We also define compactly supported completed cohomology $\tilde{H}_c^*(K^p)$ and completed Borel-Moore homology $\tilde{H}_*^{BM}(K^p)$ by the obvious variants on these recipes.

By construction, these spaces admit commuting actions of $G(\mathbb{Q}_p)$ and $\mathbb{T}(K^p)$, and the $G(\mathbb{Q}_p)$-actions are continuous for the natural topologies. Moreover, these spaces are not "too big". In particular, they are all p-adically separated and complete with bounded p^∞-torsion. Additionally, $\tilde{H}_*$ and $\tilde{H}_*^{BM}$ are finitely generated as modules over the completed group ring $\mathbb{Z}_p[[K_p]]$ for any open compact subgroup $K_p \subset G(\mathbb{Q}_p)$, while $\tilde{H}^*(K^p)[\frac{1}{p}]$ and $\tilde{H}_c^*(K^p)[\frac{1}{p}]$ are naturally admissible unitary $\mathbb{Q}_p$-Banach space representations of $G(\mathbb{Q}_p)$.

The main motivations for considering completed (co)homology are summarized in the following conjecture, which we don't attempt to formulate precisely. For a more careful discussion, we refer the reader to [CE12] and [Eme14].

Hope 1.2. *Let $\psi : \mathbb{T}(K^p) \to \overline{\mathbb{Q}_p}$ be a system of Hecke eigenvalues occurring in $\tilde{H}^*(K^p)[\frac{1}{p}]$. Then there exists a continuous, odd, almost everywhere unramified Galois representation $\rho_\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^{C}G(\overline{\mathbb{Q}_p})$ which matches ψ in the usual sense. Moreover, the ψ-isotypic part of $\tilde{H}^*(K^p)[\frac{1}{p}]$, as a $\mathbb{Q}_p$-Banach space representation of $G(\mathbb{Q}_p)$, should (up to multiplicities) depend only on $\rho_\psi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$.*

Finally, every *(suitable) continuous, odd, almost everywhere unramified Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^{C}G(\overline{\mathbb{Q}_p})$ should occur in this way.*

Here ${}^C G$ denotes the C-group of G as defined in [BG14], which is a mild extension of ${}^L G$. When $G = \mathrm{GL}_2/\mathbb{Q}$, this is a theorem of Emerton [Eme11]. However, in general, very little is known. As mentioned, the precise formulation of this conjecture should not be taken too seriously. The reader wondering about the appearance of the C-group and what "suitable" might mean might consider the case $G = \mathrm{PGL}_2/\mathbb{Q}$.

1.2. **Main results.** In this paper, we study the qualitative properties of completed (co)homology, which are encapsulated in a beautiful conjecture of Calegari–Emerton. To state this conjecture, we need a small amount of additional notation. If $G/\mathbb{Q}$ is a connected reductive group, we define nonnegative integers $l_0 = \operatorname{rank} G(\mathbb{R}) - \operatorname{rank} A(\mathbb{R})K_\infty$ and $q_0 = \frac{\dim X^G - l_0}{2}$. Roughly speaking, l_0 measures the failure of the semisimple group $G^{der}(\mathbb{R})$ to admit discrete series representations, while q_0 is the lowest degree in which the locally symmetric spaces X_K should have "interesting" cohomology.

Conjecture 1.3 (Calegari–Emerton)**.** *Let $G/\mathbb{Q}$ be a connected reductive group. Let q_0 and l_0 be the invariants of G defined above. Let $K^p \subset G(\mathbb{A}_f^p)$ be any open compact subgroup. Then*

1) For all $i > q_0$, $\tilde{H}_c^i(K^p) = \tilde{H}^i(K^p) = 0$.

2) For all $i > q_0$, $\tilde{H}_i^{BM}(K^p) = \tilde{H}_i(K^p) = 0$, and $\tilde{H}_{q_0}^{BM}(K^p)$ and $\tilde{H}_{q_0}(K^p)$ are p-torsion-free.

3) For any compact open pro-p subgroup $K_p \subset G(\mathbb{Q}_p)$, the groups $\tilde{H}_i(K^p)$ and $\tilde{H}_i^{BM}(K^p)$ have codimension $\geq q_0 + l_0 - i$ over the completed group ring $\mathbb{Z}_p[[K_p]]$ for any $i < q_0$.

4) The groups $\tilde{H}_{q_0}(K^p)$ and $\tilde{H}_{q_0}^{BM}(K^p)$ have codimension exactly l_0.

The individual portions of this conjecture are far from independent, and in fact there are natural implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3). Amusingly, these implications are "asymmetric" in the sense that 1) for $\tilde{H}^*$ implies 2) for $\tilde{H}_*$ implies 3) for $\tilde{H}_*^{BM}$, and similarly 1) for $\tilde{H}_c^*$ implies 2) for $\tilde{H}_*^{BM}$ implies 3) for $\tilde{H}_*$.

Let us discuss what was previously known about this conjecture.

- For some groups of small rank (e.g. GL_2, or $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_2$ for $K/\mathbb{Q}$ quadratic, or GSp_4), one can prove Conjecture 1.3 by hand using various tricks involving the congruence subgroup property, the cohomological dimension bounds of [BS], Poincaré duality, etc. However, these methods quickly run out of steam.
- When $l_0 = 0$, part 4) of the conjecture was proved by Calegari-Emerton [CE09], as a consequence of Matsushima's formula and limit multiplicity results for discrete series representations.
- When G admits a Shimura datum of Hodge type, Scholze proved part 1) of Conjecture 1.3, *but for $\tilde{H}_c^*$ only*, by perfectoid methods [Sch15].

The main result of this paper is the following theorem (cf. Theorems 4.4, 4.5, and 4.9).

Theorem 1.4. *Let $G/\mathbb{Q}$ be a semisimple group such that X is a Hermitian symmetric space and (G, X) is a connected Shimura datum of pre-abelian type. Then Conjecture 1.3 is true for G.*

More generally, let $G/\mathbb{Q}$ be a connected reductive group such that $Z(G)$ satisfies the Leopoldt conjecture and such that G^{der} admits a connected Shimura datum of pre-abelian type. Then Conjecture 1.3 is true for G.

The assumptions on G here guarantee that $l_0(G^{der}) = 0$, which allows us to prove part 4) of Conjecture 1.3 by a fairly straightforward analysis combining the results of [CE09] with the Leopoldt conjecture for $Z(G)$. By our previous remarks, the whole conjecture now follows if we can prove part 1). Note that when $l_0 = 0$ and X is a Hermitian symmetric domain, part 1) of the conjecture simply asserts that $\tilde{H}_c^i = \tilde{H}^i = 0$ for all $i > d = \dim_{\mathbb{C}} X$. It is this vanishing conjecture which we focus on.

Our proof of the vanishing conjecture builds on Scholze's methods and combines them with some new ideas. Roughly speaking, we first reduce to the case where (G, X) is a connected Shimura datum of pre-abelian type, and then proceed in two steps:

Step One. We prove the vanishing of $\tilde{H}_c^i$ for $i > d$ by pushing Scholze's methods to their limit.

Step Two. We prove the vanishing of $\tilde{H}^i$ for $i > d$ by a careful analysis of boundary cohomology, using Step One for G *and* for various auxiliary Levi subgroups related to the boundary strata of the minimal compactification.

Let us now describe these steps in more detail.

1.3. **Step One: p-adic methods.** As described above, the proof of Theorem 1.4 proceeds in two essentially distinct steps. In the first step, we prove the vanishing of $\tilde{H}_c^i(K^p)$ for i above the middle degree, using the p-adic geometry of Shimura varieties. For Shimura data of Hodge type, this is one of the main results of [Sch15], where it is deduced from the existence of perfectoid Shimura varieties of Hodge type.

We thus need to generalize the geometric results of [Sch15] to a wider class of Shimura data. To this end, we prove the following theorem.

Theorem 1.5. *Let (G, X) be a Shimura datum of* pre-abelian type*, with reflex field E. Fix a complete algebraically closed field $C/\mathbb{Q}_p$ and an embedding $E \to C$. Fix any open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$. For any open compact subgroup $K_p \subset G(\mathbb{Q}_p)$, let $\mathcal{X}^*_{K^pK_p}$ denote the adic space over* $\operatorname{Spa} C$ *associated with the base change of the minimal compactification* $\mathrm{Sh}_{K^pK_p}(G, X)^*$ *along $E \to C$.*

Then there is a perfectoid space $\mathcal{X}^*_{K^p}$ such that

$$\mathcal{X}^*_{K^p} = \varprojlim_{K_p \subset G(\mathbb{Q}_p)} \mathcal{X}^*_{K^pK_p},$$

as diamonds over $\operatorname{Spd} C$*. Moreover, the Hodge-Tate period map $\pi_{\mathrm{HT}} : \mathcal{X}^*_{K^p} \to \mathscr{F}\!\ell_{G,\mu}$ exists as a map of adic spaces over C and has all expected properties. Finally, the boundary of $\mathcal{X}^*_{K^p}$ is Zariski-closed.*

Recall that a Shimura datum (G, X) is of pre-abelian type if there exists a Shimura datum (G', X') of Hodge type admitting an isomorphism of connected Shimura data $(G^{ad}, X^+) \simeq (G'^{ad}, X'^+)$. This is slightly more general than the (somewhat more well-known) notion of a Shimura datum of abelian type. While it is probably true that every tower of minimally compactified Shimura varieties with infinite level at p is perfectoid, we expect that Theorem 1.5 is the most general result which can be proved via current technology. We also state and prove a similar result for connected Shimura varieties, cf. Theorem 5.20.

While the idea behind the proof of Theorem 1.5 is clear, the argument is unfortunately somewhat technical.[1] Roughly speaking, there are two key ingredients:

- "Perfectoidization results" à la Bhatt-Scholze, building in particular on [BS19, Theorem 1.16(1)]. Roughly speaking, these techniques let us prove that if $(X_i)_{i\in I} \overset{(f_i)_{i\in I}}{\to} (Y_i)_{i\in I}$ is a (pro-)finite morphism between two reasonable inverse systems of rigid analytic spaces, and $\varprojlim_{i\in I} Y_i$ is perfectoid, then $\varprojlim_{i\in I} X_i$ is also perfectoid. For a precise statement, see Lemma 5.10.
- A general and user-friendly existence result for quotients of perfectoid spaces by finite groups, cf. Theorem 5.8.

We also note that for open Shimura varieties of abelian type, the problem of proving perfectoidness at infinite level was previously considered by Shen [She17].

[1] A glance at the proof of the key Proposition 5.19 should convince the reader of this.

1.4. **Step Two: Topological methods.** The second step is totally disjoint from the first, and doesn't use any p-adic geometry. We content ourselves with a somewhat impressionistic sketch here. In what follows, assume G is a semisimple group such that (G, X) is a connected Shimura datum of pre-abelian type, and set $d = \dim_{\mathbb{C}} X$ as before.

First, we prove an isomorphism of the form $\tilde{H}^i(K^p) \cong H^i(X_{K^pK_p}, \mathscr{C}(K_p, \mathbb{Z}_p))$ for any choice of open compact subgroup $K_p \subset G(\mathbb{Q}_p)$. Here $\mathscr{C}(K_p, \mathbb{Z}_p)$ denotes the K_p-module of continuous $\mathbb{Z}_p$-valued functions on K_p. This is essentially a version of Shapiro's lemma, and goes back to a paper of Hill [Hil10]. Next, by standard properties of manifolds with boundary, this isomorphism induces an isomorphism $\tilde{H}^i(K^p) \cong H^i(\overline{X}_{K^pK_p}, \mathscr{C}(K_p, \mathbb{Z}_p))$, where $\overline{X}_{K^pK_p}$ denotes the Borel-Serre compactification of $X_{K^pK_p}$.

By repeated use of excision for compactly supported cohomology, it now suffices to prove that for *some* stratification $\overline{X}_{K^pK_p} = \cup_{Z\in\mathcal{Z}} Z$, we have $H^i_c(Z, \mathscr{C}(K_p, \mathbb{Z}_p)|_Z) = 0$ for all $i > d$ and all $Z \in \mathcal{Z}$. Let us say that $\mathcal{Z}$ is an *adapted* stratification if this holds. The key idea can now be phrased as follows:

(†) If we take $\mathcal{Z}$ to be the stratification of $\overline{X}_{K^pK_p}$ obtained by pulling back the usual stratification of $X^*_{K^pK_p}$ along the canonical map $\pi : \overline{X}_{K^pK_p} \to X^*_{K^pK_p}$, then $\mathcal{Z}$ is an adapted stratification.

The idea that (†) is both true and provable is perhaps the most novel contribution of this paper. Let us give a sketch of the key ideas. Let $S \subset X^*_{K^pK_p}$ be a boundary stratum, with preimage $Z = \pi^{-1}(S) \subset \overline{X}_{K^pK_p}$. By the structure theory of the minimal compactification, the strata S are indexed by (equivalence classes of) pairs (Q, α) where $Q \subset G$ is a $\mathbb{Q}$-rational parabolic subgroup whose projection to each simple factor of G^{ad} is maximal or equal to G^{ad}, and α is some auxiliary data depending on the level structure. (We will suppress all dependences on level structures in the following discussion.) Moreover, the parabolic Q comes equipped with a canonically defined almost direct product decomposition $Q = U \cdot L \cdot H$. Here U is the unipotent radical of Q, L is a reductive group, H is a semisimple group whose associated symmetric space is Hermitian, and $L \cdot H$ is the full Levi subgroup of Q.

In parallel with this decomposition of Q, the stratum Z almost admits a direct product decomposition $Z \approx Z_U \times Z_L \times Z_H$, where Z_U is a torus, Z_L is the Borel-Serre compactification of a locally symmetric space for the group L, and $Z_H \cong S$ is a locally symmetric space for the group H. The key idea now is that $H^i_c(Z, \mathscr{C}(K_p, \mathbb{Z}_p)|_Z)$ can also be decomposed accordingly, by a Künneth-like formula, into contributions coming from each of these three factors, which can each be controlled:

- The contribution of Z_U is trivial, which follows from a well-known vanishing principle for completed cohomology of unipotent groups.
- The contribution of Z_H can be expressed in terms of compactly supported completed cohomology for H, which can be controlled by Step One.
- The contribution of Z_L can be expressed in terms of completed cohomology for L, which can be controlled using the bounds in [BS73], or even using the trivial bound.

The critical observation here is that Step One gives such good control over the contribution of Z_H that we need very little control over the contribution of Z_L.

In reality, the above sketch is somewhat oversimplified, because Z does not really admit a direct product decomposition; rather, it has the structure of an iterated fibration whose fibers are as described above. This leads to a number of irritating complications in the proof. Nevertheless, the essential idea follows the outline given above.

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2. Preliminaries

In this section we collect some facts and definitions from topology and algebraic groups that we will need. We make no attempt to state results in maximal generality and none of them are original, but we have often had difficulties locating the precise statements that we need in the literature. We hope that collecting this material here is of sufficient aid to the reader to justify its inclusion.

The topological spaces that we will work with will mostly be smooth manifolds with boundary; we will simply write "manifold with boundary" to mean a smooth manifold with boundary. Any smooth manifold with boundary admits a combinatorial triangulation for which the boundary is a subsimplicial complex (see e.g. [Mun63, Theorem 10.6]). We recall that if $\overline{X}$ is a manifold with boundary with interior X and $U \subseteq \overline{X}$ is an open subset containing X, then the inclusion $j : U \to \overline{X}$ is homotopy equivalence. In a very similar vein, if $\mathcal{F}$ is a local system on $\overline{X}$, then a simple local calculation shows that $Rj_*j^{-1}\mathcal{F} = \mathcal{F}$. In particular, we obtain canonical isomorphisms $H^i(U, \mathcal{F}) \cong H^i(\overline{X}, \mathcal{F})$ which we will often treat as equalities.

All actions of groups on topological spaces will be left actions in this section. Of course, all results have natural analogues for right actions (and we will use them).

2.1. **Local systems.** Let X be a topological space and let Γ be a group acting from the left on X. In this paper most of our actions will be *free*[2], by which we mean that every point $x \in X$ has an open neighborhood U such that $U \cap \gamma U \neq \emptyset$ only if $\gamma = 1$. The quotient map $\pi : X \to X_\Gamma := \Gamma\backslash X$ is then a covering map, and we recall that any left Γ-module[3] M defines a local system $\widetilde{M}$ on X_Γ given by

$$\widetilde{M}(U) = \mathrm{Map}_{lc,\Gamma}(\pi^{-1}(U), M)$$

where the right hand side denotes the locally constant functions $f : \pi^{-1}(U) \to M$ satisfying $f(\gamma x) = \gamma.f(x)$ for all $\gamma \in \Gamma$ and all $x \in \pi^{-1}(U)$. When X is a manifold with boundary, this may be written as

$$\widetilde{M}(U) = \mathrm{Map}_\Gamma(\pi_0(\pi^{-1}(U)), M),$$

where Map simply denotes set-theoretic functions (as $\pi_0(\pi^{-1}(U))$ is discrete). The following theorem is well known, and follows directly from the fact that the singular chain complex $C_\bullet(X)$ is a resolution of $\mathbb{Z}$ by free Γ-modules.

Theorem 2.1. *Let X is a contractible manifold with boundary with a free action of Γ. Then*

$$H^*(X_\Gamma, \widetilde{M}) \cong \mathrm{Ext}^*_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, M) \cong H^*(\Gamma, M)$$

canonically for every Γ-module M.

We now consider a relative version of Theorem 2.1. Let $p : E \to B$ be a fibre bundle with contractible fibre F (all spaces are manifolds with boundary). Assume that we have a group Γ acting (from the left) on both E and B, making p Γ-equivariant. We assume further that the action of Γ is free on E, and that the action of Γ on B factors through a quotient Δ which acts freely on B. Set $N = \mathrm{Ker}(\Gamma \to \Delta)$; N then acts freely on the fibres of p. Consider the induced map

$$q : E_\Gamma \to B_\Delta$$

on quotients.

[2]The most common terminology for this seems to be a free and properly discontinuous action, but we find this terminology rather cumbersome.

[3]By which we always mean a (left) $\mathbb{Z}[\Gamma]$-module, unless otherwise stated.

Corollary 2.2. *Let M be a Γ-module and let $i \geq 0$. Then $R^i q_* \widetilde{M}$ is the local system on B_Δ given by the Δ-module $H^i(N, M)$.*

Proof. We begin by proving the case $i = 0$. Write $\pi_E : E \to E_\Gamma$ and $\pi_B : B \to B_\Delta$ for the quotient maps and let $U \subseteq B$ be open. From the definitions, one sees that

$$q_* \widetilde{M}(U) = \mathrm{Map}_{lc,\Gamma}(p^{-1}\pi_B^{-1}(U), M).$$

Since the fibres of p are connected and the action of N preserves the fibres, we have $\mathrm{Map}_{lc,\Gamma}(p^{-1}\pi_B^{-1}(U), M) = \mathrm{Map}_{lc,\Delta}(\pi_B^{-1}(U), M^N)$, which is the desired statement.

This proves that the diagram of functors

$$\begin{tikzcd}
\mathrm{Mod}_\Gamma \arrow[r] \arrow[d, "M \mapsto M^N"] & \mathrm{Sh}(E_\Gamma) \arrow[d, "q_*"] \\
\mathrm{Mod}_\Delta \arrow[r] & \mathrm{Sh}(B_\Delta)
\end{tikzcd}$$

commutes up to natural isomorphism, where the horizontal functors are the local systems functors $M \mapsto \widetilde{M}$. The horizontal functors are exact (by looking at stalks), so it suffices to show that $M \mapsto \widetilde{M}$ sends injective Γ-modules to q_*-acyclic sheaves on E_Γ (then the diagram above commutes also after passing to derived categories and derived functors, which is what we want).

So, let M be an injective Γ-module, and let $i \geq 1$. $R^i q_* \widetilde{M}$ is the sheafification of the presheaf $U \mapsto H^i(q^{-1}(U), \widetilde{M})$ on B_Δ. There is a basis of open subsets U of B_Δ which are contractible and for which the fibre bundle $q : q^{-1}(U) \to U$ is trivial. In this case $q^{-1}(U) \cong U \times N\backslash F$ and hence

$$H^i(q^{-1}(U), \widetilde{M}) \cong H^i(N, M)$$

by Theorem 2.1. But M is an injective N-module since the restriction functor from Γ-modules to N-modules has an exact left adjoint $V \mapsto \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[N]} V$. Thus $H^i(q^{-1}(U), \widetilde{M}) = 0$ for all such U, and hence $R^i q_* \widetilde{M} = 0$ as desired. □

We will also use a (less precise but more general) version for pushforwards with proper support.

