# Local Fields Ex Sheet 1, solutions

1. $xy = 0,\ y \neq 0 \implies$

$\implies \underset{|x||y|}{\underset{\shortparallel}{|xy|}} = 0$ & $|y| \neq 0 \implies$

$\implies |x| = 0 \implies x = 0.$

$\therefore$ R integral domain.

$K = \mathrm{Frac}(R),\quad \alpha = x/y \quad x, y \in R,\ y \neq 0.$

Define $\quad |\alpha| = \dfrac{|x|}{|y|}$

Welldefined: Multiplicativity.

$|\alpha| = 0 \iff x = 0 \iff \alpha = 0.$

Multiplicativity: By multiplicativity on R.

Triangle inequality:

$$\left|\frac{x}{y} + \frac{z}{w}\right| = \left|\frac{xw + yz}{yw}\right| = \frac{|xw + yz|}{|y||w|} \leq$$

$$\leq \frac{|x||w| + |y||z|}{|y||w|} = \frac{|x|}{|y|} + \frac{|z|}{|w|} = \left|\frac{x}{y}\right| + \left|\frac{z}{w}\right|.$$

Note also that the strong triangle inequality holds on $K$ if it holds on $R$ (same calculation).

2. Part (i) is clear.

Part (ii) $x_n \to 0$, $y_n \to 0$ $\Rightarrow$

$\Rightarrow x_n + y_n \to 0$

$z_n$ Cauchy $\Rightarrow z_n$ bounded, so

$x_n z_n \to 0$.

$\therefore$ $I$ ideal.

$I$ is prime: $x_n \not\to 0$, $y_n \not\to 0$.

$x_n \not\to 0 \Rightarrow \exists \varepsilon > 0$ & $N$ s.t.

$|x_n| \geq \varepsilon \ \forall n > 0$.

Here we use that $(x_n)$ is Cauchy, otherwise we only have it for a subsequence.

Same for $y_n$; wlog same $\varepsilon$ & $N$.

Then $|x_n y_n| \geq \varepsilon^2 > 0 \ \forall n \geq N$ so $x_n y_n \notin I$.

$\therefore$ I prime.

$j$ injective: $j: R \to C/I$

is a homomorphism & $(x)_n \in I$ iff

$x \to 0 \iff x = 0$.

$\therefore$ $j$ injective.

<u>Part (iii)</u>: $|\cdot| : R \to \mathbb{R}_{\geq 0}$ satisfies

$\big| |x| - |y| \big| \leq |x-y|$, so is uniformly cts.

$\therefore$ sends Cauchy sequences to Cauchy sequences

$\Rightarrow$ (by completeness of $\mathbb{R}_{\geq 0}$) that

$\lim_{n\to\infty} |x_n|$ exists.

If $(z_n) \in I$, then $\forall \varepsilon > 0$ $\exists N$ s.t

$|z_n| \leq \varepsilon$ $\forall n \geq N$. It follows that

$|x_n + z_n| \leq |x_n| + \varepsilon$ $\forall n \geq N$ $\Rightarrow$

taking limit

$\Rightarrow$ $|(x_n) + (z_n)|' \leq |(x_n)|' + \varepsilon$.

$\varepsilon$ arbitrary $\Rightarrow$ $|(x_n) + (z_n)| \leq |(x_n)|'$.

Symmetry $\Rightarrow$ equality.

$\therefore\ |\cdot|' : C/I \longrightarrow \mathbb{R}_{\geq 0}$.

<u>Part (iv)</u>: $|\cdot|'$ absolute value gets inherited from $|\cdot|$,

and $|j(x)|' = |x|$ is clear.

<u>$j(R)$ dense in $\hat{R}$</u>: Let $(x_n) \in \hat{R}$.

$\forall \varepsilon > 0\ \exists N : |x_m - x_n| \leq \varepsilon\ \ \forall m, n \geq N$.

In particular, $|x_n - x_N| \leq \varepsilon\ \ \forall n \geq N$, so

$$\left|(x_N)_n - (x_n)_n\right|' = \lim_{n \to \infty} |x_n - x_N| \leq \varepsilon .$$

$\therefore\ j(R)$ dense in $\hat{R}$.