Proposition 2.3. *Let $f : X \to Y$ be a fibre bundle of manifolds with boundary, with fibre Z (also a manifold with boundary). Let $\mathcal{F}$ be a local system on X. Then, for any $i \geq 0$, $R^i f_! \mathcal{F}$ is a local system on Y with fibre $H^i_c(Z, \mathcal{F})$.*

Proof. We will use the commutation of derived pushforward with proper support with (arbitrary) pullbacks; see [KS94, Proposition 2.6.7]. Let $U \subseteq Y$ be a contractible open subset such that f is trivial over U, i.e. isomorphic to the canonical projection $p_U : U \times Z \to U$. These form an open cover of Y, so since $R^i f_!$ commutes with pullback it suffices to show that $R^i p_{U,!}\mathcal{F}$ is a constant sheaf. Since U is contractible, the restriction of $\mathcal{F}$ to $U \times Y$ $\mathcal{F}$ comes by pullback from a local system on Y, which we will call $\mathcal{F}_Z$. Consider the cartesian diagram

$$\begin{tikzcd}
U \times Z \arrow[r, "p_Z"] \arrow[d, "p_U"] & Z \arrow[d, "g"] \\
U \arrow[r, "f"] & pt,
\end{tikzcd}$$

where *pt* denotes the point and f and g are the canonical maps. Then we have

$$R^i p_{U,!}\mathcal{F} = R^i p_{U,!} p_Z^{-1} \mathcal{F}_Z \cong f^{-1} R^i g_! \mathcal{F}_Z.$$

In other words, $R^i p_{U,!}\mathcal{F}$ is the pullback of $H^i_c(Z, \mathcal{F}_Z)$ via the canonical map $U \to pt$. This proves the proposition. $\square$

Next, let X be a manifold with boundary with a free left action of a group Γ, and assume that $\Gamma' \subseteq \Gamma$ is a finite index subgroup. Consider the natural map $q : X_{\Gamma'} \to X_\Gamma$. If M is a Γ'-module, we put

$$\mathrm{Ind}_{\Gamma'}^{\Gamma} M = \{f : \Gamma \to M \mid f(\gamma'\gamma) = \gamma'.f(\gamma) \ \ \forall \gamma' \in \Gamma', \gamma \in \Gamma\},$$

which is a left Γ-module under right translation $(\gamma.f)(x) = f(x\gamma)$. We then have the following.

Proposition 2.4. *With notation and assumptions as above, $R^i q_* \widetilde{M} = 0$ for $i \geq 1$, and $q_*\widetilde{M}$ is the local system attached to $\mathrm{Ind}_{\Gamma'}^{\Gamma} M$.*

Proof. The map q is proper, so if $x \in X_\Gamma$, then $(R^i q_* \widetilde{M})_x = H^i(q^{-1}(x), \widetilde{M})$ (by [KS94, Proposition 2.6.7]), and $q^{-1}(x)$ has no higher cohomology since it is a finite set. This proves the first part. To compute $q_*\widetilde{M}$, let $U \subseteq X_\Gamma$ be open and write $\pi : X \to X_\Gamma$ for the quotient map. Unwinding the definitions, we see that

$$q_*\widetilde{M}(U) = \mathrm{Map}_{\Gamma'}(\pi_0(\pi^{-1}(U)), M),$$

and the right hand side is easily seen to be equal to $\mathrm{Map}_\Gamma(\pi_0(\pi^{-1}(U)), \mathrm{Ind}_{\Gamma'}^{\Gamma} M)$ functorially in U, as desired. $\square$

We move on to results on the commutation of $M \mapsto \widetilde{M}$ with direct limits. First, let $\overline{X}$ be a manifold with boundary, with a free left action of a group Γ. Write $\overline{X}_\Gamma := \Gamma\backslash\overline{X}$; we assume that $\overline{X}_\Gamma$ is compact, so it has a finite triangulation. Fix such a triangulation and pull it back to $\overline{X}$; this gives a triangulation whose corresponding complex of simplicial chains $C^\Delta_\bullet(\overline{X})$ is a bounded complex of finite free $\mathbb{Z}[\Gamma]$-modules. Let $(M_i)_{i\in I}$ be a directed system of Γ-modules with direct limit $M = \varinjlim_i M_i$.

Lemma 2.5. *The natural map*

$$\varinjlim_i H^*(\overline{X}_\Gamma, \widetilde{M_i}) \to H^*(\overline{X}_\Gamma, \widetilde{M})$$

is an isomorphism.

Proof. The canonical map

$$i : C^\Delta_\bullet(\overline{X}) \to C_\bullet(\overline{X})$$

is Γ-equivariant and a quasi-isomorphism; since the terms of both complexes are projective $\mathbb{Z}[\Gamma]$-modules the map is therefore a chain homotopy equivalence. This then gives us a commutative diagram of complexes

$$\begin{array}{ccc} \varinjlim_i \mathrm{Hom}_\Gamma(C_\bullet(\overline{X}), M_i) & \longrightarrow & \mathrm{Hom}_\Gamma(C_\bullet(\overline{X}), M) \\ \downarrow & & \downarrow \\ \varinjlim_i \mathrm{Hom}_\Gamma(C^\Delta_\bullet(\overline{X}), M_i) & \longrightarrow & \mathrm{Hom}_\Gamma(C^\Delta_\bullet(\overline{X}), M) \end{array}$$

where the vertical maps are induced by i and the horizontal maps are the natural maps. The vertical maps are then quasi-isomorphisms since they are induced from i, and the lower horizontal map is an isomorphism since $C^\Delta_\bullet(\overline{X})$ is bounded complex of finite free $\mathbb{Z}[\Gamma]$-modules. The top horizontal map is therefore a quasi-isomorphism as well, and taking cohomology gives the desired result. $\square$

We can then prove the result in greater generality. With $\overline{X}$ and Γ as above, let $U \subseteq \overline{X}$ be a Γ-invariant open subset containing the interior of $\overline{X}$. Set $U_\Gamma := \Gamma\backslash U$, $Z := \overline{X} \setminus U$ and $Z_\Gamma := \Gamma\backslash Z$.

Proposition 2.6. *The natural map*

$$\varinjlim_i H^*_?(X_\Gamma, \widetilde{M}_i) \to H^*_?(X_\Gamma, \widetilde{M})$$

is an isomorphism, for $? \in \{\emptyset, c\}$.

Proof. For $? = \emptyset$ this reduces directly to Lemma 2.5 by our setup, so assume that $? = c$. By naturality of the excision sequence and exactness of direct limits we have a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \varinjlim_i H^j_c(U_\Gamma, \widetilde{M}_i) \arrow[r] \arrow[d] & \varinjlim_i H^j(\overline{X}_\Gamma, \widetilde{M}_i) \arrow[r] \arrow[d] & \varinjlim_i H^j(Z_\Gamma, \widetilde{M}_i) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^j_c(U_\Gamma, \widetilde{M}) \arrow[r] & H^j(\overline{X}_\Gamma, \widetilde{M}) \arrow[r] & H^j(Z_\Gamma, \widetilde{M}) \arrow[r] & \dots
\end{tikzcd}$$

with exact rows. The result then follows from Lemma 2.5 (since it is applicable to both $\overline{X}_\Gamma$ and Z_Γ) and the five lemma. □

We also state an analogous result for inverse limits.

Proposition 2.7. *Keep the setup above, and assume additionally that all M_i are finite (as sets). Then natural map*

$$H^*_?(X_\Gamma, \widetilde{\varprojlim_i M_i}) \to \varprojlim_i H^*_?(X_\Gamma, \widetilde{M}_i)$$

is an isomorphism, for $? \in \{\emptyset, c\}$.

Proof. The proof is the same as for direct limits, using the finiteness of the M_i to ensure that the inverse limits occurring are exact. □

2.2. **"Completed cohomology".** In this subsection we make some definitions and recall a theorem of Hill which we will use to handle completed cohomology later. To begin with, we make the following general definition. Let $R = \varprojlim_i R/I^n$ be an adic ring, with I a finitely generated ideal of definition.

Definition 2.8. *Let $(X_i)_{i\in I}$ be an inverse system of topological spaces, with inverse limit X. We define the completed cohomology groups $\widetilde{H}^*_?(X, R)$ of $(X_i)_{i\in I}$ with coefficients in R, to be*

$$\widetilde{H}^*_?(X, R) = \varprojlim_n \varinjlim_i H^*_?(X_i, R/I^n).$$

Here $? \in \{\emptyset, c\}$, i.e. we consider either usual or compactly supported cohomology, when the latter makes sense.

Remark 2.9. *A few remarks on this definition:*

1. *The notation is chosen for simplicity; we make no assertion that $\widetilde{H}^*_?(X, R)$ only depends on X. One weak form of independence is clear though: We may replace I with a cofinal subsystem J. In particular, we may always assume that I contains an initial element $0 \in I$.*
2. *We will almost exclusively work with discrete R, where the inverse limit in the definition of $\widetilde{H}^*_?(X, R)$ disappears.*

We now recall the computation of completed cohomology as the cohomology of a "big" local system at finite level in some circumstances, which first appeared in [Hil10]. Let $\overline{X}$ be a manifold with boundary, equipped with a left action of a group G. We assume that there is a subgroup

$\Gamma \subseteq G$ which acts freely on $\overline{X}$, and suppose that $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots$ is a sequence of finite index subgroups of Γ. Let $X \subseteq \overline{X}$ be a Γ-stable open subset containing the interior of $\overline{X}$. Set

$$\widehat{X} := \varprojlim \left(\cdots \to \Gamma_2 \backslash X \to \Gamma_1 \backslash X \to \Gamma_0 \backslash X \right).$$

and define

$$K = \varprojlim_i \Gamma_i \backslash \Gamma;$$

this is a profinite set with a right action of Γ. Assume that $\overline{X}_\Gamma$ is compact. Then we get the following formula for completed cohomology of $\widehat{X}$ (cf. [Hil10, Corollary 1]):

Proposition 2.10. *With assumptions as above, let R be a discrete ring and let $? \in \{\emptyset, c\}$. Then there is a canonical isomorphism*

$$\widetilde{H}^*_?(\widehat{X}, R) \cong H^*_? \left(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(K, R)} \right),$$

where Γ acts on $\mathrm{Map}_{cts}(K, R)$ via right translation.

Proof. By Lemma 2.4 and the definition, we have

$$\widetilde{H}^*_?(\widehat{X}, R) \cong \varinjlim_i H^*_? \left(X_\Gamma, \widetilde{\mathrm{Map}(\Gamma_i \backslash \Gamma, R)} \right).$$

Our setup implies that we may apply Proposition 2.6 to the right hand side, so it remains to show that

$$\varinjlim_i \mathrm{Map}(\Gamma_i \backslash \Gamma, R) = \mathrm{Map}_{cts}(K, R)$$

as Γ-modules. But this is immediate from the definition of K. $\square$

We will also encounter local systems slightly bigger than the one appearing in Proposition 2.10. We keep the notation and assumptions of Proposition 2.10, except that we forget the groups denoted by K and Γ_i, $i \geq 1$. Let G be a profinite group with closed subgroups $K \subseteq H \subseteq G$, and assume that K is normal in H. For simplicity, we assume that there is a countable basis of neighborhoods of $1 \in G$. Suppose that we have a group homomorphism $\Gamma \to H/K$; then $\mathrm{Map}_{cts}(H/K, R)$ and $\mathrm{Map}_{cts}(G/K, R)$ become left Γ-modules via right translation, and hence induce local systems on the space X_Γ. Then we have the following simple but useful lemma.

Lemma 2.11. *Fix an integer $q \geq 0$ and let $? \in \{\emptyset, c\}$.*

1. $H^q_?(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(H/K, R)}) = 0$ *if and only if* $H^q_?(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(G/K, R)}) = 0$*;*
2. $H^q_c(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(H/K, R)}) \to H^q(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(H/K, R)})$ *is injective (or surjective, or bijective) if and only if* $H^q_c(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(G/K, R)}) \to H^q(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(G/K, R)})$ *is injective (or surjective, or bijective).*

Proof. Choose a continuous splitting of the natural map $G/K \to G/H$ (the existence of which is easy to prove using the assumption that $1 \in G$ has a countable basis of neighborhoods); this gives a homeomorphism

$$G/K \cong G/H \times H/K$$

of right H/K-spaces (where H/K acts on the right hand side through the second factor). Then

$$\mathrm{Map}_{cts}(G/K, R) \cong \mathrm{Map}_{cts}(G/H \times H/K, R) \cong \mathrm{Map}_{cts}(G/H, R) \otimes_R \mathrm{Map}_{cts}(H/K, R)$$

as H/K-modules (and hence as Γ-modules), where the action is trivial on $\mathrm{Map}_{cts}(G/H,R)$. Now $\mathrm{Map}_{cts}(G/H,R)$ is a direct limit of finite free R-modules, so using Proposition 2.6 we have an isomorphism

$$H^q_?(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(G/K,R)}) \cong \mathrm{Map}_{cts}(G/H,R) \otimes_R H^q_?(X_\Gamma, \widetilde{\mathrm{Map}_{cts}(H/K,R)})$$

which respects the maps in part (2). The lemma follows from this (since $\mathrm{Map}_{cts}(G/H,R)$ is a free R-module). $\square$

2.3. **Arithmetic and congruence subgroups.** Here we quickly recall some material on arithmetic and congruence subgroups. Let G be a connected linear algebraic group over $\mathbb{Q}$. Congruence subgroups of $G(\mathbb{Q})$ are subgroups of the form $G(\mathbb{Q}) \cap K$, where $K \subseteq G(\mathbb{A}_f)$ is a compact open subgroup and the intersection is taken inside $G(\mathbb{A}_f)$. A subgroup in $G(\mathbb{Q})$ is arithmetic if it is commensurable with one (equivalently any) congruence subgroup. Let H be another connected linear algebraic group, and let $\Gamma \subseteq G(\mathbb{Q})$ be an arithmetic subgroup. If $H \subseteq G$ is a subgroup, then directly from the definitions we see that $\Gamma \cap H(\mathbb{Q})$ is an arithmetic subgroup in $H(\mathbb{Q})$, which is congruence if Γ is. If we instead have a surjection $f : G \to H$, then $f(\Gamma)$ is an arithmetic subgroup (see [PR94, Theorem 4.1]); this will be important in this paper and we will use it freely. We note, however, that $f(\Gamma)$ need not be a congruence subgroup even if Γ is. Before moving on, we recall that group cohomology for any torsion-free arithmetic subgroup Γ commutes with direct limits.

We recall the notion of neatness from [Bor69, §17.1]. An element $\gamma \in G(\mathbb{Q})$ is called neat if there is a faithful representation $r : G \to \mathrm{GL}(V)$ such that the multiplicative group generated by the eigenvalues of $r(\gamma)$ (in one, or equivalently any, algebraically closed field containing $\mathbb{Q}$) is torsion-free. A neat element cannot have finite order. An arithmetic subgroup $\Gamma \subseteq G(\mathbb{Q})$ is called neat if all its elements are neat; such subgroups are in particular torsion-free. From the definitions, we see that if $H \subseteq G$ is a connected linear algebraic subgroup and $\Gamma \subseteq G(\mathbb{Q})$ is neat, then $\Gamma \cap H(\mathbb{Q})$ is neat. If an element γ is neat, then for any representation $\rho : G \to GL(W)$, the subgroup generated by the eigenvalues of $\rho(\gamma)$ is torsion-free [Bor69, Corollaire 17.3]. An easy consequence of this that if $f : G \to H$ is a surjection of linear algebraic groups and Γ is neat, then $f(\Gamma)$ is neat.

For language reasons, let us also introduce notions of neatness for adelic and p-adic groups. The notion of neatness for an element $g = (g_p)_p \in G(\mathbb{A}_f)$ and a subgroup $K \subseteq G(\mathbb{A}_f)$ is defined in [Pin90, §0.6]. For p-adic groups, we make the definition analogous to the case of arithmetic groups: An element $g \in G(\mathbb{Q}_p)$ is called neat if there is a faithful representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(W)$ over $\mathbb{Q}_p$ such that the multiplicative group generated by the eigenvalues of $\rho(g)$ (in one, or equivalently any, algebraically closed field containing $\mathbb{Q}_p$) is torsion-free. Again, this is independent of the choice of ρ. A subgroup $K_p \subseteq G(\mathbb{Q}_p)$ is called neat if all of its elements are neat. We note the following implications among these concepts: If $K_p \subseteq G(\mathbb{Q}_p)$ is a neat compact open subgroup, then $K^pK_p \subseteq G(\mathbb{A}_f)$ is neat for any compact open $K^p \subseteq G(\mathbb{A}_f^p)$. If a compact open $K \subseteq G(\mathbb{A}_f)$ is neat, then $\Gamma = \Gamma(\mathbb{Q}) \cap K$ is a neat congruence subgroup of G.

We record the following version of the standard result that "sufficiently small" congruence subgroups are neat; it will be important for us to be able to only impose congruence conditions at a fixed prime p.

Proposition 2.12. *Let p be a prime. Then sufficiently small compact open subgroups of $G(\mathbb{Q}_p)$ are neat. In particular, if $K^p \subseteq G(\mathbb{A}_f^p)$ is compact open, then $K = K^pK_p$ and $\Gamma = G(\mathbb{Q}) \cap K$ are neat for sufficiently small $K_p \subseteq G(\mathbb{Q}_p)$.*

Proof. By choosing a faithful representation $\rho : G \to \mathrm{GL}_n$ (and remembering that any compact subgroup of a locally profinite group is contained in a compact open subgroup), we may reduce to $G = \mathrm{GL}_n$. In this case, set $K_{r,p} = \mathrm{Ker}(\mathrm{GL}_n(\mathbb{Z}_p) \to GL_n(\mathbb{Z}/p^r))$; we will prove that if $r > n/(p-1)$, then $K_{r,p}$ is neat, so we assume this condition on r from now on. To show neatness, it suffices to show that if $\gamma \in K_{r,p}$, then the group generated by the eigenvalues of γ is torsion-free. Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of γ (in some choice of $\overline{\mathbb{Q}}_p$, with valuation v_p normalized so that $v_p(p) = 1$). The characteristic polynomial of γ reduces to $(X-1)^n$ modulo p^r, so by looking at Newton polygons $v_p(\alpha_i - 1) \geq r/n$ for all i. Thus, if α is any element in the multiplicative group generated by the α_i, $v_p(\alpha - 1) \geq r/n$. In particular, since $r > n/(p-1)$, α cannot be a nontrivial root of unity. This finishes the proof of the proposition. □

We also recall another fact about "sufficiently small" congruence subgroups, and set up some notation. For any real Lie group J, we write J^+ for the identity component of J. The following is [Del79, Corollaire 2.0.14].

Proposition 2.13. *Let G be a connected reductive group over $\mathbb{Q}$. Then there exists a congruence subgroup $\Gamma \subseteq G(\mathbb{Q})$ which is contained in $G(\mathbb{R})^+$. In particular, if $\Delta \subseteq G(\mathbb{Q})$ is any congruence subgroup, then $\Delta \cap G(\mathbb{R})^+$ is also a congruence subgroup.*

We remark that, unlike neatness, the condition $\Gamma \subseteq G(\mathbb{R})^+$ cannot be enforced only by congruence conditions at a single prime (chosen independently of G). For a simple example, consider $G = \mathrm{Res}^F_{\mathbb{Q}}\mathbb{G}_m$ with $F := \mathbb{Q}(\sqrt{3})$, and consider the totally negative unit $\alpha = -2 + \sqrt{3} \in F$. One checks easily that $\alpha^{3^n} \equiv 1$ modulo 3^n for all n but all the α^{3^n} are totally negative. For an example with a semisimple G, consider $G = \mathrm{Res}^F_{\mathbb{Q}}\mathrm{PGL}_2$ and the matrices

$$\begin{pmatrix} \alpha^{3^n} & 0 \\ 0 & 1 \end{pmatrix}, \quad n \geq 1;$$

again these tend to the identity 3-adically but they all lie in a non-identity component since they have totally negative determinant.

2.4. **Cohomology of unipotent groups.** From now on we fix a prime number p. Let N be a unipotent algebraic group over $\mathbb{Q}$. The goal in this subsection is to prove the following theorem (we remark that N satisfies strong approximation and that all arithmetic subgroups of $N(\mathbb{Q})$ are congruence subgroups):

Theorem 2.14. *If $\Gamma \subseteq N(\mathbb{Q})$ is a congruence subgroup with closure $K_p \subseteq N(\mathbb{Q}_p)$ and V is a smooth K-representation over $\mathbb{F}_p$, then the natural map*

$$H^i_{cts}(K_p, V) \to H^i(\Gamma, V)$$

is an isomorphism for all i.

We start with some recollections. First, in the situation above, $\Gamma = N(\mathbb{Q}) \cap K$ for some open compact subgroup $K \subseteq N(\mathbb{A}_f)$, and Γ is dense in K by strong approximation for N. In particular, K_p is the image of K under the projection map $N(\mathbb{A}_f) \to N(\mathbb{Q}_p)$, and hence open. We have a natural forgetful functor

$$\mathrm{Mod}_{sm}(K_p, \mathbb{F}_p) \to \mathrm{Mod}(\Gamma)$$

and if $V \in \mathrm{Mod}_{sm}(K_p, \mathbb{F}_p)$, then $V^\Gamma = V^{K_p}$ by smoothness of V and density of Γ in K_p. In light of this, Theorem 2.14 follows directly from the following special case, which is in fact all we will need.

Proposition 2.15. *Let V be an injective smooth K_p-representation over $\mathbb{F}_p$. Then $H^i(\Gamma, V) = 0$ for all $i \geq 1$.*

We will prove this by induction on $\dim N$. Before the main argument, we will discuss the structure of injective K_p-representations. Let W be any $\mathbb{F}_p$-vector space, which we give the discrete topology. We can form $\mathrm{Map}_{cts}(K_p, W)$, where K_p acts by right translation. This is the smooth induction of W, viewed as a representation of the trivial group, to K_p. Since smooth induction has an exact left adjoint (restriction), $\mathrm{Map}_{cts}(K_p, W)$ is injective for any W. We will refer to these representations as "standard injectives". Now if $V \in \mathrm{Mod}_{sm}(K_p)$ is arbitrary, there is a K_p-equivariant injection

$$V \to \mathrm{Map}_{cts}(K_p, V)$$

given by $v \mapsto (k \mapsto kv)$, where K_p acts on $\mathrm{Map}_{cts}(K_p, V)$ by right translation. Thus there are enough standard injectives, and any injective is a direct summand of a standard injective. In particular, it suffices to prove Proposition 2.15 for standard injectives. Moreover, since group cohomology of Γ commutes with direct limits, it suffices to prove Proposition 2.15 for $\mathrm{Map}_{cts}(K_p, \mathbb{F}_p)$.

We now begin the induction. First assume that $\dim N = 1$, i.e. that $N = \mathbb{G}_a$. Then (up to isomorphism) $\Gamma = \mathbb{Z}$ and $K_p = \mathbb{Z}_p$. There are a number of ways of proving that $H^i(\mathbb{Z}, \mathrm{Map}_{cts}(\mathbb{Z}_p, \mathbb{F}_p)) = 0$ for $i \geq 1$. For example, by Proposition 2.10,

$$H^i(\mathbb{Z}, \mathrm{Map}_{cts}(\mathbb{Z}_p, \mathbb{F}_p)) = \varinjlim_n H^i(\mathbb{R}/p^n\mathbb{Z}, \mathbb{F}_p) = \varinjlim_n H^i(S^1, \mathbb{F}_p)$$

where on the right the transition maps come from pullback along the maps $S^1 \to S^1$, $z \mapsto z^p$. All groups are 0 for $i \geq 2$, and for $i = 1$ one easily sees that the transition maps are all 0, so this proves Proposition 2.15 for $N = \mathbb{G}_a$.

We move on to the induction step. By the structure of unipotent groups, we can choose a proper non-trivial normal subgroup $U \subseteq N$. Set $H = N/U$ and let $f : N \to H$ denote the natural map. Put $\Gamma_U = \Gamma \cap U(\mathbb{Q})$, $\Gamma_H = f(\Gamma)$, $K_{U,p} = K_p \cap U(\mathbb{Q}_p)$ and $K_{H,p} = f(K_p) \subseteq H(\mathbb{Q}_p)$. Then $K_{U,p}$ is the closure of Γ_U in $U(\mathbb{Q}_p)$ and $K_{H,p}$ is the closure of Γ_H in $H(\mathbb{Q}_p)$. Let V be an injective smooth K_p-representation over $\mathbb{F}_p$. We have the Hochschild–Serre spectral sequence

$$H^i(\Gamma_H, H^j(\Gamma_U, V)) \implies H^{i+j}(\Gamma, V).$$

The restriction of V to $K_{U,p}$ is still injective by [Eme10b, Proposition 2.1.11]. Thus, by the induction hypothesis, $H^j(\Gamma_U, V) = 0$ for $j \geq 1$, and hence the spectral sequence degenerates to $H^i(\Gamma, V) = H^i(\Gamma_H, V^{\Gamma_U})$. By above, $V^{\Gamma_U} = V^{K_{U,p}}$, which is an injective[4] $K_{H,p}$-module. By the induction hypothesis again we get

$$H^i(\Gamma, V) = H^i(\Gamma_H, V^{K_{U,p}}) = 0$$

for $i \geq 1$, as desired. This finishes the proof of Proposition 2.15, and hence the proof of Theorem 2.14.

3. Completed cohomology of locally symmetric spaces

We continue to fix a prime number p.

3.1. **Locally symmetric spaces.**

In this section we recall some material on locally symmetric spaces and their Borel–Serre compactifications. Let G be a connected linear algebraic group over $\mathbb{Q}$, let $A = A_G \subseteq G$ be a maximal torus in the $\mathbb{Q}$-split part of the radical of G and let $K_\infty = K_{G,\infty} \subseteq G(\mathbb{R})$ be a maximal compact subgroup. We will work with the (connected) symmetric space

$$X = X^G := G(\mathbb{R})^+/A(\mathbb{R})^+K_\infty^+ = G(\mathbb{R})/A(\mathbb{R})K_\infty,$$

which is the symmetric space part of any space of type $S - \mathbb{Q}$ for G, in the terminology of [BS73]. If $\Gamma \subseteq G(\mathbb{Q})$ is a torsion-free arithmetic subgroup, then Γ acts freely on X and the quotient $\Gamma\backslash X$ is a locally symmetric space. If $K \subseteq G(\mathbb{A}_f)$ is a compact (not necessarily open) subgroup, we will set

$$X_K^G := G(\mathbb{Q})^+\backslash X \times G(\mathbb{A}_f)/K,$$

[4] $M \mapsto M^{K_{U,p}}$ preserves injectives, since inflation from $K_{H,p}$ to K_p provides an exact left adjoint.

where $G(\mathbb{Q})^+ := G(\mathbb{Q}) \cap G(\mathbb{R})^+$ and K and $G(\mathbb{A}_f)$ carry their usual adelic topologies. When K is additionally open and $g \in G(\mathbb{A}_f)$, set $\Gamma_g = \Gamma_{g,K} := G(\mathbb{Q})^+ \cap gKg^{-1}$; these are congruence subgroups by Proposition 2.13. We have the following decomposition as topological spaces

$$X_K^G \cong \coprod_{g \in G(\mathbb{Q})^+ \backslash G(\mathbb{A}_f)/K} \Gamma_g \backslash X,$$

where the set $\Sigma_K := G(\mathbb{Q})^+ \backslash G(\mathbb{A}_f)/K$ is finite by [Bor63, Theorem 5.1]. If K is neat, then all the Γ_g are neat and in particular torsion-free, so X_K^G is a (possibly disconnected) manifold of dimension $\dim_{\mathbb{R}} X$.

Recall the Borel–Serre bordification $\overline{X} = \overline{X}^G$ of $X = X^G$ from [BS73]. $\overline{X}$ has a natural structure of a manifold with corners, with interior X. We write $\partial X = \overline{X} \setminus X$. The action of $G(\mathbb{Q})$ on X extends to an action of $\overline{X}$, and again any torsion-free arithmetic subgroup $\Gamma \subseteq G(\mathbb{Q})$ acts freely on $\overline{X}$. As a set,

$$\overline{X} = \coprod_Q X^Q$$

where Q runs through the (rational) parabolic subgroups of G. The closure of X^Q inside $\overline{X}$ is $\overline{X}^Q = \coprod_{P' \subseteq Q} X^{P'}$. Write C_Q for the set of parabolics Q' of G which are conjugate to Q (over $\mathbb{Q}$); C_Q carries a (tautological) left $G(\mathbb{Q})$-action by conjugation. Fix a minimal parabolic P of G over $\mathbb{Q}$ for simplicity. We can then write

$$\overline{X} = \coprod_Q X^Q = \coprod_{Q \supseteq P} \coprod_{Q' \in C_Q} X^{Q'},$$

and the subsets $X^{G,Q} = \coprod_{Q' \in C_Q} X^{Q'}$ are stable under $G(\mathbb{Q})$. If $g \in G(\mathbb{Q})$, then $gX^{Q'} = X^{gQ'g^{-1}}$ and hence the stabilizer of $X^{Q'}$ is $Q'(\mathbb{Q})$. In particular, if $\Gamma \subseteq G(\mathbb{Q})$ is an arithmetic subgroup, we see that

$$\Gamma \backslash \overline{X} = \coprod_{Q \supseteq P} \coprod_{Q' \in C_{Q,\Gamma}} \Gamma_{Q'} \backslash X^{Q'},$$

where $C_{Q,\Gamma} = \Gamma \backslash C_Q$ and $\Gamma_{Q'} = \Gamma \cap Q'(\mathbb{Q})$. If Γ is neat, then $\Gamma_{Q'}$ is neat for all Q'. The space $\Gamma \backslash \overline{X}$ is a compact manifold with corners, which in particular implies that it is homeomorphic to a manifold with boundary [BS73, Appendix], so the results of §2 apply to it.

3.2. **The vanishing conjecture for completed cohomology.** In this subsection we assume that G is reductive. Fix a compact open subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. Let R be an adic ring with finitely generated ideal of definition I. We define completed cohomology of G (with respect to K^p) to be

$$\widetilde{H}^*_?(K^p, R) := \widetilde{H}^*_?(X_{K^p}, R) = \varprojlim_n \varinjlim_{K_p} H^*_?\left(X_{K^pK_p}, R/I^n\right),$$

where $? \in \{\emptyset, c\}$ and K_p runs through the compact open subgroups of $G(\mathbb{Q}_p)$. We recall the quantities

$$l_0 = l_0(G) := \operatorname{rank}(G(\mathbb{R})) - \operatorname{rank}(A(\mathbb{R})K_\infty)$$

and

$$q_0 = q_0(G) := \frac{\dim_{\mathbb{R}} X - l_0}{2},$$

where rank denotes the rank as a Lie group. With these preparations, we may state the main vanishing conjecture of Calegari–Emerton:

Conjecture 3.1. *Let $? \in \{\emptyset, c\}$. Then $\widetilde{H}^n_?(K^p, \mathbb{Z}_p) = 0$ for all $n > q_0$.*

Remark 3.2. *While Conjecture 3.1 is not explicitly stated in* [CE12]*, it is a direct consequence of* [CE12, Conjecture 1.5(5)-(8) and Theorem 1.1(3)]*. We will discuss* [CE12, Conjecture 1.5] *in §3.4.*

We will focus on the following equivalent version, which is also implicit in [CE12].

Conjecture 3.3. *Let* $? \in \{\emptyset, c\}$*. Then* $\widetilde{H}^n_?(K^p, \mathbb{F}_p) = 0$ *for all* $n > q_0$*.*

Proposition 3.4. *Conjecture 3.1 is equivalent to Conjecture 3.3.*

Proof. That Conjecture 3.1 implies Conjecture 3.3 follows from [CE12, Theorem 1.16(5)]. For the converse, note first that we have long exact sequences

$$\cdots \to \widetilde{H}^i_?(K^p, \mathbb{Z}/p^{r-1}) \to \widetilde{H}^i_?(K^p, \mathbb{Z}/p^r) \to \widetilde{H}^i_?(K^p, \mathbb{F}_p) \to \dots$$

coming from the the corresponding long exact sequences at finite level, so by induction on r we see that Conjecture 3.3 implies that $\widetilde{H}^i_?(K^p, \mathbb{Z}/p^r) = 0$ for all r and $n > q_0$. Conjecture 3.1 then follows since $\widetilde{H}^i_?(K^p, \mathbb{Z}_p) = \varprojlim_r \widetilde{H}^i_?(K^p, \mathbb{Z}/p^r)$. □

As usual in the Langlands program, adelic double quotients have the advantage that they make the Hecke actions and group actions transparent. These actions will, however, play essentially no role in this paper, and we found it simpler to work non-adelically. The rest of this subsection will discuss a version of Conjecture 3.3 in this language that we will treat. To this end, let us define $G(\mathbb{R})_+$ to be the inverse image of $G^{\mathrm{ad}}(\mathbb{R})^+$ under the natural map $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$, where G^{ad} is the adjoint group of G. Then we set $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$ and

$$C_p = C_p(K^p) := \{G(\mathbb{Q})_+ \cap K^p K_p \mid K_p \subseteq G(\mathbb{Q}_p) \text{ compact open}\}.$$

Informally, this is the set of congruence subgroups of $G(\mathbb{Q})$ with fixed tame level K^p contained in $G(\mathbb{R})_+$. Armed with this definition, we set

$$\widehat{X} = \widehat{X}^G = \widehat{X}(K^p) = \widehat{X}^G(K^p) := \varprojlim_{\Gamma \in C_p} \Gamma \backslash X.$$

We can then state the following conjecture.

Conjecture 3.5. *Let* $? \in \{\emptyset, c\}$*. Then we have* $\widetilde{H}^n_?(\widehat{X}, \mathbb{F}_p) = 0$ *for all* $n > q_0$*.*

This is the conjecture that we will focus on. A priori, it is slightly stronger than Conjecture 3.3 because we are looking at congruence subgroups inside $G(\mathbb{R})_+$ instead of $G(\mathbb{R})^+$. We give a general discussion of the passage between disconnected spaces and their components, and formalize the implication relevant to this paper. To simplify notation, we drop the notation $\widetilde{M}$ used in §2 to denote the local system associated with a representation M, simply writing M for the local system as well in the rest of this paper.

First, for any compact subgroup $K \subseteq G(\mathbb{A}_f)$, define

$$\mathfrak{X}_K := \mathfrak{X}_K^G := G(\mathbb{Q})^+ \backslash X \times G(\mathbb{A}_f)/K,$$

where now we give $G(\mathbb{A}_f)$ the *discrete* topology. Note that $\mathfrak{X}_K = X_K$ when K is open. In general, $\mathfrak{X}_K$ is a manifold when K is neat. If $K_1 \subseteq K_2$ are neat, with K_1 normal in K_2, then K_2/K_1 acts freely on $\mathfrak{X}_{K_1}$ with quotient $\mathfrak{X}_{K_2}$. We similarly define $\overline{\mathfrak{X}}_K$, replacing X by $\overline{X}$. In particular, using $\mathfrak{X}_{K^p}$ and $\overline{\mathfrak{X}}_{K^p}$, we may apply Theorem 2.10 to deduce that

$$\widetilde{H}^i_?(K^p, \mathbb{F}_p) \cong H^i_?(X_K, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p))$$

where $K = K^p K_p$ with K_p neat. Using the decomposition into connected components, we see that

$$\widetilde{H}^i_?(K^p, \mathbb{F}_p) \cong \bigoplus_{g \in G(\mathbb{Q})^+ \backslash G(\mathbb{A}_f)/K} H^i_?(\Gamma_g \backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)).$$

Here, the right K_p-module $\mathrm{Map}_{cts}(K_p, \mathbb{F}_p)$ (action via left translation) becomes a right $\Gamma_g = G(\mathbb{Q})^+ \cap gKg^{-1}$-module via the composition $\Gamma_g \to K \to K_p$ where the first map is conjugation by g^{-1} and the second is the projection, and then a left Γ_g-module by inversion. In particular, we have an isomorphism

$$\mathrm{Map}_{cts}(K_p, \mathbb{F}_p) \cong \mathrm{Map}_{cts}(g_p K_p g_p^{-1}, \mathbb{F}_p)$$

of Γ_g-modules (with the obvious Γ_g-structure on the right hand side). Then, note that the left Γ_g-module $\mathrm{Map}_{cts}(g_p K_p g_p^{-1}, \mathbb{F}_p)$, where the action is via inverting the left translation action, is isomorphic to the left Γ_g-module $\mathrm{Map}_{cts}(g_p K_p g_p^{-1}, \mathbb{F}_p)$ where the action is the right translation action (the isomorphism is given by inversion on $g_p K_p g_p^{-1}$). This proves the following:

Proposition 3.6. *Fix i and K^p. Choose K_p sufficiently small to make $K = K^p K_p$ neat. For any other $K' \subseteq G(\mathbb{A}_f)$ compact open, set $\Gamma' = G(\mathbb{Q})^+ \cap K'$. Then $\widetilde{H}^i_?(K^p, \mathbb{F}_p) = 0$ if and only if $H^i_?(\Gamma' \backslash X, \mathrm{Map}_{cts}(K'_p, \mathbb{F}_p)) = 0$ for all conjugates K' of K in $G(\mathbb{A}_f)$, where Γ' acts on $\mathrm{Map}_{cts}(K'_p, \mathbb{F}_p))$ either via right translation or by inverting the left translation action.*

As a corollary we get the implication between Conjecture 3.5 and Conjecture 3.3.

Proposition 3.7. *Choose $? \in \{\emptyset, c\}$. Then Conjecture 3.5 (for all conjugates of a fixed K^p) for $?$ implies Conjecture 3.3 (for K^p) for $?$.*

Proof. Let $K^p \subseteq G(\mathbb{A}_f^p)$ be compact open and let $n > q_0$. By Proposition 3.6, it suffices to show that $H^n_?(\Gamma \backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)) = 0$ for some sufficiently small K_p, where $\Gamma = G(\mathbb{Q})^+ \cap K^p K_p$ acts on $\mathrm{Map}_{cts}(K_p, \mathbb{F}_p))$ via right translation. Γ is a congruence subgroup by Proposition 2.13, so $\Gamma = K_1^p K_{1,p} \cap G(\mathbb{Q})$ for some $K_1^p \subseteq K^p$, $K_{1,p} \subseteq K_p$. In particular $\Gamma = K_1^p K_{1,p} \cap G(\mathbb{Q})_+$ as well. By shrinking K_p, we may assume that $K_p = K_{1,p}$. To simplify notation, we then choose a cofinal sequence

$$K_p = K_{p,0} \supseteq K_{p,1} \supseteq K_{p,2} \supseteq \dots$$

of compact open subgroups of $G(\mathbb{Q}_p)$ with $K_{p,n}$ normal in $K_{p,0}$ for all n, and set $\Gamma_i = G(\mathbb{Q})_+ \cap K_1^p K_{p,i}$, $X_i = \Gamma_i \backslash X$ and $\widehat{X} = \varprojlim_i X_i$. By Conjecture 3.5, $\widetilde{H}^n_?(\widehat{X}, \mathbb{F}_p) = 0$. By Theorem 2.10,

$$H^n_?(\Gamma \backslash X, \mathrm{Map}_{cts}(H, \mathbb{F}_p)) = \widetilde{H}^n_?(\widehat{X}, \mathbb{F}_p) = 0$$

where $H = \varprojlim_i \Gamma/\Gamma_i$ is the closure of Γ in K_p and Γ acts on $\mathrm{Map}_{cts}(H, \mathbb{F}_p)$ via right translation. An application of Lemma 2.11 then gives that $H^n_?(\Gamma \backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)) = 0$, as desired. □

3.3. **The case of Hermitian symmetric domains.** In this subsection, we assume that G is semisimple and that X is a Hermitian symmetric domain. In this case, $l_0 = 0$ and $q_0 = (\dim_{\mathbb{R}} X)/2 = \dim_{\mathbb{C}} X$; we will simply write d for this quantity. We briefly recall some material from the theory of hermitian symmetric domains and their boundary components; some references for this material are [AMRT10, BB66, Hel78]. We do not assume that G has no $\mathbb{R}$-anisotropic $\mathbb{Q}$-simple factors.