<u>$\hat{R}$ complete</u>. To reduce the notational nightmare, we use the following abstract lemma:

<u>Lemma</u>: Let $X$ be a metric space & let $A \subseteq X$ be a dense subset. Assume that every Cauchy sequences in $A$ has a limit in $X$. Then $X$ is complete.

<u>Pf</u>: $(x_n)$ Cauchy in $X$.

Pick $a_n \in A$ s.t. $d(a_n, x_n) \leq \frac{1}{2^n}$.

$(a_n)$ is Cauchy: $\forall \varepsilon > 0 \ \exists N$ s.t.

$m, n \geq N \Rightarrow d(x_n, x_m) \leq \varepsilon$ and $\frac{1}{2^N} \leq \varepsilon$

then $d(a_m, a_n) \leq d(a_m, x_m) + d(x_m, x_n) +$

$+ d(a_n, x_n) \leq 3\varepsilon \quad \forall m, n \geq N$.

$\therefore$ $(a_n)$ Cauchy

thus $\exists$ limit $x \in X$. then

$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x) \longrightarrow 0$

so $x$ is a limit of $(x_n)$.

$\therefore$ $X$ complete.

every element in
Thus, it suffices to prove that $j(R)$ ~~has~~ ~~complete~~ to a limit in $\widehat{R}$.

Let $(z_n)$ be a Cauchy sequence in $R$, which is the same thing as a Cauchy sequence in $j(R)$.

Let $z^* = (z_n)$ as an element of $C$.

We claim that $(z_n) \longrightarrow z^*$.

Proof: $\forall \varepsilon > 0 \ \exists N : m,n \geq N \Rightarrow |z_m - z_n| \leq \varepsilon$

then $|z^* - z_n|' = \lim_{k \to \infty} |z_k - z_n| \leq \varepsilon$

for all $n \geq N$.

$\therefore \ (z_n) \longrightarrow z^*$.

with $|\cdot|_p$
Part (v): $R = \mathbb{Z}[\frac{1}{p}]$; not a field, but the completion is $\mathbb{Q}_p$, a field.

If $R$ is a field, we claim that $\widehat{R}$ is ~~maximal~~ a field.

Let $(x_n) \notin I$. Then $\exists \varepsilon > 0$ & $N$ s.t

~~$(x_n)$~~ $|x_n| \geq \varepsilon \ \forall n \geq N$, and so

changing $(x_n)$ by an element in $I$ wlog

$x_n \neq 0 \ \forall n$.

<u>Claim</u>: $(x_n^{-1})$ is Cauchy.

This would do it.

We have $\varepsilon$ s.t $|x_n| \geq \varepsilon \ \forall n$.

Let $\delta > 0$ & pick $N$ s.t $|x_m - x_n| \leq \delta$

for $m, n \geq N$.

Then $|x_m^{-1} - x_n^{-1}| = |x_m^{-1} x_n^{-1}| \, |x_m - x_n| \leq$

$$\leq \varepsilon^{-2} \delta$$

$\forall m, n \geq N$, so $(x_n^{-1})$ is Cauchy.

3. $a \in \mathbb{Z}$ s.t $|4a-1|_5 \leq 5^{-10}$

$$\left|a - \frac{1}{4}\right|_5 \leq 5^{10}$$

$$\frac{1}{4} = -\frac{1}{1-5} = -\left(1 + 5 + 5^2 + \dots + 5^9 + 5^{10} + \dots\right)$$

Set $a = -(1 + 5 + \dots + 5^9) \Rightarrow$

$\Rightarrow$ done.

4. (ii) Induction.

$$v_p(0!) = v_p(1) = 0 = \frac{0-0}{p-1}.$$

$\cancel{v_p((x+1)!)}$

$$x = (p-1) + (p-1)p + \dots + (p-1)p^n + a_{n+1}p^{n+1} + \dots + a_r p^r$$

$$x+1 = (a_{n+1}+1)p^{n+1} + a_{n+2}p^{n+2} + \dots + a_r p^r.$$

$$\frac{s_p(x+1) - s_p(x+1)}{p-1} = \frac{-\left(\sum_{i=n+1}^{r} a_i\right) + 1 + (x+1)}{p-1} =$$

$$= \frac{\left(x - \sum_{i=n+1}^{r} a_i - (p-1)n+1\right) + (p-1)(n+1)}{p-1} =$$

$$= v_p(x!) + n - 1 = v_p(x!) + v_p(x+1) =$$

$$= v_p((x+1)!) .$$

(i) Let $x \in \mathbb{Q}^\times$. Multiplying by powers of $p$ & adding integers, wlog $|x|_p = 1$ & $-1 \leq x < 0$.