First, let us recall that an element $g \in G(\mathbb{R})$ acts holomorphically on X if and only if $g \in G(\mathbb{R})_+$; see [BB66, Proposition 11.3] (note that G is assumed to be adjoint in this reference). The space $X = X^G$ has a bordification $X^* = X^{G,*}$ obtained by adding the rational boundary components of X, see [BB66]. To describe it, we make a definition. If G is $\mathbb{Q}$-simple, we call a parabolic subgroup Q *maximal* if there is no parabolic subgroup Q' with $Q \subsetneq Q' \subsetneq G$. For general G, we will call a parabolic subgroup Q maximal if its projection to every $\mathbb{Q}$-simple factor is maximal in the previous sense. Let Q be such a maximal parabolic subgroup of G; we write N_Q for its unipotent radical and M_Q for its Levi quotient. M_Q decomposes into an almost direct product $M_Q = M_{Q,\ell} M_{Q,h}$; see [AMRT10, Item (5), p. 142] (in the notation of that reference, we take

$M_{Q,\ell} = \mathscr{G}_\ell$ and $M_{Q,h} = \mathscr{G}_h \cdot \mathscr{M}$). $M_{Q,\ell}$ is called the *linear* part; it is a connected reductive group. $M_{Q,h}$ is called the *Hermitian* part and it is a semisimple group whose symmetric space is a Hermitian symmetric domain. Our main result in the topological part of this paper is the following.

Theorem 3.8. *With assumptions as above, assume that Conjecture 3.5 holds for $M_{Q,h}$ for all maximal parabolics Q of G (including $Q = G$) and $? = c$. Then Conjecture 3.5 holds for G and $? = \emptyset$.*

The proof will occupy the rest of this subsection. Let us now describe the bordification X^*. Set-theoretically,

$$X^* = \bigsqcup_{Q \text{ maximal}} X^{M_{Q,h}} = \bigsqcup_{Q \supseteq P \text{ maximal}} X^{G,M_{Q,h}},$$

where $X^{G,M_{Q,h}} := \bigsqcup_{Q' \in C_Q} X^{M_{Q',h}}$ and we recall that P is a fixed choice of a minimal parabolic subgroup. The action of $G(\mathbb{Q})$ on X extends to an action on X^*, but torsion-free arithmetic subgroups will no longer act freely (in general). The spaces $X^{G,M_{Q,h}}$ are stable under $G(\mathbb{Q})$. If $\Gamma \subseteq G(\mathbb{Q})_+$ is a torsion-free arithmetic subgroup, then $\Gamma\backslash X^*$ has a canonical structure of a projective algebraic variety over $\mathbb{C}$. Let us now assume that Γ is in addition neat, and let $\Gamma_{M_{Q',h}}$ be the image of $\Gamma_{Q'}$ in $M_{Q',h}(\mathbb{Q})$; this is a neat arithmetic subgroup. We have a stratification

$$\Gamma\backslash X^* = \bigsqcup_{Q \supseteq P \text{ maximal}} \bigsqcup_{Q' \in C_{Q,\Gamma}} \Gamma_{M_{Q',h}} \backslash X^{M_{Q',h}}$$

of the quotient. By construction $\Gamma_{M_{Q',h}}$ acts holomorphically on $X^{M_{Q',h}}$, so $\Gamma_{M_{Q',h}} \subseteq M_{Q',h}(\mathbb{Q})_+$.

In [Zuc83], Zucker constructs a $G(\mathbb{Q})$-equivariant continuous map $\pi : \overline{X} \to X^*$ that we will make use of.[5] With Q as above, let us write $Y(Q) = \pi^{-1}(X^{M_{Q,h}})$. By [Zuc83, (3.8), Proposition], we have a natural homeomorphism

$$Y(Q) \cong X^{M_{Q,h}} \times \overline{X}^{M_{Q,\ell}} \times X^{N_Q}$$

and the projection maps

$$Y(Q) \to Y(M_Q) := X^{M_{Q,h}} \times \overline{X}^{M_{Q,\ell}} \to \overline{X}^{M_{Q,\ell}}$$

are $Q(\mathbb{Q})$-equivariant (and fibre bundles). Write $L_Q = M_{Q,\ell}/(M_{Q,\ell} \cap M_{Q,h})$; the natural map $M_{Q,\ell} \to L_Q$ is a central isogeny and $\overline{X}^{M_{Q,\ell}} = \overline{X}^{L_Q}$. Then we remark that, in the displayed equation above, $Q(\mathbb{Q})$ acts via the projection map $Q(\mathbb{Q}) \to M(\mathbb{Q})$ on $Y(M_Q)$ and via the projection map $Q(\mathbb{Q}) \to L_Q(\mathbb{Q})$ on $\overline{X}^{M_{Q,\ell}}$. In particular, we note that $Y(Q)$ is contractible and that if Γ is torsion-free, then Γ_Q acts freely on $Y(Q)$.

We now begin the proof of Theorem 3.8. Fix a compact open subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. Our goal is to understand $\widetilde{H}^*(\widehat{X}, \mathbb{F}_p) = \widetilde{H}^*(\widehat{\overline{X}}, \mathbb{F}_p)$ in terms of the $\widetilde{H}_c^*(\widehat{X}^{M_{Q,h}}, \mathbb{F}_p)$, where

$$\widehat{\overline{X}} = \widehat{\overline{X}}^G = \widehat{\overline{X}}(K^p) = \widehat{\overline{X}}^G(K^p) := \varprojlim_{\Gamma \in C_p} \Gamma\backslash\overline{X},$$

[5]It is, strictly speaking, not necessary for us to use minimal compactifcations and Zucker's work [Zuc83], as all we need is the resulting stratification of the Borel–Serre compactification which one may describe directly. Nevertheless, we have opted to include the minimal compactification in our discussion as it gives a conceptual way of understanding the stratification that we use, and why we use it.

and we recall that $C_p = C_p(K^p)$ is (informally) the collection of congruence subgroups of $G(\mathbb{Q})$ with fixed tame level K^p contained in $G(\mathbb{Q})_+$. We choose once and for all a neat $\Gamma \in C_p$; this is possible by Proposition 2.12. Set

$$S = \varprojlim_{\Gamma \supseteq \Gamma' \in C_p} \Gamma' \backslash \Gamma;$$

S is the closure of Γ in $G(\mathbb{Q}_p)$. Proposition 2.10 then gives us the following description of $\widetilde{H}^*(\widehat{\overline{X}}, \mathbb{F}_p)$.

Proposition 3.9. *We have a canonical isomorphism*

$$\widetilde{H}^*(\widehat{\overline{X}}, \mathbb{F}_p) \cong H^*(\Gamma \backslash \overline{X}, \mathrm{Map}_{cts}(S, \mathbb{F}_p)).$$

The 'stratification' $(Y(Q))_Q$ of $\overline{X}$ induces a finite stratification $(\Gamma_Q \backslash Y(Q))_Q$ of $\Gamma \backslash \overline{X}$ into locally closed subsets, parametrized by Γ-conjugacy classes of maximal parabolic subgroups Q. By repeated use of the excision sequence, it suffices for us to prove that

$$H^i_c(\Gamma_Q \backslash Y(Q), \mathrm{Map}_{cts}(S, \mathbb{F}_p)) = 0$$

for $i > d$ and for all Q. From now on we fix Q and drop the subscripts $-_Q$ from all associated algebraic groups for simplicity. Consider the proper map $f : \Gamma_Q \backslash Y(Q) \to \Gamma_M \backslash Y(M)$, which is a fibre bundle with fibre $\Gamma_N \backslash X^{N_Q}$. Here $\Gamma_N = N(\mathbb{Q}) \cap \Gamma_Q$ and Γ_M is the image of Γ_Q under $Q(\mathbb{Q}) \to M(\mathbb{Q})$; these are both congruence subgroups. Set $S_N = S \cap N(\mathbb{Q}_p)$; by strong approximation this is the closure of Γ_N in $N(\mathbb{Q}_p)$ (and hence open). Then we have

$$H^*_c(\Gamma_Q \backslash Y(Q), \mathrm{Map}_{cts}(S, \mathbb{F}_p)) = H^*_c(\Gamma_M \backslash Y(M), Rf_* \mathrm{Map}_{cts}(S, \mathbb{F}_p)).$$

Since $\Gamma \cap S_N = \Gamma_N$, $\Gamma_M = \Gamma_Q / \Gamma_N$ acts by right translation on S/S_N.

Proposition 3.10. $f_* \mathrm{Map}_{cts}(S, \mathbb{F}_p) = \mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p)$ *with* Γ_M *acting by right translation, and* $R^i f_* \mathrm{Map}_{cts}(S, \mathbb{F}_p) = 0$ *for all* $i \geq 1$.

Proof. By Corollary 2.2, $R^i f_* \mathrm{Map}_{cts}(S, \mathbb{F}_p)$ is the local system on $\Gamma_M \backslash Y(M)$ corresponding to the Γ_M-representation $H^i(\Gamma_N, \mathrm{Map}_{cts}(S, \mathbb{F}_p))$. When $i = 0$, the description is clear since Γ_N is dense in S_N. In general, choose a continuous section $S \to S_N$ of the inclusion, which gives a homeomorphism $S \cong S/S_N \times S_N$ of right S_N-spaces. Arguing as in Lemma 2.11, we see that

$$H^i(\Gamma_N, \mathrm{Map}_{cts}(S, \mathbb{F}_p)) \cong \mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^i(\Gamma_N, \mathrm{Map}_{cts}(S_N, \mathbb{F}_p)).$$

By Proposition 2.15 and the injectivity of $\mathrm{Map}_{cts}(S_N, \mathbb{F}_p)$ (discussed in §2.4), the right hand side is 0 when $i \geq 1$. □

So, we are down to computing $H^*_c(\Gamma_M \backslash Y(M), \mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p))$, for which we use the fibre bundle

$$g : \Gamma_M \backslash Y(M) \to \Gamma_L \backslash \overline{X}^L,$$

with fibre $\Gamma_h \backslash X^{M_h}$. Here $\Gamma_h = M_h(\mathbb{Q}) \cap \Gamma_M$ and $\Gamma_L = r(\Gamma_M)$, where $r : M \to L$ denotes the canonical map. Γ_h is a neat congruence subgroup of $M_h(\mathbb{Q})$, and Γ_L is a neat arithmetic subgroup of $L(\mathbb{Q})$. The Leray spectral sequence reads

$$H^i(\Gamma_l \backslash \overline{X}^L, R^j g_! \mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p)) \implies H^{i+j}_c(\Gamma_M \backslash Y(M), \mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p)).$$

The key is then the following.

Proposition 3.11. $R^j g_! \mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p))$ *is a local system on* $\Gamma_L \backslash \overline{X}^L$ *and vanishes for* $j > \dim_{\mathbb{C}} X^{M_h}$.

Proof. $R^j g_! \operatorname{Map}_{cts}(S/S_N, \mathbb{F}_p))$ is a local system with fibre $H_c^j(\Gamma_h \backslash X^{M,h}, \operatorname{Map}_{cts}(S/S_N, \mathbb{F}_p))$ by Proposition 2.3. Consider the closure T_h of Γ_h in $M_h(\mathbb{Q}_p)$, which we may also view as the closure of Γ_h in S/S_N. Write S_h for the preimage of T_h under $S \to S/S_N$. S_h is a group containing S_N as a normal subgroup, and $T_h = S_h/S_N$. Applying Lemma 2.11 with $G = S$, $H = S_h$, $K = S_N$ and $\Gamma = \Gamma_h$, $H_c^j(\Gamma_h \backslash X^{M,h}, \operatorname{Map}_{cts}(S/S_N, \mathbb{F}_p))$ vanishes if

$$H_c^j(\Gamma_h \backslash X^{M_h}, \operatorname{Map}_{cts}(T_h, \mathbb{F}_p)).$$

But $H_c^j(\Gamma_h \backslash X^{M,h}, \operatorname{Map}_{cts}(T_h, \mathbb{F}_p))$ is compactly supported completed $\mathbb{F}_p$-cohomology for M_h by Proposition 2.10, so this vanishes for $j > \dim_{\mathbb{C}} X^{M_h}$ by assumption. □

Before we put everything together, we need to relate d to $\dim_{\mathbb{C}} X^{M_h}$ and $\dim_{\mathbb{R}} X^L$. Recall that A_L is the maximal $\mathbb{Q}$-split torus in the center of L, and write $Z(N)$ for the center of N. The result is then the following.

Lemma 3.12. $\dim_{\mathbb{C}} X^{M_h} + \dim_{\mathbb{R}} X^L = d - \frac{1}{2}(\dim N - \dim Z(N)) - \dim A_L$.

Proof. The symmetric space X^G has a decomposition

$$X^G \cong X^{M_h} \times C(L) \times N(\mathbb{R})$$

as real manifolds[6] by [AMRT10, Equation (4.1)]. This gives

$$\dim_{\mathbb{C}} X^{M_h} = d - \frac{1}{2}\left(\dim_{\mathbb{R}} C(L) + \dim N\right).$$

The space $C(L)$, called $C(F)$ in [AMRT10], is an open subset of $Z(N)(\mathbb{R})$ and diffeomorphic to $L(\mathbb{R})/K_{L,\infty}$ by [AMRT10, Theorem 4.1(2)], where $K_{L,\infty}$ denotes a maximal compact subgroup of $L(\mathbb{R})$. Thus $\dim_{\mathbb{R}} X^L = \dim_{\mathbb{R}} C(L) - \dim A_L$ and $\dim_{\mathbb{R}} C(L) = \dim Z(N)$. Combining this with the displayed equation above and rearranging gives the desired result. □

We may now put everything together to prove a more precise version of Theorem 3.8. From now on we let Q denote an arbitrary maximal parabolic of G again, and set

$$\gamma(Q) = \frac{1}{2}\left(\dim N_Q - \dim Z(N_Q)\right) + \dim A_{L_Q} + \operatorname{ss.rank}_{\mathbb{Q}}(L_Q)$$

whenver $Q \neq G$. Here $\operatorname{ss.rank}_{\mathbb{Q}}(H)$, for H a reductive group over $\mathbb{Q}$, denotes the $\mathbb{Q}$-rank of the derived group of H (the 'semisimple $\mathbb{Q}$-rank' of H). Note that $\gamma(Q)$ is non-negative and only depends on the conjugacy class of Q. In fact, $\dim A_{L_Q}$, and hence $\gamma(Q)$, is always positive. This follows, for example, from [BS73, §4.2, Equation (2)], upon noting that $\dim A_{L_Q} = \dim A_Q$. More precisely, this shows that $\dim A_{L_Q}$ is equal to the number of $\mathbb{Q}$-simple adjoint factors of $G \to H$ in which the projection of Q is not equal to H.

Theorem 3.13. *Assume that Conjecture 3.5 holds for $M_{Q,h}$ for all maximal parabolics Q of G (including $Q = G$) and $? = c$. Then the natural map*

$$H_c^i(\widehat{X}, \mathbb{F}_p) \to H^i(\widehat{X}, \mathbb{F}_p)$$

is an isomorphism when $i > d + 1 - \inf_{Q \neq G} \gamma(Q)$, and surjective for $i = d + 1 - \inf_{Q \neq G} \gamma(Q)$. In particular, Conjecture 3.5 holds for G and $? = \emptyset$, and $H_c^d(\widehat{X}, \mathbb{F}_p) \to H^d(\widehat{X}, \mathbb{F}_p)$ is an isomorphism.

[6]This is written as $D \cong F \times C(F) \times W(F)$ in [AMRT10]; with respect to our notation $D = X^G$, $F = X^{M_h}$, $C(F) = C(L)$ and $W(F) = N(\mathbb{R})$.

Proof. This merely summarizes the work done above, so we will be rather brief. By Proposition 3.9 and repeated use of the excision sequence, it suffices to show that, for all $Q \neq G$, $H^i_c(\Gamma_Q\backslash Y(Q), \mathrm{Map}_{cts}(S, \mathbb{F}_p)) = 0$ for $i > d - \gamma(Q)$. Propositions 3.10 and 3.11 then give us a spectral sequence

$$H^j(\Gamma_l\backslash \overline{X}^L, R^k g_! \,\mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p) \implies H^{j+k}_c(\Gamma_Q\backslash Y(Q), \mathrm{Map}_{cts}(S, \mathbb{F}_p))$$

and shows that $R^k g_! \,\mathrm{Map}_{cts}(S/S_N, \mathbb{F}_p)$ is a local system which is 0 for $k > \dim_{\mathbb{C}} X^{M_h}$. By [BS73, Corollary 11.4.3] the cohomology of local systems on $\Gamma_l\backslash\overline{X}^L$ vanishes in degrees $> \dim_{\mathbb{R}} X^L - \mathrm{ss.rank}_{\mathbb{Q}}(L)$, so we see that $H^i_c(\Gamma_Q\backslash Y(Q), \mathrm{Map}_{cts}(S, \mathbb{F}_p)) = 0$ for $i > \dim_{\mathbb{C}} X^{M_h} + \dim_{\mathbb{R}} X^L - \mathrm{ss.rank}_{\mathbb{Q}}(L)$. Finally, by Lemma 3.12, this quantity is equal to $d - \gamma(Q)$ as desired, finishing the proof. □

3.4. **The Calegari–Emerton conjectures on completed homology.** We return to the setting of §3.2. We recall from [CE12] that completed homology of G with tame level $K^p \subseteq G(\mathbb{A}^p_f)$ values in an adic ring R is defined as

$$\widetilde{H}_i(K^p, R) := \varprojlim_{K_p} H_i(X_{K^pK_p}, R),$$

where K_p runs through the compact open subgroups of $G(\mathbb{Q}_p)$. One may define completed Borel–Moore homology $\widetilde{H}^{BM}_i(K^p, R)$ similarly (again see [CE12]). Let $? \in \{\emptyset, BM\}$. For any compact open subgroup $K_p \subseteq G(\mathbb{Q}_p)$, $\widetilde{H}^?_i(K^p, \mathbb{Z}_p)$ is a finitely generated right module for the Iwasawa algebra $\mathbb{Z}_p[\![K_p]\!]$, which is an Auslander–Gorenstein ring and has well-defined codimension (or grade) function on its finitely generated right modules, defined by

$$cd(M) = \inf\{j \mid \mathrm{Ext}^j_{\mathbb{Z}_p[\![K_p]\!]}(M, \mathbb{Z}_p[\![K_p]\!]) \neq 0\}.$$

We refer to [AW13, §2.5] for more details on the properties of the codimension function. In particular we remark that by general properties, $cd(\widetilde{H}^?_i(K^p, \mathbb{Z}_p))$ is independent of the choice of K_p. Recall the quantities q_0 and l_0 from §3.2. We may then state a slightly weaker version of [CE12, Conjecture 1.5]. For simplicity, from now on we write $\widetilde{H}^?_i$ for $\widetilde{H}^?_i(K^p, \mathbb{Z}_p)$.

Conjecture 3.14 (Calegari–Emerton)**.** *Let $? \in \{\emptyset, BM\}$. Then the following holds:*

(1) *If $i < q_0$, then $cd(\widetilde{H}^?_i) \geq q_0 + l_0 - i$.*
(2) *$\widetilde{H}^?_{q_0}$ has codimension l_0.*
(3) *$\widetilde{H}^?_{q_0}$ is p-torsionfree.*
(4) *$\widetilde{H}^?_i = 0$ for $i > q_0$.*

The difference between this conjecture and [CE12, Conjecture 1.5] is that the latter predicts $cd(\widetilde{H}^?_i) > q_0 + l_0 - i$ when $i < q_0$. Completed (Borel–Moore) homology is closely related to completed (compacty supported) cohomology via [CE12, Theorem 1.1]. Moreover, completed homology and completed Borel–Moore homology are related via the two Poincaré duality spectral sequences

$$E_2^{ij} = \mathrm{Ext}^i_A(\widetilde{H}_j, A) \implies \widetilde{H}^{BM}_{D-i-j};$$

$$E_2^{ij} = \mathrm{Ext}^i_A(\widetilde{H}^{BM}_j, A) \implies \widetilde{H}_{D-i-j},$$

where $A = \mathbb{Z}_p[\![K_p]\!]$ and $D = \dim_{\mathbb{R}} X = 2q_0 + l_0$; see [CE12, §1.3]. We have the following relation between Conjecture 3.1 and Conjecture 3.14.

Proposition 3.15. *Conjecture 3.1 for compactly supported completed cohomology implies Conjecture 3.14(3)-(4) for completed Borel–Moore homology and Conjecture 3.14(1) for completed homology. Similarly, Conjecture 3.1 for completed cohomology implies Conjecture 3.14(3)-(4) for completed homology and Conjecture 3.14(1) for completed Borel–Moore homology.*

Proof. The first part is essentially [Sch15, Corollary 4.2.3]; the proof there works verbatim (note that there is a small typo in that proof; the quantity c there should be chosen to be minimal, not maximal, with respect to the given property). For the second part the proof is the same, swapping the roles of completed cohomology and compactly supported completed cohomology, and completed homoology and completed Borel–Moore homology. □

Let us also indicate that Conjecture 3.14(2) is known when G is semisimple and $l_0 = 0$; this is part of [CE12, Theorem 1.4] (and follows from [CE09] and known limit multiplicity formulas for discrete series).

Theorem 3.16. *Assume that G is semisimple with no compact $\mathbb{Q}$-factors, that $l_0 = 0$, and let $? \in \{\emptyset, BM\}$. Then the codimension of $\widetilde{H}^?_{q_0}$ is equal to 0.*

4. Shimura varieties

In this section we discuss Shimura varieties of Hodge and (pre-) abelian type, and how the conditional results of §3 together with the results §5 give many unconditional cases of Conjectures 3.1 and 3.14.

4.1. **Recollections on Shimura varieties.** We use the definition and conventions for Shimura data, morphisms of Shimura data, and connected Shimura data from [Del79]; see also [Mil05]. Given a Shimura datum (G, X), there are three other data which one can attach to it, one Shimura datum and two connected Shimura data. They are as follows

- The connected Shimura datum (G^{der}, X^+);
- The connected Shimura datum (G^{ad}, X^+);
- The Shimura datum (G^{ad}, X^{ad}).

Here $X^+ \subseteq X$ is any choice of a connected component, and if $h \in X$, then X^{ad} is the $G^{ad}(\mathbb{R})$-conjugacy class of the composition of h with $G_{\mathbb{R}} \to G^{ad}_{\mathbb{R}}$ (this is independent of the choice of h). The Shimura datum (G^{ad}, X^{ad}) will only feature when we discuss the Hodge–Tate period map later, the other two will feature throughout the rest of this article. We recall that a Shimura datum (G, X) is said to be of Hodge type if there exists a Siegel Shimura datum (G', X') and a closed immersion $(G, X) \to (G', X')$ of Shimura data. A Shimura datum (G, X) is said to be of abelian type if there exists a Shimura datum (G_1, X_1) of Hodge type and a central isogeny $G_1^{der} \to G^{der}$ which induces an isomorphism $(G_1^{ad}, X_1^+) \cong (G^{ad}, X^+)$. We make the following slightly more general definition, following [Moo98, 2.10].

Definition 4.1. *Let (G, X) be a connected Shimura datum. We say that (G, X) is of pre-abelian type if there exists a Shimura datum $(\widetilde{G}, \widetilde{X})$ of Hodge type such that $(G^{ad}, X) \cong (\widetilde{G}^{ad}, \widetilde{X}^+)$. We say that a Shimura datum (G, X) is of pre-abelian type if (G^{der}, X^+) is of pre-abelian type.*

Remark 4.2. *Recall that, if G is semisimple, then by the convential definition G admits a connected Shimura datum (G, X) if and only if G has no compact $\mathbb{Q}$-factors and X^G is a hermitian symmetric domain; in this case $X \cong X^G$. The assumption that G has no compact $\mathbb{Q}$-factors could be dropped, but we will keep phrasing our results in terms of Shimura data for simplicity.*

To be able to apply the inductive arguments from §3, we will need the following lemma.

Lemma 4.3. *Assume that G admits a connected Shimura datum of pre-abelian type and let $Q \subseteq G$ be a maximal parabolic with hermitian part M_h. Then M_h admits a connected Shimura datum of pre-abelian type.*

Proof. The assertion does not depend on the choice of G inside the isogeny class of G, so we may assume that $(G, X) = (G_1^{der}, X_1^+)$ with (G_1, X_1) a Shimura datum of Hodge type. The assertion then follows from the well known fact that the rational boundary components of (G_1, X_1) are of Hodge type. $\square$

4.2. **Results for semisimple groups.** The following is the main theorem of this paper on the Calegari–Emerton conjectures; at this point the proof is simply a summary of the results so far.

Theorem 4.4. *Let G be a semisimple group which admits a connected Shimura datum of pre-abelian type. Then Conjectures 3.1, 3.5 and 3.14 hold for G. Moreover, for any K^p, the natural map $\widetilde{H}^i_c(K^p, \mathbb{Z}_p) \to \widetilde{H}^i(K^p, \mathbb{Z}_p)$ is an isomorphism for $i > d+1-\inf_{Q \neq G} \gamma(Q)$, where $d = \dim_{\mathbb{C}} X^G$, Q is a maximal parabolic subgroup of G and we recall that the quantities $\gamma(G)$ are defined in §3.3.*

Proof. We start with Conjecture 3.5. For $? = c$, this Corollary 5.21. For $? = \emptyset$, it then follows from Lemma 4.3 and Theorem 3.13. The more precise statement about the map $\widetilde{H}^i_c(K^p, \mathbb{Z}_p) \to \widetilde{H}^i(K^p, \mathbb{Z}_p)$ follows from Theorem 3.13, Lemma 2.11 and an analysis of components as in the proof of Proposition 3.6. Conjecture 3.1 then follows, and as does Conjecture 3.14 (using Proposition 3.15, Theorem 3.16 and the fact that $\widetilde{H}_d(K^p, \mathbb{Z}_p) = \widetilde{H}_d^{BM}(K^p, \mathbb{Z}_p)$). $\square$

4.3. **Results for reductive groups.** Here we will briefly indicate what type of results can be proved towards the Calegari–Emerton conjectures for more general reductive groups. Recall that if (G, X) is a Shimura datum, then X^+ need not equal the symmetric space X^G in general. Indeed, $X^+ \cong G(\mathbb{R})/Z(\mathbb{R})K_\infty$, where $Z \subseteq G$ is the center and $K_\infty \subseteq G(\mathbb{R})$ is a maximal compact subgroup. Recall that $A \subseteq Z$ is the maximal $\mathbb{Q}$-split subtorus and set

$$Z^a = \bigcap_{\chi} \operatorname{Ker} \chi,$$

where χ runs over the characters of Z defined over $\mathbb{Q}$. Then $Z = Z^a A$ with $A \cap Z^a$ finite, and $X^G \to X^+$ is a (trivial) fibration with fiber $Z^a(\mathbb{R})/(Z^a(\mathbb{R}) \cap K_\infty)$. In particular, $X^G \cong X^+$ if and only if $Z^a(\mathbb{R})$ is compact. Note that this is equivalent to all arithmetic subgroups of Z being finite, and to $l_0(Z) = 0$. When this happens, we get clean results. Let $d = \dim_{\mathbb{C}} X$.

Theorem 4.5. *Assume that G admits a Shimura datum of pre-abelian type and that $Z^a(\mathbb{R})$ is compact. Then Conjectures 3.1, 3.5 and 3.14 hold for G. Moreover, the natural map $\widetilde{H}^d_c(K^p, \mathbb{Z}_p) \to \widetilde{H}^d(K^p, \mathbb{Z}_p)$ is an isomorphism.*

Proof. We start with Conjecture 3.5. Fix K^p and $n > q_0$. Let $T = G/G^{der}$ be the cocenter of G. Since $Z \to T$ is an isogeny, all arithmetic subgroups of T are finite as well. For sufficiently small K_p, $\Gamma = G(\mathbb{Q})_+ \cap K^p K_p$ is neat and therefore its image in $T(\mathbb{Q})$ is neat, hence trivial. So Γ is contained in $G^{der}(\mathbb{Q})_+$, and one readily sees that Conjecture 3.5 for G is equivalent to Conjecture 3.5 for G^{der}, which follows from Theorem 4.4. Conjecture 3.1 then follows, and as does Conjecture 3.14 (using Proposition 3.15) apart from part (2). For a proof of this we refer to Corollary 4.10 below, though we also note that one could give an easier proof in this special case. The last statement follows from the corresponding statement for G^{der} by the same arguments as in Theorem 4.4. $\square$

Remark 4.6. *We have elected to state the isomorphism $\widetilde{H}^i_c(K^p, \mathbb{Z}_p) \to \widetilde{H}^i(K^p, \mathbb{Z}_p)$ only in degree $i = d$ for simplicity, but of course the proof also shows that we get an isomorphism in (possibly) more degrees as in Theorem 4.4. We will continue to do so throughout this section.*

Corollary 4.7. *Assume that G admits a Shimura datum of Hodge type. Then Conjectures 3.1, 3.5 and 3.14 hold. Moreover, the natural map $\widetilde{H}^d_c(K^p,\mathbb{Z}_p) \to \widetilde{H}^d(K^p,\mathbb{Z}_p)$ is an isomorphism.*

Proof. If G admits a Shimura datum of Hodge type, then $Z^a(\mathbb{R})$ is compact, so Theorem 4.5 applies. □

When $Z^a(\mathbb{R})$ is non-compact, the Leopoldt conjecture interferes in deducing the Calegari–Emerton conjectures for G from G^{der} or G^{ad}. Indeed, if $G = T$ is a torus, then the Leopoldt conjecture for T is equivalent to Conjecture 3.1 for T; see [Hil10, §4.3.3] (note that Hill uses the symmetric spaces $G(\mathbb{R})/K_\infty$ instead of our X^G). We recall this briefly. Let $K = K^pK_p$ be a compact open subgroup of $T(\mathbb{A}_f)$ with K^p arbitrary and K_p neat. Set $\Gamma = T(\mathbb{Q}) \cap K$; this is a finitely generated torsion-free abelian group. Let $\widehat{\Gamma}$ be the p-adic completion of Γ and consider the natural map $f : \widehat{\Gamma} \to K_p$; set $\Delta = \operatorname{Ker} f$ and $I = \operatorname{Im} f$. Δ is a finite free $\mathbb{Z}_p$-module and the Leopoldt conjecture asserts that $\Delta = 0$ (this assertion is independent of the choice of K). An application of [Hil10, Lemma 14] gives that

$$H^i(\Gamma, \mathrm{Map}_{cts}(I,\mathbb{F}_p)) = \mathrm{Hom}_{\mathbb{Z}_p}(\wedge^i_{\mathbb{Z}_p}\Delta, \mathbb{F}_p),$$

and by Lemma 2.11 $H^i(\Gamma, \mathrm{Map}_{cts}(I,\mathbb{F}_p))$ vanishes simultaneously with $H^i(\Gamma, \mathrm{Map}_{cts}(K_p,\mathbb{F}_p))$, so by Proposition 3.6 the vanishing of Δ is equivalent to Conjecture 3.3. In fact, the Leopoldt conjecture is also equivalent to Conjecture 3.14(2) for T (note that $q_0(T) = 0$). This is certainly well known; we give a very brief sketch of the proof.

Proposition 4.8. *Let $K^p \subseteq T(\mathbb{A}^p_f)$ be compact open. Then the codimension of $\widetilde{H}_0(K^p,\mathbb{Z}_p)$ is $l_0 - \mathrm{rank}_{\mathbb{Z}_p}\Delta$. In fact, the projective dimension of $\widetilde{H}_0(K^p,\mathbb{Z}_p)$ is $l_0 - \mathrm{rank}_{\mathbb{Z}_p}\Delta$.*

Proof. We give a very brief sketch. Choose K_p neat and set $\Gamma = T(\mathbb{Q})^+ \cap K^pK_p$. As a right K_p-module, a straightforward computation (using the commutativity of T) shows that

$$\widetilde{H}_0(K^p,\mathbb{Z}_p) \cong \bigoplus_t \mathbb{Z}_p[\![I\backslash K_p]\!]$$

where t runs over the finite set $T(\mathbb{Q})^+\backslash T(\mathbb{A}_f)/K^pK_p$ and I denotes the closure of Γ in K_p. Set $M = \mathbb{Z}_p[\![I\backslash K_p]\!]$, $A = \mathbb{Z}_p[\![I]\!]$ and $B = \mathbb{Z}_p[\![K_p]\!]$; B is a projective (left and right) A-module by [Bru66, Lemma 4.5] and M is a finitely generated right B-module, which is isomorphic to $\mathbb{Z}_p \otimes_A B$. A computation then shows that $\mathrm{Ext}^i_B(M,B) \cong B \otimes_A \mathrm{Ext}^i_A(\mathbb{Z}_p, A)$, so the codimension of M as a right B-module is equal to the codimension of $\mathbb{Z}_p$ as a right A-module. Since $I \cong \mathbb{Z}_p^{l_0 - \mathrm{rank}_{\mathbb{Z}_p}\Delta}$, a computation usng the Koszul complex shows that the codimension of $\mathbb{Z}_p$ is $l_0 - \mathrm{rank}_{\mathbb{Z}_p}\Delta$. This finishes the proof of the first part. For the second part about the projective dimension, note that the Koszul complex of A is a resolution $P_\bullet$ of $\mathbb{Z}_p$ of length $l_0 - \mathrm{rank}_{\mathbb{Z}_p}\Delta$ by finite free A-modules. It follows that $P_\bullet \otimes_A B$ is a resolution of M of length $l_0 - \mathrm{rank}_{\mathbb{Z}_p}\Delta$ by finite free B-modules. Together with the first part, this finishes the proof of the second part. □

We may now give the most general result for reductive groups that we can prove.

Theorem 4.9. *Let G be a connected reductive group over $\mathbb{Q}$ with center Z. Assume that the Leopoldt conjecture holds for Z and that G^{ad} admits a Shimura datum of abelian type. Then Conjecture 3.5 holds for G.*

Proof. We sketch a proof, which requires a minor extension of the results stated in this paper that is easily proved by the same methods. Fix K^p and choose a sufficiently small K_p which is a product $K_p = K_p^Z \times K_p^{ad}$ of a compact open $K_p^Z \subseteq Z(\mathbb{Q}_p)$ and a compact open $K_p^{ad} \subseteq G^{der}(\mathbb{Q}_p)$; note that the image of K_p^{ad} in $G^{ad}(\mathbb{Q}_p)$ is open and isomorphic to K_p^{ad}; we will conflate the two (this explains the notation). Set $\Gamma = G(\mathbb{Q})_+ \cap K^p K_p$. By Proposition 3.6 and Lemma 2.11, it suffices to show that

$$H^n_?(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)) = 0$$

for $n > q_0 = q_0(G) = q_0(G^{ad})$. Let $\Gamma^Z = Z(\mathbb{Q}) \cap \Gamma$ and let Γ^{ad} be the image of Γ in $G^{ad}(\mathbb{Q})^+$. Consider the proper fibration $\pi : \Gamma\backslash X \to \Gamma^{ad}\backslash X^{ad}$ with fiber $\Gamma^Z\backslash X^Z$ (here $X^{ad} = X^{G^{ad}}$) and the corresponding Leray spectral sequence

$$H^r_?(\Gamma^{ad}\backslash X^{ad}, R^s\pi_* \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)) \implies H^{r+s}_?(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)).$$

By Corollary 2.3, $R^s\pi_* \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)$ is the local system corresponding to $H^s(\Gamma^Z, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p))$. Using the discussion on Leopoldt's conjecture above, the assumption that Leopoldt holds for Z, and Lemma 2.11, we see that $H^s(\Gamma^Z, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)) = 0$ for $s > 0$. We then compute

$$\mathrm{Map}_{cts}(K_p, \mathbb{F}_p)^{\Gamma^Z} \cong \mathrm{Map}_{cts}(K_p^Z/\Gamma^Z, \mathbb{F}_p) \otimes \mathrm{Map}_{cts}(K_p^{ad}, \mathbb{F}_p)$$

as Γ^{ad}-modules, where Γ^{ad} acts trivially on the first factor, which is an $\mathbb{F}_p$-vector space that we call V. So, the Leray spectral sequence reduces to

$$H^n_?(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{F}_p)) \cong H^n_?(\Gamma^{ad}\backslash X^{ad}, V \otimes \mathrm{Map}_{cts}(K_p^{ad}, \mathbb{F}_p)) \cong H^n_?(\Gamma^{ad}\backslash X^{ad}, \mathrm{Map}_{cts}(K_p^{ad}, \mathbb{F}_p)) \otimes V.$$

If Γ^{ad} was a congruence subgroup of G^{ad}, then the vanishing of $H^n_?(\Gamma^{ad}\backslash X^{ad}, \mathrm{Map}_{cts}(K_p^{ad}, \mathbb{F}_p))$ for $n > q_0$ would follow from Theorem 4.4 and Lemma 2.11. As it is, we only know that Γ^{ad} is an arithmetic subgroup which is contained in a congruence subgroup. However, the generalization of Conjecture 3.5 for towers of the form $(\Gamma'\backslash X^{ad})_{\Gamma'}$, where Γ' runs through the subgroups of the form $\Gamma \cap K_p^{ad,\prime}$ with $K_p^{ad,\prime} \subseteq G^{ad}(\mathbb{Q}_p)$ compact open, still holds by the methods of this paper. To prove it when $? = c$, we only need to observe that the tower of rigid spaces corresponding to $(\Gamma'\backslash X^{ad,*})_{\Gamma'}$ is perfectoid in the limit by Theorem 5.20 and Lemma 5.10. For $? = \emptyset$, note that the proof of Theorem 3.8 goes through without changes once one assumes that the corresponding generalization for $? = c$ (for all boundary components). This finishes the sketch of proof. ∎

Corollary 4.10. *Keep the notation and assumptions of Theorem 4.9. Then Conjectures 3.1 and 3.14 hold for G. Moreover, the natural map $\widetilde{H}^{q_0}_c(K^p, \mathbb{Z}_p) \to \widetilde{H}^{q_0}(K^p, \mathbb{Z}_p)$ is an isomorphism.*

Proof. We keep the notation from the proof of Theorem 4.9. Note that $l_0 = l_0(G) = l_0(Z)$ and $q_0 = q_0(G) = q_0(G^{ad})$. We will use the notions of [Eme10a, Eme10b] freely in this proof. Fix $K^p \subseteq G(\mathbb{A}_f^p)$. Using Theorem 4.9, everything apart from Conjecture 3.14(2) follows as before, and additionally $\widetilde{H}_{q_0}(K^p, \mathbb{Z}_p) = \widetilde{H}^{BM}_{q_0}(K^p, \mathbb{Z}_p)$. The argument in Proposition 3.15 also shows that $\widetilde{H}_{q_0}(K^p, \mathbb{Z}_p)$ has codimension $\geq l_0$, so we need to show the opposite inequality. We have $\widetilde{H}_{q_0}(K^p, \mathbb{Z}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\widetilde{H}^{q_0}(K^p, \mathbb{Z}_p), \mathbb{Z}_p)$ by [CE12, Theorem 1.1(3)] and the vanishing of $\widetilde{H}^{q_0+1}(K^p, \mathbb{Z}_p)$, so it suffices to prove that $\widetilde{H}^{q_0+1}(K^p, \mathbb{Z}_p)$ has a sub K_p-representation of injective dimension $\leq l_0$. Since $H^{q_0}_?(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{Z}_p))$ is a direct summand of $\widetilde{H}^{q_0}(K^p, \mathbb{Z}_p)$, it suffices to show that $H^{q_0}_?(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{Z}_p))$ has a submodule of injective dimension $\leq l_0$. Here we view $\mathrm{Map}_{cts}(K_p, \mathbb{Z}_p)$ as a left Γ-module via inverting the left translation action; it has a commuting left K_p-action via right translation which gives $H^{q_0}_?(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{Z}_p))$ its structure of a left K_p-module.

Using the computations in the proof of Theorem 4.9 with $\mathbb{F}_p$ replaced by $\mathbb{Z}/p^r$ and taking inverse limits over r (using Proposition 2.7), we see that

$$H^{q_0}_?(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{Z}_p)) \cong H^{q_0}_?(\Gamma^{ad}\backslash X^{ad}, \mathrm{Map}_{cts}(K_p^{ad}, \mathbb{Z}_p)) \widehat{\otimes} \mathrm{Map}_{cts}(K_p^Z, \mathbb{Z}_p)^{\Gamma^Z},$$

as left $K_p = K_p^{ad} \times K_p^Z$-representations. By Proposition 4.8 and the assumption on Z, $\mathrm{Map}_{cts}(K_p^Z, \mathbb{Z}_p)^{\Gamma^Z}$ has injective dimension l_0. By Theorem 3.16, $H_?^{q_0}(\Gamma^{ad}\backslash X^{ad}, \mathrm{Map}_{cts}(K_p^{ad}, \mathbb{Z}_p))$ contains an injective admissible K_p^{ad}-subrepresentation W. It follows that $W \widehat{\otimes} \mathrm{Map}_{cts}(K_p^Z, \mathbb{Z}_p)^{\Gamma^Z}$ is a sub K_p-representation of $H_?^{q_0}(\Gamma\backslash X, \mathrm{Map}_{cts}(K_p, \mathbb{Z}_p))$ of injective dimension $\leq l_0$, as desired. □

Remark 4.11. *We make a few additional remarks on these results.*

1. *Examples of cases when Theorem 4.9 and Corollary 4.10 are unconditional include* $G = \mathrm{Res}_{\mathbb{Q}}^F \mathrm{GSp}_{2g}$ *for abelian totally real fields F, since the Leopoldt conjecture is known for tori which split over an abelian extension of $\mathbb{Q}$. One could also get weaker unconditional results by assuming the known bounds for the Leopoldt defect.*
2. *Conjecture 3.14 has a natural analogue for $\mathbb{F}_p$-coefficients, stated in* [CE12, §1.7]. *Our methods prove this conjecture too under the same assumptions. We content ourselves by noting that the arguments to prove Proposition 3.15 and Corollary 4.10 go through with only superficial changes for $\mathbb{F}_p$-coefficients (though one could simplify the argument in Corollary 4.10 for $\mathbb{F}_p$-coefficients). Note here that Theorem 3.16 implies its $\mathbb{F}_p$-version if one knows p-torsionfreeness of $\widetilde{H}_{q_0}$, using the results of* [CE12, §1.7].

5. Perfectoid Shimura varieties

5.1. **Preparations in p-adic geometry.** In this long preliminary section, we prove a number of loosely related results in p-adic geometry. We continue to fix a prime p. Group actions on spaces will mostly be right actions throughout this section.

Until further notice, "adic space" means "analytic adic space over $\mathbb{Z}_p$". In what follows, we freely use the language of diamonds and some standard notation from [Sch17]. Recall that a diamond is a pro-étale sheaf on the site Perf of characteristic p perfectoid spaces with certain properties. If X is an adic space, the corresponding diamond $X^\diamondsuit$ comes equipped with a natural map $X^\diamondsuit \to \mathrm{Spd}\ \mathbb{Z}_p$; since $\mathrm{Perf}_{/\,\mathrm{Spd}\ \mathbb{Z}_p}$ is naturally equivalent to the category Perfd of all perfectoid spaces, one is free to think of $X^\diamondsuit$ as a functor on Perfd. If X is a diamond with a $\underline{G}$-action for some profinite group G, we write $X/\underline{G}$ for the quotient sheaf computed as a *pro-étale* sheaf.

Lemma 5.1. *Let X be a spatial diamond with a $\underline{G}$-action for some profinite group G. Suppose that G acts with finitely many orbits on $\pi_0 X$, and that each connected component of X is a perfectoid space. Then X is a perfectoid space.*

Proof. Let X_0 be some connected component of X, and let $x \in X_0$ be any point. Choose some open affinoid perfectoid neighborhood $U \subseteq X_0$ of x. This spreads out (e.g. by [Sch17, Proposition 11.23(iii)]) to a small open spatial subdiamond $\tilde{U} \subseteq X$ with $\tilde{U} \cap X_0 = U$. Let $K \subseteq G$ be the open subgroup stabilizing $\tilde{U}$. Then for any $k \in K$, $\tilde{U} \cap X_0 k = \tilde{U}k \cap X_0 k = (\tilde{U} \cap X_0)k = Uk$ is an affinoid perfectoid space. Since our assumptions on the group action guarantee that the orbit $X_0 K$ is an open spatial subdiamond of X, we deduce that $\tilde{U} \cap X_0 K$ is an open spatial subdiamond of X containing x, with the property that each connected component of $\tilde{U} \cap X_0 K$ is affinoid perfectoid. By [Sch17, Lemma 11.27], we deduce that $\tilde{U} \cap X_0 K$ itself is affinoid perfectoid. Since X_0 and x were arbitrary, we get the result. □

We now turn to some general results on group quotients. Let X be an adic space equipped with an action of a finite group G. The coarse quotient X/G always exists in Huber's category $\mathcal{V}$, but in general it may not be an adic space. We need some general results showing that if X is a rigid analytic space or a perfectoid space, then so is X/G. The first author already considered this problem in [Han16], but the results there can be difficult to apply, since they included the assumption that X admits a G-invariant affinoid covering, and such coverings can be hard to exhibit in "real-life" situations. Here we obtain much more satisfying and user-friendly results, which don't assume the a priori existence of G-invariant affinoid covers. In the rigid analytic situation we obtain a very general result, cf. Theorem 5.3 below. In the perfectoid situation, we need slightly stronger hyptheses, cf. Theorem 5.8, but the result is sufficient for our intended applications to Shimura varieties.

Let X be a topological space with an action of a *finite* group G by continuous automorphisms. Let $x \in X$ be any point, with stabilizer $H_x \subseteq G$. We say an open neighborhood U of x is *G-clean* if $Uh = U$ for all $h \in H_x$ and moreover $U \cap Ug = \emptyset$ for all $g \in G \smallsetminus H_x$. Note in particular that if U is a G-clean neighborhood of x, then the natural map

$$U \times^{H_x} G \overset{(u,g)\mapsto ug}{\longrightarrow} X$$

is an open embedding, and its image is just the union inside X of $[G : H_x]$ many disjoint translates of U, so this is an especially pleasant type of G-stable open containing the orbit xG.

Lemma 5.2. *Let X be a Hausdorff topological space with a G-action. Then every point $x \in X$ admits a G-clean open neighborhood.*

Proof. Fix $x \in X$, with stabilizer H. Choose coset representatives $G = \coprod_{1\leq i\leq n} Hg_i$ with $g_1 = 1$; the orbit of x is then $\{x_1, \dots, x_n\}$, with $x_i = xg_i$. Since X is Hausdorff we may choose pairwise disjoint open neighborhoods U_i' of the x_i's. Clearly $g_i^{-1}Hg_i$ is the stabilizer of x_i, so the open set

$$U_i = \bigcap_{k\in g_i^{-1}Hg_i} U_i'k$$

contains x_i and is stable under $g_i^{-1}Hg_i$; moreover the U_i's are pairwise disjoint. Now set $V_i = U_ig_i^{-1}$, so $x \in V_i$ and V_i is H-stable. Finally, set $W = \bigcap_i V_i$; we claim that W is a G-clean open neighborhood of x. Indeed, W is H-stable since the V_i's are, so it remains to check that if $i \neq j$, then $Wg_i \cap Wg_j = \emptyset$. But $Wg_i \subseteq V_ig_i = U_i$ and similarly for Wg_j, so $Wg_i \cap Wg_j \subseteq U_i \cap U_j = \emptyset$, as desired. □

Theorem 5.3. *Let X be a rigid analytic space over some nonarchimedean field K with an action of a finite group G. Assume that X is separated, and that for every rank one point $x \in X$, the closure $\overline{\{x\}} \subseteq X$ is contained in some open affinoid subspace $U = \mathrm{Spa}(A, A^\circ) \subseteq X$. Then the categorical quotient $X/G = (|X|/G, (q_*\mathcal{O}_X)^G, \cdots)$ is a rigid analytic space, and the natural map $X \to X/G$ is finite. Moreover, the canonical map $X^\diamondsuit/\underline{G} \to (X/G)^\diamondsuit$ is an isomorphism.*

The auxiliary conditions on X in this theorem are satisfied e.g. if X is affinoid, or if X is partially proper. In particular, the theorem applies whenever X is the analytification of a separated K-scheme of finite type. We would like to emphasize that these auxiliary conditions do not involve the G-action in any way. In particular, we are *not* assuming a priori that X admits a covering by G-stable affinoid subsets (though, a posteriori, the theorem shows that this is the case).

Proof. Let $x \in |X|$ be any rank one point, with stabilizer H_x and closure $\overline{\{x\}} \subseteq |X|$. Let $|X|^h$ be the maximal Hausdorff quotient of $|X|$, and let $\pi : |X| \to |X|^h$ be the natural map, so if $x \in |X|$ is any rank one point, then $\overline{\{x\}} \subseteq \pi^{-1}(\pi(x))$.[7] By functoriality of the maximal Hausdorff quotient,

[7] One might guess that in fact $\overline{\{x\}} = \pi^{-1}(\pi(x))$, but this is not clear to us. Indeed, let $|X|^\nu$ be the quotient of $|X|$ by the transitive closure of the pre-relation "$x \sim y$ if $U \cap V \neq \emptyset$ for all open neighborhoods $x \in U, y \in V$". Then π naturally factors as a composition of quotient maps $|X| \overset{\tau}{\to} |X|^\nu \overset{q}{\to} |X|^h$. By some standard structure theory of analytic adic spaces, τ induces a bijection from the rank one points of $|X|$ onto $|X|^\nu$, and $\tau^{-1}(\tau(x)) = \overline{\{x\}}$ for any rank one point $x \in |X|$. However, the map q may not be a homeomorphism: for a general topological space T, T^h can be obtained by transfinitely iterating the construction $T \rightsquigarrow T^\nu$. When $|X|$ is *taut*, one can prove that q is a homeomorphism by combining [Hub96, Lemmas 5.3.4 and 8.1.5].

G naturally acts on $|X|^h$ and π is G-equivariant. By Lemma 5.2 we can choose a G-clean open neighborhood $U_x \subseteq |X|^h$ of $\pi(x)$. Set $\tilde{U}_x = \pi^{-1}(U_x) \subseteq |X|$, so $\tilde{U}_x$ is a G-clean open neighborhood of x containing $\overline{\{x\}}$.

By assumption, we can choose an open affinoid subspace $V_x = \mathrm{Spa}(A, A^\circ) \subseteq X$ containing $\overline{\{x\}}$. Since X is separated, the intersection $\cap_{h \in H_x} V_x h$ is still affinoid, so after replacing V_x by $\cap_{h \in H_x} V_x h$, we can assume that V_x is H_x-stable. The intersection $W_x = \tilde{U}_x \cap V_x$ is still a G-clean open neighborhood of x containing $\overline{\{x\}}$. Now, observe that $W_x \times^{H_x} G \subseteq X$ is a G-stable open subspace of X containing $\overline{\{x\}}G$ with the crucial property that

$$W_x/H_x \cong \left(W_x \times^{H_x} G\right)/G \subseteq X/G$$

is naturally a rigid analytic space, because $V_x/H_x \cong \mathrm{Spa}(A^{H_x}, A^{\circ H_x})$ is an affinoid rigid space and $|W_x|/H_x$ is an open subset of $|V_x|/H_x$. Varying over all rank one points $x \in X$, the spaces W_x/H_x give an open covering of X/G by rigid analytic spaces, so X/G is a rigid analytic space, as desired.

For finiteness of the map $X \to X/G$, note that $f : W_x \to W_x/H_x$ is finite, since it's the pullback of the finite map $V_x \to V_x/H_x$ along $W_x/H_x \to V_x/H_x$. It then suffices to observe that the pullback of $X \to X/G$ along the open embedding $W_x/H_x \to X/G$ is given by the map

$$W_x \times^{H_x} G \simeq \coprod_{1 \leq i \leq n} W_x g_i \overset{\coprod f \circ g_i^{-1}}{\longrightarrow} W_x/H_x,$$

which is clearly finite.

For the last point, it suffices to prove that the canonical maps $V_x^\lozenge/\underline{H_x} \to (V_x/H_x)^\lozenge$ are isomorphisms of pro-étale sheaves. We claim that in fact for any Tate $\mathbb{Z}_p$-algebra A with an action of a finite group G and a G-stable subring of integral elements A^+, the canonical map $\mathrm{Spd}(A, A^+)/\underline{G} \to \mathrm{Spd}(A^G, A^{+G})$ is an isomorphism. It suffices to check that $\mathrm{Spd}(A, A^+) \times \underline{G} \rightrightarrows \mathrm{Spd}(A, A^+)$ is a presentation of $\mathrm{Spd}(A^G, A^{+G})$ as a pro-étale sheaf. Arguing as in [CGJ19, Proposition 2.1.1], this reduces to the fact that the maps $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(A^G, A^{+G})$ and $\mathrm{Spd}(A, A^+) \times \underline{G} \to \mathrm{Spd}(A, A^+) \times_{\mathrm{Spd}(A^G, A^{+G})} \mathrm{Spd}(A, A^+)$ are quasi-pro-étale. Since the morphisms in question are separated, this can be checked on rank one geometric points by [Sch17, Proposition 13.6], where it is obvious. □

Unfortunately, the perfectoid variant of the previous theorem is not so clean, primarily because of "problems" with the notion of a "separated" perfectoid space. For example, for perfectoid spaces over a perfectoid field, the notion introduced in [Sch17, Definition 5.10] is too weak for our purposes. The following notion of separation is more than sufficient for our purposes. In what follows, we will frequently use the fact that if X and Y are perfectoid spaces over $\mathrm{Spa}(K, K^+)$ for some affinoid field (K, K^+), then the fiber product $X \times_{\mathrm{Spa}(K,K^+)} Y$ is naturally a perfectoid space. By gluing, this reduces to the claim that this fiber product is naturally affinoid perfectoid if X and Y are each affinoid perfectoid, which is [KL15, Corollary 3.6.18].

Definition 5.4. (1) *A map of perfectoid spaces $Z \to X$ is a* Zariski-closed embedding *if for any open affinoid perfectoid subset $U \subseteq X$, the map $Z \times_X U \to U$ is a Zariski-closed embedding of affinoid perfectoid spaces in the sense of* [Sch15, §2.2]. *We say that an open subset U of a perfectoid space X is* Zariski open *if the inclusion $X \setminus U \to X$ is a Zariski closed embedding.*

(2) *A perfectoid space X over a nonarchimedean field* $\mathrm{Spa}(K, K^+)$ *is* analytically separated *if the diagonal map $X \to X \times_{\mathrm{Spa}(K,K^+)} X$ is a Zariski-closed embedding.*

We caution the reader this definition of being a Zariski-closed embedding is rather delicate: among other things, it's not clear whether this property can be checked locally on a single affinoid cover of X, or whether this property is stable under base change. The key property of analytically separated perfectoid spaces that we will use is part (2) of the following lemma.

Lemma 5.5. (1) *If a perfectoid space X is analytically separated, then it is separated in the sense of* [Sch17], *i.e. $X^{\lozenge} \to \mathrm{Spd}(K, K^+)$ is a separated map of v-sheaves.*
(2) *If X is analytically separated, then for any two open affinoid perfectoid subsets $U, V \subseteq X$, the intersection $U \cap V$ is affinoid perfectoid.*

Proof. Part (1) is straightforward and left to the reader (and we won't need it anyway). Part (2) is immediate upon writing $U \cap V = (U \times_{\mathrm{Spa}(K,K^+)} V) \times_{X \times_{\mathrm{Spa}(K,K^+)} X, \Delta} X$. □

In practice, analytic separation can often be checked via the following lemma.

Lemma 5.6. *Let $(X_i)_{i \in I}$ be a cofiltered inverse system of separated rigid analytic spaces over some $\mathrm{Spa}(K, K^\circ)$, and suppose there is some perfectoid space X_∞ such that $X_\infty = \varprojlim_i X_i$ as diamonds. Suppose moreover that each X_i is an open subset of the analytification of a projective variety over K. Then X_∞ is analytically separated.*

Proof. By assumption, we can choose open immersions $X_i \to V_i^{\mathrm{an}}$ for some projective varieties V_i. Let $U \subseteq X_\infty \times_{\mathrm{Spa}(K,K^\circ)} X_\infty$ be some open affinoid perfectoid subset. Set

$$W_i = U \times_{X_i \times_{\mathrm{Spa}(K,K^\circ)} X_i, \Delta} X_i \cong U \times_{V_i^{\mathrm{an}} \times_{\mathrm{Spa}(K,K^\circ)} V_i^{\mathrm{an}}, \Delta} V_i^{\mathrm{an}}.$$

A priori, we are computing this fiber product as diamonds. However, by the subsequent lemma, W_i is affinoid perfectoid and the resulting map $W_i \to U$ is a Zariski-closed embedding. Then $U \times_{X_\infty \times_{\mathrm{Spa}(K,K^\circ)} X_\infty, \Delta} X_\infty = \varprojlim_i W_i$ is affinoid perfectoid, and $\varprojlim_i W_i \to U$ is a cofiltered limit of Zariski-closed embeddings. Since any cofiltered limit of Zariski-closed embeddings with fixed target is a Zariski-closed embedding, we get the result. □

Lemma 5.7. *Let $Y \to X$ be a closed immersion of quasi-projective varieties over a nonarchimedean field K, and let Z be any perfectoid space equipped with a map $f : Z \to X^{\mathrm{an}}$. Then the diamond $W = Z \times_{X^{\mathrm{an}}} Y^{\mathrm{an}}$ is a perfectoid space, and the natural map $W \to Z$ is a Zariski-closed embedding.*

Proof. Unwinding the definitions, it suffices to prove that if Z is affinoid perfectoid, then $W = Z \times_{X^{\mathrm{an}}} Y^{\mathrm{an}} \to Z$ is a Zariski-closed embedding of affinoid perfectoid spaces.

Replacing X by its closure in some projective space, and replacing Y by its closure in X, we can assume that $Y \to X$ is a closed immersion of projective varieties. Let $\mathcal{I} \subseteq \mathcal{O}_{X^{\mathrm{an}}}$ be the ideal sheaf cutting out Y^{an}. By rigid GAGA and the projectivity of X, we can choose a vector bundle $\mathcal{E}$ on X^{an} together with a surjection $\mathcal{E} \twoheadrightarrow \mathcal{I}$. Then $f^*\mathcal{E}$ is naturally a vector bundle on Z, and the image of the natural map $f^*\mathcal{E} \to \mathcal{O}_Z$ is just the ideal sheaf generated by $f^{-1}\mathcal{I}$. However, Z is affinoid perfectoid, so $f^*\mathcal{E}$ is generated by its global sections, which are just a finitely generated projective $\mathcal{O}_Z(Z)$-module. In paticular, if $e_1, \ldots, e_n \in H^0(Z, f^*\mathcal{E})$ is any set of generators, then their images in $\mathcal{O}_Z(Z)$ generate an ideal I corresponding to the ideal sheaf generated by $f^{-1}\mathcal{I}$. Let $W \subseteq Z$ be the Zariski-closed subset cut out by I. It is then easy to see that W represents the fiber product claimed in the statement of the lemma. □

Theorem 5.8. *Let X be a perfectoid space over a nonarchimedean field, with an action of a finite group G. Assume that X is analytically separated, and that for every rank one point $x \in X$, the closure $\overline{\{x\}} \subseteq X$ is contained in some open affinoid perfectoid subspace $U = \mathrm{Spa}(A, A^+) \subseteq X$.*

Then the categorical quotient X/G is a perfectoid space, and the natural map $q : X \to X/G$ is affinoid in the (weak) sense that any point $y \in X/G$ admits a neighborhood basis of open affinoid *perfectoid subsets $Y \subseteq X$ whose preimages $q^{-1}(Y)$ are affinoid perfectoid. Moreover, the canonical morphism $X^{\diamondsuit}/\underline{G} \to (X/G)^{\diamondsuit}$ is an isomorphism.*

Proof. The first portion of the proof is nearly identical to the proof of Theorem 5.3, but we repeat the details for the reader's convenience.

Let $x \in X$ be any rank one point, with stabilizer H_x and closure $\overline{\{x\}} \subset X$. Let $|X|^h$ be the maximal Hausdorff quotient of $|X|$, and let $\pi : |X| \to |X|^h$ be the natural map, so if $x \in |X|$ is any rank one point, then $\overline{\{x\}} \subseteq \pi^{-1}(\pi(x))$. By functoriality of the maximal Hausdorff quotient, G naturally acts on $|X|^h$ and π is G-equivariant. By Lemma 5.2 we can choose a G-clean open neighborhood $U_x \subseteq |X|^h$ of $\pi(x)$. Let $\tilde{U}_x$ be the preimage of U_x in $|X|$, so $\tilde{U}_x$ is a G-clean open neighborhood of x containing $\overline{\{x\}}$.

By assumption, we can choose an open affinoid perfectoid subspace $V_x = \mathrm{Spa}(A, A^+) \subseteq X$ containing $\overline{\{x\}}$. Since X is analytically separated, the intersection $\cap_{h \in H_x} V_x h$ is affinoid perfectoid by Lemma 5.5.(2), so after replacing V_x by $\cap_{h \in H_x} V_x h$, we can assume that V_x is H_x-stable. The intersection $W_x = \tilde{U}_x \cap V_x$ is still a G-clean open neighborhood of x containing $\overline{\{x\}}$. Now, observe that $W_x \times^{H_x} G \subset X$ is a G-stable open subspace of X containing $\overline{\{x\}}G$ with the crucial property that

$$W_x/H_x \cong \left(W_x \times^{H_x} G\right)/G \subseteq X/G$$

is naturally a perfectoid space, because $V_x/H_x \cong \mathrm{Spa}(A^{H_x}, A^{+H_x})$ is an affinoid perfectoid space by [Han16, Theorem 1.4] and $|W_x|/H_x$ is an open subset of $|V_x|/H_x$. Varying over all rank one points $x \in X$, the spaces W_x/H_x give an open covering of X/G by perfectoid spaces, so X/G is a perfectoid space, as desired.

To see that q is affinoid, let $y \in X/G$ be any point, so y is contained in some W_x/H_x. Let $Y \subseteq W_x/H_x \subseteq X/G$ be any open subset containing y such that Y is a rational subset of V_x/H_x. The set of such Y's is clearly a neighborhood basis of y. Moreover, $q^{-1}(Y)$ is a finite disjoint union of copies of the preimage of Y in V_x, but the latter preimage is a rational subset of V_x, and hence is affinoid perfectoid, so $q^{-1}(Y)$ is affinoid perfectoid. Varying y, we get the claim.

The last point follows exactly as in the proof of Theorem 5.3. □

The next lemma will allow us to extend the Hodge-Tate period map across the boundary of the minimal compactification, in situations where we already know it extends on some finite cover of the Shimura variety.

Lemma 5.9. *Let $\tilde{X}$ be a perfectoid space over an affinoid field (K, K^+) with an action of a finite group G satisfying the hypotheses of the previous theorem. Let $q : \tilde{X} \to X = \tilde{X}/G$ be the natural map. Let $U \subseteq X$ be a dense Zariski-open subset, with preimage $\tilde{U} \subseteq \tilde{X}$. Finally, let Z be a quasiseparated adic space over (K, K^+) such that any finite subset of $|Z|$ is contained in an open affinoid, and let $\tilde{f} : \tilde{X} \to Z$ be any map such that $\tilde{f}|_{\tilde{U}} = f \circ q$ for some $f : U \to Z$. Then f extends uniquely to a map $f : X \to Z$ such that $\tilde{f} = f \circ q$.*

The conditions on Z hold, for example, if Z is the analytification of a quasiprojective variety. This is the only case we will need.

Proof. Let $x \in \tilde{X}$ be any point. Let $W \subseteq Z$ be any open affinoid subset containing $f(xG)$, so $\tilde{f}^{-1}(W) \subseteq \tilde{X}$ is a retrocompact open subset[8] containing xG. Now let $T \subseteq X$ be any open affinoid perfectoid subset with $q(x) \in T$ and with $\tilde{T} = q^{-1}(T)$ affinoid perfectoid. Then $\tilde{Y} =$

[8] If $T \subseteq \tilde{X}$ is a quasicompact open subset, then the quasiseparatedness of Z implies that $T \cap f^{-1}(W) \cong T \times_Z W$ is quasicompact.

$\tilde{T} \cap \bigcap_{g \in G} f^{-1}(W)g$ is a G-stable quasicompact open subset of $\tilde{T}$ containing xG such that $\tilde{f}$ maps $\tilde{Y}$ into W. Shrinking $Y = q(\tilde{Y}) \subset T$ further if necessary, we can assume that Y and $\tilde{Y}$ are affinoid perfectoid. It's enough to show that the map $f|_{U \cap Y} : U \cap Y \to W$ extends uniquely to a map $f : Y \to W$ such that $\tilde{f}|_{\tilde{Y}} = f \circ q$.

Write $W = \mathrm{Spa}(R, R^+)$ and $\tilde{Y} = \mathrm{Spa}(A, A^+)$, so $Y = \mathrm{Spa}(A^G, A^{+G})$. Then $\tilde{f}|_{\tilde{Y}}$ corresponds to a map $\varphi : (R, R^+) \to (A, A^+)$, and we need to show that the latter map factors through a map $(R, R^+) \to (A^G, A^{+G})$. Since $A^{+G} = A^G \cap A^+$, it's enough to show that $\varphi : R \to A$ has image contained in A^G. Let $r \in R$ be any element, with image $\varphi(r) \in A$. Then for any $g \in G$, $|g\varphi(r) - \varphi(r)|_x = 0$ for all $x \in \tilde{U} \cap \tilde{Y}$. Since the vanishing locus of any element $a \in A$ is closed and $\tilde{U} \cap \tilde{Y}$ is dense in $\tilde{Y}$, this shows that $|g\varphi(r) - \varphi(r)|_x = 0$ for all $x \in \tilde{Y}$. Since A is uniform, this implies that $g\varphi(r) = \varphi(r)$, and so $\varphi : R \to A$ factors over $\varphi : R \to A^G$, as desired. □

In the next section, we will often be in a situation where we have a morphism between two inverse systems of Shimura varieties for some closely related Shimura data. In the remainder of this section, we prove some results which will allow us to transfer information from one inverse system to the other.

Lemma 5.10. *Let $(X_i)_{i \in I} \xrightarrow{f_i} (Y_i)_{i \in I}$ be a morphism of cofiltered inverse systems of locally Noetherian adic spaces. Assume moreover that the maps f_i and the transition maps in the inverse systems are all finite maps, and that $Y_\infty = \varprojlim_i Y_i$ is perfectoid.*

Then $X_\infty = \varprojlim_i X_i$ is perfectoid, and the morphism $f_\infty : X_\infty \to Y_\infty$ is quasicompact. Moreover, if $U \subseteq Y_\infty$ is an open affinoid perfectoid subset which arises as the preimage of an open affinoid $U_i \subseteq Y_i$ for some i, then $f_\infty^{-1}(U) \subseteq X_\infty$ is also affinoid perfectoid.

With more effort, one can show that the morphism f_∞ is proper and quasi-pro-étale in the sense of [Sch17]. We will not need this.

Proof. Without loss of generality, we may assume that I contains an initial element 0. Next, observe that

$$\begin{aligned} X_\infty &\cong \varprojlim_j X_\infty \times_{Y_j} Y_\infty \\ &\cong \varprojlim_{i \geq j} X_i \times_{Y_j} Y_\infty \\ &\cong \varprojlim_i X_i \times_{Y_i} Y_\infty \end{aligned}$$

using the cofinality of the diagonal to get the last line. Choose an open affinoid subset $U_0 \subseteq Y_0$ with preimages $U_i \subseteq Y_i$, $W_i \subseteq X_i$, $U_\infty \subseteq Y_\infty$, $W_\infty \subseteq X_\infty$. To prove the first part of the theorem, it suffices to prove that W_∞ is a perfectoid space. This can be checked locally on some covering of U_∞ by open affinoid perfectoid subsets $V = \mathrm{Spa}(R, R^+) \subseteq U_\infty$. By our assumptions, the natural maps $W_i \to U_i$ are finite maps of affinoid adic spaces, so in particular $\mathcal{O}^+(U_i) \to \mathcal{O}^+(W_i)$ is an integral ring map. By general nonsense, the fiber product $X_i \times_{Y_i} V = W_i \times_{U_i} V$ is computed as $\mathrm{Spd}(S, S^+)$, where $S = R \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W_i)$ (topologized in the usual way) and S^+ is the integral closure of $\mathrm{im}(R^+ \otimes_{\mathcal{O}^+(U_i)} \mathcal{O}^+(W_i) \to S)$ in S. In particular, $R^+ \to S^+$ is an integral ring map, so the subsequent lemma implies that $W_i \times_{U_i} V$ is an affinoid perfectoid space. Passing to the limit over i, we deduce that $W_\infty \times_{U_\infty} V$ is an affinoid perfectoid space, and then varying over all choices of $U_0 \subseteq Y_0$ and $V \subseteq U_\infty$ as above, we conclude that X_∞ is a perfectoid space.

Quasicompactness of f_∞ is clear. For the final claim of the theorem, choose some $U_i \subseteq Y_i$ and $U \subseteq Y_\infty$ as in the statement of the claim, and let $U_j \subseteq Y_j$ and $W_j \subseteq X_j$ denote the evident preimages for all $j \geq i$. Arguing as in the first part of the proof, we see that $f_\infty^{-1}(U) = \varprojlim_{j \geq i} W_j \times_{U_j} U$ and that $W_j \times_{U_j} U$ is an affinoid perfectoid space for any $j \geq i$. Passing to the limit over j gives the claim. $\square$

In the course of this proof, we crucially used the following result, which is essentially just a rephrasing of a theorem of Bhatt-Scholze.

Lemma 5.11. *Let $(R, R^+) \to (S, S^+)$ be a map of Tate-Huber pairs such that R is a perfectoid Tate ring and the ring map $R^+ \to S^+$ is integral. Then the diamond $\mathrm{Spd}(S, S^+)$ is an affinoid perfectoid space.*

Proof. Choose a pseudouniformizer $\varpi \in R^+$. Since R^+ is integral perfectoid and $R^+ \to S^+$ is an integral ring map, [BS19, Theorem 1.16(1)] guarantees the existence of an integral perfectoid S^+-algebra S^+_{perfd} such that any map from S^+ to an integral perfectoid ring factors uniquely through the map $S^+ \to S^+_{\mathrm{perfd}}$. Set $T = S^+_{\mathrm{perfd}}[1/\varpi]$, and let $T^+ \subset T$ be the integral closure of S^+_{perfd} in T. Then T is a perfectoid Tate ring, and the natural map $(S, S^+) \to (T, T^+)$ induces a bijection $\mathrm{Hom}((T, T^+), (A, A^+)) \cong \mathrm{Hom}((S, S^+), (A, A^+))$ for any perfectoid Tate-Huber pair (A, A^+). This shows that $\mathrm{Spd}(S, S^+) \cong \mathrm{Spd}(T, T^+)$ is affinoid perfectoid, as desired. $\square$

In applications, we will usually care about inverse systems with the following restrictive properties.

Definition 5.12. *Fix a nonarchimedean field K. A* good tower *is a cofiltered inverse system of locally Noetherian adic spaces $(X_i)_{i \in I}$ over $\mathrm{Spa}\, K$ with the following properties.*

(1) *Each X_i is the analytification of a projective variety over K, and the transition maps are finite.*
(2) *The inverse limit $X = \varprojlim_i X_i$ is a perfectoid space.*
(3) *There exists a pair of coverings of X by open affinoid perfectoid subsets U_j, V_j such that $\overline{U_j} \subseteq V_j$ for all j, and such that for each j, U_j and V_j occur as the preimages of some open affinoids $U_{j,i_j}, V_{j,i_j} \subseteq X_{i_j}$ for some $i_j \in I$.*

The point of this definition is captured in the following proposition.

Proposition 5.13. (1) *Let $(Y_i)_{i \in I}$ be a good tower. If $(X_i)_{i \in I} \xrightarrow{f_i} (Y_i)_{i \in I}$ is any map of cofiltered inverse systems such that the morphisms f_i are finite, then $(X_i)_{i \in I}$ is a good tower.*
(2) *If $(X_i)_{i \in I}$ is a good tower with an action of a finite group G, then the categorical quotient X/G is a perfectoid space and $X/G \cong \varprojlim_i X_i/G$.*

Note that in part (2), we are not claiming that $(X_i/G)_{i \in I}$ is a good tower: it's not clear to us whether condition (3) is preserved.

Proof. For part (1), let $f : X \to Y$ denote the map between the limits of the towers. Note that since $X_i \to Y_i$ is finite, the tower $(X_i)_{i \in I}$ satisfies condition (1) of Definition 5.12 by rigid GAGA. Conditions (2) and (3) then follow from Lemma 5.10. Indeed, (2) is immediate, and (3) follows from the observation that if $U_j \subseteq V_j \subseteq Y$ are open affinoid perfectoid subsets pulled back from some finite-level affinoids $U_{j,i_j}, V_{j,i_j} \subseteq Y_i$, then $f^{-1}(U_j)$ is affinoid perfectoid by Lemma 5.10 and is clearly the preimage of the affinoid $f_{i_j}^{-1}(U_{j,i_j}) \subseteq X_{i_j}$ (and similarly for the V_j's). Finally, the condition on closures follows from the inclusions $\overline{f^{-1}(U_j)} \subseteq f^{-1}(\overline{U_j}) \subseteq f^{-1}(V_j)$.

For part (2), X/G is perfectoid by Theorem 5.8, since by design the limit of a good tower satisfies the conditions of that Theorem. Indeed, the limit of any good tower is analytically separated by

Lemma 5.6. Moreover, if $U_j, V_j \subseteq X$ are as in the definition of a good tower, then any rank one point $x \in X$ is contained in some U_j, in which case $\overline{\{x\}} \subseteq \overline{U_j} \subset V_j$.

It remains to check that the natural map $f : X/G \to \varprojlim_i X_i/G$ is an isomorphism of diamonds. The source and target of this map are spatial diamonds, so the map is automatically qcqs. Thus, by [Sch17, Lemma 11.11], it suffices to prove that f induces a bijection on (C, C^+)-points for every algebraically closed perfectoid field C with an open and bounded valuation subring $C^+ \subseteq C$. In what follows, we will freely use the fact that (C, C^+)-points can be computed "naively": if X is a pro-étale sheaf with a $\underline{G}$-action for some profinite group G and $X/\underline{G}$ denotes the quotient as pro-étale sheaves, then $X(C, C^+)/G \cong (X/\underline{G})(C, C^+)$. This is an easy consequence of the fact that any pro-étale cover of a geometric point (C, C^+) has a section.[9]

For surjectivity, let $(x_i \in X_i(C, C^+)/G)_{i \in I}$ be any inverse system of points. Let $W_i \subseteq X(C, C^+)$ be the preimage of x_i. Since $W_i \cong \varprojlim_j W_{i,j}$ where $W_{i,j} \subseteq X_j(C, C^+)$ is the preimage of x_i, and each $W_{i,j}$ is finite and nonempty (use that $X_j \to X_i$ is finite), W_i naturally has the structure of a (non-empty) profinite set. Then $W = \varprojlim_i W_i$ is an inverse limit of non-empty compact Hausdorff spaces, and thus is non-empty. Any choice of $x \in W \subseteq X(C, C^+)$ maps to the inverse system $(x_i)_{i \in I}$.

For injectivity, let $x, y \in X(C, C^+)$ be two elements with the same image in $\varprojlim_i X_i(C, C^+)/G$. Let $x_i, y_i \in X_i(C, C^+)$ be the images of x and y, and let $G_i \subset G$ be the set $g \in G$ with $gx_i = y_i$. Then G_i is nonempty by assumption, and $G_j \to G_i$ is injective for all $j \geq i$, so $\varprojlim_i G_i$ is nonempty. Choosing any $g \in \varprojlim_i G_i$, we then have $gx = y$, as desired. $\square$

5.2. **Perfectoid Shimura varieties of Hodge type.** We now return to Shimura varieties. Let (G, X) be a Shimura datum of Hodge type, with reflex field E and Hodge cocharacter μ. For any open compact subgroup $K \subseteq G(\mathbb{A}_f)$, we write $Sh_K(G, X)$ for the canonical model of the associated Shimura variety; this is a normal quasi-projective scheme over E. This has a canonical projective minimal compactification $Sh_K^*(G, X)$, which is also normal. Fix a prime $\mathfrak{p}$ of E lying over p, and let $\mathcal{X}_K$, resp. $\mathcal{X}_K^*$ denote the rigid analytic space over $E_\mathfrak{p}$ associated with $Sh_K(G, X) \otimes_E E_\mathfrak{p}$, resp. $Sh_K^*(G, X) \otimes_E E_\mathfrak{p}$. As K varies, these spaces form a pair of inverse systems with finite transition maps, and compatible open immersions $\mathcal{X}_K \to \mathcal{X}_K^*$. Recall the (rigid analytic) flag variety $\mathscr{F}\ell_{G,\mu}$ attached to (G, X), as defined over $E_\mathfrak{p}$ in [CS17, §2.1].

Proposition 5.14. *Fix any open compact subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. Then $\mathcal{X}_{K^p}^* = \varprojlim_{K_p} \mathcal{X}_{K^p K_p}^*$ is a perfectoid space, and there is a $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map $\pi_{HT} : \mathcal{X}_{K^p}^* \to \mathscr{F}\ell_{G,\mu}$ with all expected properties.*

Moreover, $\mathcal{X}_{K^p}^*$ is analytically separated, and we can find a pair of coverings by finitely many open affinoid perfectoid subsets $U_i, V_i \subseteq \mathcal{X}_{K^p}^*$ such that $\overline{U_i} \subseteq V_i$ for all i and such that U_i and V_i arise as the preimages of some open affinoid subsets of some $\mathcal{X}_{K^p K_p}^*$.

In particular, for any cofinal system of open compact subgroups $K_p \subseteq G(\mathbb{Q}_p)$, $(\mathcal{X}_{K^p K_p}^*)_{K_p}$ is a good tower (over $E_\mathfrak{p}$) in the sense of Definition 5.12.

Note that $\mathcal{X}_{K^p}^*$ may not coincide with the "ad hoc" compactification $\mathcal{X}_{K^p}^{\underline{*}}$ constructed in [Sch15], although by construction there is certainly a map $\mathcal{X}_{K^p}^* \to \mathcal{X}_{K^p}^{\underline{*}}$.

Proof. Fix a closed embedding $\iota : (G, X) \to (\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm})$ into a Siegel Shimura datum. For any open compact subgroup $K \subseteq \mathrm{GSp}_{2g}(\mathbb{Q}_p)$, let $\mathcal{S}_K$, resp. $\mathcal{S}_K^*$ denote the rigid analytic space over $E_\mathfrak{p}$

[9]More generally, if $\mathcal{F}$ is a presheaf of sets on a site $\mathcal{C}$, and $X \in \mathcal{C}$ is any object with the property that every covering of X admits a section, then the natural map $\mathcal{F}(X) \to \mathcal{F}^{sh}(X)$ is a bijection, where $(-)^{sh}$ denotes sheafification. This is easy and left to the reader.

associated with $Sh_K(\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm}) \otimes_{\mathbb{Q}} E_{\mathfrak{p}}$, resp. $Sh_K^*(\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm}) \otimes_{\mathbb{Q}} E_{\mathfrak{p}}$. By [Sch15, Theorem 3.3.18], $\varprojlim_{K_p} \mathcal{S}^*_{K^pK_p}$ is a perfectoid space for any open compact subgroup $K^p \subseteq \mathrm{GSp}_{2g}(\mathbb{Q}_p)$ contained in some conjugate of a principal congruence subgroup of level ≥ 3. However, this last condition can easily be removed using [Han16, Theorem 1.4], noting in particular that $\mathcal{S}^*_{K^p}$ is covered by finitely many $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$-translates of a certain open affinoid perfectoid subset $\mathcal{S}^*_{K^p}(\epsilon)_a$,[10] and that these subsets are invariant under the action of K'^p/K^p for any normal inclusion $K^p \subseteq K'^p$ of tame level groups.

The chosen embedding ι gives rise to compatible finite maps $\mathcal{X}_{K \cap G(\mathbb{A}_f)} \to \mathcal{S}_K$ for any $K \subseteq \mathrm{GSp}_{2g}(\mathbb{A}_f)$ as above, which naturally extend to compatible finite morphisms $\mathcal{X}^*_{K \cap G(\mathbb{A}_f)} \to \mathcal{S}^*_K$. Now, choose any $K^p \subseteq G(\mathbb{A}_f^p)$ as in the proposition, and choose an open compact $K'^p \subseteq \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ such that $K^p \subseteq K'^p$. Choosing a cofinal set of (neat) open compact subgroups $K_0 \supseteq K_1 \supseteq K_2 \cdots$ in $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$, we get a map of inverse systems $(\mathcal{X}^*_{K^p\iota^{-1}(K_n)})_{n \geq 0} \to (\mathcal{S}^*_{K'^pK_n})_{n \geq 0}$ satisfying all the hypotheses of Lemma 5.10. Applying that lemma, we deduce that $\mathcal{X}^*_{K^p}$ is a perfectoid space and the natural map $f : \mathcal{X}^*_{K^p} \to \mathcal{S}^*_{K'^p}$ is quasicompact. Moreover, $\mathcal{X}^*_{K^p}$ is analytically separated by Lemma 5.6.

Now choose some $0 < \epsilon < \epsilon' < 1/2$ and some finitely many $g_i \in \mathrm{GSp}_{2g}(\mathbb{Q}_p)$ such that the translates $\mathcal{S}^*_{K'^p}(\epsilon)_a \cdot g_i$ cover $\mathcal{S}^*_{K'^p}$. Note that any such translate is the preimage of an open affinoid subset of some $\mathcal{S}^*_{K'^pK_n}$, so again by Lemma 5.10 we see that the preimages

$$U_i = f^{-1}(\mathcal{S}^*_{K'^p}(\epsilon)_a \cdot g_i) \subseteq V_i = f^{-1}(\mathcal{S}^*_{K'^p}(\epsilon')_a \cdot g_i)$$

are affinoid perfectoid and give open covers of $\mathcal{X}^*_{K^p}$, and arise by pullback from some finite level. Moreover, $\overline{\mathcal{S}^*_{K'^p}(\epsilon)_a \cdot g_i} \subset \mathcal{S}^*_{K'^p}(\epsilon')_a \cdot g_i$ for any $\epsilon < \epsilon' < 1/2$, and clearly $\overline{U_i} \subseteq f^{-1}(\overline{\mathcal{S}^*_{K'^p}(\epsilon)_a \cdot g_i})$, so we conclude that $\overline{U_i} \subseteq V_i$ as desired.

The Hodge-Tate period map is the composition of the natural map $\mathcal{X}^*_{K^p} \to \mathcal{X}^{\underline{*}}_{K^p}$ with the (previously known) Hodge-Tate period map $\mathcal{X}^{\underline{*}}_{K^p} \to \mathscr{F}\ell_{G,\mu}$, cf. [CGH+18, Theorem 3.3.1] for a discussion of the latter (the argument there also works to construct $\pi_{HT} : \mathcal{X}^*_{K^p} \to \mathscr{F}\ell_{G,\mu}$ without the use of ad hoc compactifications). □

For later use, we also record an extremely mild generalization of this result.

Corollary 5.15. *For any open compact subgroup $K \subseteq G(\mathbb{A}_f)$ and any cofinal system of open compact subgroups $K_p \subseteq G(\mathbb{Q}_p)$, $(\mathcal{X}^*_{K \cap K_p})_{K_p}$ is a good tower (over $E_{\mathfrak{p}}$) in the sense of Definition 5.12.*

Here and in what follows, we adopt the following notational convention: if G is an algebraic group over $\mathbb{Q}$, H is a subgroup of $G(\mathbb{A}_f)$, and K_p is a subgroup of $G(\mathbb{Q}_p)$, then $H \cap K_p$ denotes the group of elements $h \in H$ whose image in $G(\mathbb{Q}_p)$ lies in K_p. In other words, $H \cap K_p$ is short for $H \cap (G(\mathbb{A}_f^p)K_p)$. We hope this doesn't cause any confusion.

Proof. Let $K^p \subseteq G(\mathbb{A}_f^p)$ denote the image of K along the natural projection. Then $K \cap K_p$ has finite index in K^pK_p, so we get natural finite morphisms $\mathcal{X}^*_{K \cap K_p} \to \mathcal{X}^*_{K^pK_p}$ which compile into a map of towers $(\mathcal{X}^*_{K \cap K_p})_{K_p} \to (\mathcal{X}^*_{K^pK_p})_{K_p}$. Since the target is a good tower by the previous proposition, we may apply Proposition 5.13(i) to conclude. □

[10] This subset is denoted $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ in [Sch15, §3].

5.3. **Perfectoid Shimura varieties of pre-abelian type.** In this section we change notation slightly. Given a Shimura datum (G, X) and an open compact subgroup $K \subseteq G(\mathbb{A}_f)$, we write $Sh_K(G, X)$ for the associated Shimura variety regarded as a quasi-projective variety *over* $\mathbb{C}$, and $Sh^*_K(G, X)$ for its projective minimal compactification. For a (usually implicit) choice of connected component $X^+ \subseteq X$, we write $Sh_K(G, X)^0$ for the connected component of $Sh_K(G, X)$ whose analytification is the image of the natural map

$$X^+ \times \{e\} \to G(\mathbb{Q})_+ \backslash (X^+ \times G(\mathbb{A}_f))/K \cong Sh_K(G, X)^{\mathrm{an}},$$

and we write $Sh^*_K(G, X)^0$ for the Zariski closure of $Sh_K(G, X)^0$ in $Sh^*_K(G, X)$. Note that since $Sh^*_K(G, X)$ is normal, the map $\pi_0 Sh_K(G, X) \to \pi_0 Sh^*_K(G, X)$ is a homeomorphism.

Now, fix once and for all an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ (for simplicity), and let $C/\mathbb{C}_p$ be a complete algebraically closed extension of nonarchimedean fields. All of the following results hold for any choice of C. We write $\mathcal{X}^*_K(G, X)$ for the rigid analytic space associated with $Sh^*_K(G, X) \otimes_{\mathbb{C}} C$. Similarly, we get rigid analytic spaces $\mathcal{X}_K(G, X)$, $\mathcal{X}_K(G, X)^0$, $\mathcal{X}^*_K(G, X)^0$ with the obvious meanings.

For any fixed open compact subgroup $K^p \subseteq G(\mathbb{A}^p_f)$, define

$$\mathcal{X}^*_{K^p}(G, X) = \varprojlim_{K_p \subseteq G(\mathbb{Q}_p) \text{ open compact}} \mathcal{X}^*_{K^p K_p}(G, X)$$

where the inverse limit is taken in the category of diamonds over $\operatorname{Spd} C$. We also write $\mathcal{X}_{K^p}(G, X)$, $\mathcal{X}^*_{K^p}(G, X)^0$, and $\mathcal{X}_{K^p}(G, X)^0$ for the obvious variants.

Proposition 5.16. *Maintain the above notation. The following conditions on a Shimura datum (G, X) are equivalent.*

(1) *The diamond $\mathcal{X}^*_{K^p}(G, X)$ is a perfectoid space for any choice of K^p.*
(2) *The diamond $\mathcal{X}^*_{K^p}(G, X)^0$ is a perfectoid space for any choice of K^p.*

We say the Shimura datum (G, X) satisfies Property $\mathcal{P}$ *if either of these equivalent conditions holds.*

Proof. (1) implies (2): In general, $\mathcal{X}^*_{K^p}(G, X)^0$ is an inverse limit of open-closed subfunctors $\mathcal{X}_i \subseteq \mathcal{X}^*_{K^p}(G, X)$. Therefore, if $\mathcal{X}^*_{K^p}(G, X)$ is perfectoid and $U \subseteq \mathcal{X}^*_{K^p}(G, X)$ is any open affinoid perfectoid subset, then $U \cap \mathcal{X}^*_{K^p}(G, X)^0 = \varprojlim_i U \cap \mathcal{X}_i$ and each $U \cap \mathcal{X}_i$ is affinoid perfectoid, so $U \cap \mathcal{X}^*_{K^p}(G, X)^0$ is affinoid perfectoid. Varying U then gives the result.

(2) implies (1): Choose any open compact subgroup $K_p \subset G(\mathbb{Q}_p)$, so the diamond $\mathcal{X}^*_{K^p}(G, X)$ has a natural K_p-action. Then K_p acts with finitely many open orbits on the profinite set $\pi_0 \mathcal{X}^*_{K^p}(G, X) \cong \overline{G(\mathbb{Q})_+} \backslash G(\mathbb{A}_f)/K^p$ (by [Bor63, Theorem 5.1]). Moreover, each connected component of $\mathcal{X}^*_{K^p}(G, X)$ is isomorphic to $\mathcal{X}^*_{gK^p g^{-1}}(G, X)^0$ for some $g \in G(\mathbb{A}^p_f)$, and in particular is perfectoid. By Lemma 5.1, we deduce that $\mathcal{X}^*_{K^p}(G, X)$ is a perfectoid space, as desired. □

We also need to work with connected Shimura varieties. Let (G, X^+) be a connected Shimura datum and let $\pi : G(\mathbb{Q})^+ \to G^{ad}(\mathbb{Q})^+$ denote the natural map. Let us say that an arithmetic subgroup $\Gamma \subseteq G^{ad}(\mathbb{Q})^+$ is a *pre-congruence subgroup* (for G) if it satisfies the equivalent conditions of [Mil05, Proposition 4.12]. In particular, Γ is pre-congruence iff $\pi^{-1}(\Gamma)$ is a congruence subgroup of $G(\mathbb{Q})^+$, and if G is an adjoint group then Γ is pre-congruence iff it is congruence.

If $\Gamma \subset G^{ad}(\mathbb{Q})^+$ is an arithmetic subgroup, then the quotient $\Gamma \backslash X^+$ is the analytification of a connected normal quasiprojective complex variety, defined uniquely up to unique isomorphism, which we denote by $Sh_\Gamma(G, X^+)$. Again, this has a canonical minimal compactification $Sh^*_\Gamma(G, X^+)$, which is a connected normal projective variety. If Γ is torsion-free, then $Sh_\Gamma(G, X^+)$ is smooth. Again, we denote the associated rigid analytic spaces over C by $\mathcal{X}^*_\Gamma(G, X^+)$, etc.

Definition 5.17. *We say a connected Shimura datum (G, X^+) satisfies Property $\mathcal{P}$ if for every pre-congruence subgroup $\Gamma \subseteq G^{ad}(\mathbb{Q})^+$, the diamond*

$$\mathcal{X}^*_{\Gamma,\infty}(G, X^+) := \varprojlim_{K_p \subset G^{ad}(\mathbb{Q}_p) \text{ open compact}} \mathcal{X}^*_{\Gamma\cap K_p}(G, X^+)$$

is a perfectoid space.

In this statement, recall our notational convention that $\Gamma \cap K_p$ is shorthand for $\Gamma \cap (G(\mathbb{A}^p_f)K_p)$ (cf. the discussion following Corollary 5.15). Note that in this case we could also view Γ as a subgroup of $G^{ad}(\mathbb{Q}_p)$ via the natural embedding $G^{ad}(\mathbb{Q})^+ \subseteq G^{ad}(\mathbb{Q}_p)$, and then $\Gamma \cap K_p$ is literally the intersection of these two subgroups inside $G^{ad}(\mathbb{Q}_p)$. The reader should also note that $G(\mathbb{Q}_p)$ and $G^{ad}(\mathbb{Q}_p)$ are locally isomorphic in a neighborhood of the identity. We are also implicitly using the easy fact that for any open compact subgroup $K_p \subseteq G^{ad}(\mathbb{Q}_p)$ and any pre-congruence subgroup Γ, $\Gamma \cap K_p$ is still a pre-congruence subgroup.

Proposition 5.18. *Let (G, X) be a Shimura datum or a connected Shimura datum. Suppose that (G^{ad}, X^+) satisfies Property $\mathcal{P}$. Then (G, X) satisfies Property $\mathcal{P}$.*

Proof. We do the case a Shimura variety (G, X) first. Let $\pi : G \to G^{ad}$ denote the natural map. It's enough to show that $\mathcal{X}^*_{K^p}(G, X)^0$ is perfectoid for any $K^p \subseteq G(\mathbb{A}^p_f)$. Let $\Gamma = G^{ad}(\mathbb{Q})^+ \cap K$ be a choice of congruence subgroup for some open compact subgroup $K \subseteq G^{ad}(\mathbb{A}_f)$ with the property that $\pi(K^p) \subseteq K \cap G^{ad}(\mathbb{A}^p_f)$. Then for any open compact subgroup $K_p \subseteq G(\mathbb{Q}_p)$, there is a natural finite morphism $\mathcal{X}^*_{K^pK_p}(G, X)^0 \to \mathcal{X}^*_{\Gamma\cap\pi(K_p)}(G^{ad}, X^+)$. Moreover, these morphisms are compatible as K_p varies, and the transition maps in the two towers are finite. Passing to the inverse limit over K_p, the result now follows from Lemma 5.10.

Now assume that (G, X) is a connected Shimura datum (then $X = X^+$)and $\Gamma \subseteq G^{ad}(\mathbb{Q})^+$ is a pre-congruence subgroup for G. Choose a congruence subgroup $\Gamma^{ad} \subseteq G^{ad}(\mathbb{Q})^+$ containing Γ. Then we have finite maps $\mathcal{X}^*_{\Gamma\cap K_p}(G, X) \to \mathcal{X}^*_{\Gamma^{ad}\cap K_p}(G^{ad}, X)$ for all $K_p \subseteq G^{ad}(\mathbb{Q}_p)$ compact open compatible with the transition maps. Lemma 5.10 then applies to give the result. $\square$

We now come to the key result in this subsection.

Proposition 5.19. *Let (G, X) be a Shimura datum of Hodge type. Then the connected Shimura datum (G^{ad}, X^+) satisfies Property $\mathcal{P}$.*

Proof. Fix any congruence subgroup $\Gamma \subseteq G^{ad}(\mathbb{Q})^+$. We need to prove that

$$\varprojlim_{K_p \subset G^{ad}(\mathbb{Q}_p)} \mathcal{X}^*_{\Gamma\cap K_p}(G^{ad}, X^+)$$

is a perfectoid space.

Let $\pi : G \to G^{ad}$ denote the natural map. Choose a congruence subgroup $\Gamma' = K \cap G(\mathbb{Q})_+ \subseteq G(\mathbb{Q})_+$ with $\pi(\Gamma') \subseteq \Gamma$, and set $\Gamma'' = \Gamma' \cap G^{der}(\mathbb{Q})$, so Γ'' is also a congruence subgroup. Choose a cofinal descending family of open compact subgroups

$$K_{p,0} \supseteq K_{p,1} \supseteq \cdots \supseteq K_{p,n} \supseteq \cdots$$

in $G(\mathbb{Q}_p)$, and write $K^{der}_{p,n} = K_{p,n} \cap G^{der}(\mathbb{Q}_p)$. Without loss of generality, we can assume that $K_{p,0} \cap G^{der}(\mathbb{Q}_p) \cap Z_G(\mathbb{Q}_p) = \{1\}$ and that $\Gamma' \subseteq K_{p,0}$, so then $\Gamma'' \subseteq K^{der}_{p,0}$ and $\Gamma'' \cap Z_G(\mathbb{Q}_p) = \{1\}$, and the map π induces isomorphisms $\pi(\Gamma'' \cap K_{p,n}) = \pi(\Gamma'' \cap K_{p,n}^{der}) = \pi(\Gamma'') \cap \pi(K_{p,n}^{der})$. Moreover, the inclusion $\Gamma'' \subseteq \Gamma'$ induces a natural map of towers

$$(\mathcal{X}^*_{\pi(\Gamma'' \cap K_{p,n})}(G^{ad}, X^+))_{n \geq 0} \to (\mathcal{X}^*_{K \cap K_{p,n}}(G, X))_{n \geq 0}$$

where the map at every level n is finite. By Corollary 5.15, the target of this map is a good tower.

Now define $\Gamma''' = \cap_{\gamma \in \Gamma/\pi(\Gamma'')} \gamma \pi(\Gamma'') \gamma^{-1}$. By design, Γ''' is an arithmetic subgroup of $G^{ad}(\mathbb{Q})^+$, and is a normal subgroup of Γ with finite index. Since $\Gamma''' \cap \pi(K_{p,n}^{der})$ is of finite index in $\pi(\Gamma'') \cap \pi(K_{p,n}^{der}) = \pi(\Gamma'' \cap K_{p,n})$, we get another natural map of towers

$$(\mathcal{X}^*_{\Gamma''' \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+))_{n \geq 0} \to (\mathcal{X}^*_{\pi(\Gamma'' \cap K_{p,n})}(G^{ad}, X^+))_{n \geq 0}$$

where the map at every level n is finite. For any $n \geq 0$, $\Gamma''' \cap \pi(K_{p,n}^{der})$ is a normal finite-index subgroup of $\Gamma \cap \pi(K_{p,n}^{der})$. Set $\Delta_n = (\Gamma''' \cap \pi(K_{p,n}^{der})) \backslash (\Gamma \cap \pi(K_{p,n}^{der}))$, so Δ_n is a finite group and the natural maps $\Delta_{n+1} \to \Delta_n$ are injective. Write $\Delta = \varprojlim_n \Delta_n$, so $\Delta = \Delta_n$ for all sufficiently large n. Then Δ operates naturally on the tower $(\mathcal{X}^*_{\pi(\Gamma''') \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+))_{n \geq 0}$, and $\mathcal{X}^*_{\Gamma''' \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+)/\Delta \cong \mathcal{X}^*_{\Gamma \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+)$ for all sufficiently large n.

Summarizing the situation so far, we have a diagram of towers

$$\begin{tikzcd}
(\mathcal{X}^*_{\pi(\Gamma''') \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+))_{n \geq 0} \arrow[r] \arrow[d] & (\mathcal{X}^*_{\pi(\Gamma'' \cap K_{p,n})}(G^{ad}, X^+))_{n \geq 0} \arrow[r] & (\mathcal{X}^*_{K \cap K_{p,n}}(G, X))_{n \geq 0} \\
(\mathcal{X}^*_{\Gamma \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+))_{n \geq 0} & &
\end{tikzcd}$$

where all the morphisms at any given level n are finite. We've already observed that the upper-right tower is a good tower, so by two applications of Proposition 5.13(i), we deduce that the upper-left tower is a good tower. Since Δ operates naturally on the upper-left tower and $\mathcal{X}^*_{\Gamma''' \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+)/\Delta \cong \mathcal{X}^*_{\Gamma \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+)$ for all sufficiently large n, we may apply Proposition 5.13(ii) to deduce that $\mathcal{X}^*_{\Gamma''',\infty}(G^{ad}, X^+)/\Delta$ is a perfectoid space and that $\mathcal{X}^*_{\Gamma''',\infty}(G^{ad}, X^+)/\Delta \cong \varprojlim_n \mathcal{X}^*_{\Gamma''' \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+)/\Delta$. But $\varprojlim_n \mathcal{X}^*_{\Gamma''' \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+)/\Delta \cong \varprojlim_n \mathcal{X}^*_{\Gamma \cap \pi(K_{p,n}^{der})}(G^{ad}, X^+) = \mathcal{X}^*_{\Gamma,\infty}(G^{ad}, X^+)$, so we conclude that $\mathcal{X}^*_{\Gamma,\infty}(G^{ad}, X^+)$ is a perfectoid space, as desired. □

We may now summarize our results in this section the following theorem.

Theorem 5.20. *Let (G, X) be a Shimura datum (resp. a connected Shimura datum) of pre-abelian type. Then, for any compact open subgroup $K^p \subseteq G(\mathbb{A}_f)$ (resp. pre-congruence subgroup $\Gamma \subseteq G^{ad}(\mathbb{Q})^+$), the diamond $\mathcal{X}^*_{K^p}(G, X)$ (resp. $\mathcal{X}^*_{\Gamma,\infty}(G, X)$) is a perfectoid space.*

Proof. Choose a Shimura datum (G_1, X_1) of Hodge type with a central isogeny $G_1^{der} \to G^{ad}$ inducing an isomorphism $(G_1^{ad}, X_1^+) \cong (G^{ad}, X^+)$. By Proposition 5.19, (G^{ad}, X^+) satisfies property $\mathcal{P}$, and then Proposition 5.18 implies that (G, X) satisfies property $\mathcal{P}$, as desired. □

This has the following consequence for compactly supported completed cohomology, which may be viewed as a generalization of [Sch15, Corollary 4.2.2].

Corollary 5.21. *Let (G, X) be a connected Shimura variety of pre-abelian type. Then Conjecture 3.5 for $? = c$ holds for G.*

Proof. Note that the towers used to formulate Conjecture 3.5 are easily seen to correspond to the towers used in this section. Once we know that the towers of minimal compactifications are perfectoid in the limit (by Theorem 5.20), the argument in the proof of [Sch15, Corollary 4.2.2] goes through verbatim. □

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