Write $x = \frac{a}{b}$, $a < 0$, $b \geq 1$, $(b, p) = 1$. Then $\exists k$: $p^k \equiv 1 \mod b$ so $\exists c > 0$ : $p^k - 1 = bc$.

Then $x = \frac{a}{b} = \frac{ac}{bc} = \frac{-ac}{1-p^k}$

Set $N = -ac > 0$. Then $N \leq p^k - 1$ since $x \geq -1$. Write $N = n_0 + n_1 p + \dots + n_{k-1} p^{k-1}$,

then $x = \frac{N}{1-p^k} = n_0 + n_1 p + \dots + n_{k-1} p^{k-1} + n_0 p^k +$
$+ n_1 p^{k+1} + \dots + n_{k-1} p^{2k-1} + n_0 p^{2k} + \dots$

periodic.

4(i) continued.

<u>Converse</u>: ~~xxx~~ We can multiply by a suitable power of $p$ so that $x \in \mathbb{Z}_p$.

Have

$$x = x_0 + x_1 p + \cdots + x_{\ell-1} p^{\ell-1} +$$
$$+ y_0 p^{\ell} + y_1 p^{\ell+1} + \cdots + y_{m-1} p^{\ell+m-1} +$$
$$+ y_0 p^{\ell+m} + \cdots$$

Then set $b = x_0 + x_1 p + \cdots + x_{\ell-1} p^{\ell-1}$,

$c = y_0 + y_1 p + \cdots + y_{m-1} p^{m-1}$.

We have $x = b + c \dfrac{p^{\ell}}{1 - p^m}$.

5. $\pi = f(a_0)$.

WTP: (by induction)

$$\begin{cases} f(a_n) \equiv 0 \mod \pi^{2^n} \\ |f'(a_n)| = 1 \end{cases}$$

$\forall n \geq 0$.

$n=0$ ✓ (by assumption)

$n \geq 1$:

$$f(x - a_{n-1}) = f(a_{n-1}) + (x - a_{n-1}) f'(a_{n-1}) + $$
$$+ b(x - a_{n-1})^2.$$

$$b \in \mathcal{O}_K.$$

$$f(a_n) = \cancel{f(a_{n-1}) - f(a_{n-1})} + $$
$$+ b\left(- \frac{f(a_{n-1})}{f'(a_{n-1})}\right)^2 \equiv$$

$$\boxed{|f(a_n)| \leq \left(|\pi|^{2^{n-1}}\right)^2} \qquad \equiv 0 \mod \pi^{2^n}$$

$$f'(x - a_{n-1}) = f'(a_{n-1}) + c(x - a_{n-1})$$

$$\Rightarrow |f'(a_n)| = 1.$$

Uniqueness & simple root: Otherwise $a_n$ wouldn't be a simple root mod $m$.

6. $\mathbb{Z}_7$: $x^3 - 3x + 4 \equiv \cancel{x^3 \ldots}$ mod 7:

~~$=22$~~ $(-3)^3 - 3(-3) + 4 = -27 + 9 + 4 =$

~~$-27+9+14$~~ $= -14 \equiv 0 \pmod 7$.

Mod 7:

$(x+3)(x^2 - 3x - 1)$

$(-3)^2 - 3(-3) - 1 = 17 \not\equiv 0 \pmod 7$.

Hensel's Lemma $\Rightarrow$ unique solution.

~~$-8+6+4$~~

$\mathbb{Z}_3$: $x^3 - 3x + 4 \equiv x^3 + 1$ mod 3, ~~so~~

~~$27 - 9 + 4$~~ $-6+4$ $=-2$ ~~$x^3$ no roots.~~ $x^3 - 3x + 4$ mod 9 ?

No roots.

| | |
|---|---|
| 0 | 4 |
| 1 | 2 |
| 2 | / |
| 3 | 4 |
| 4 | / |
| 5 | $\neq 0$ |
| 6 | / |
| 7 | $\neq 0$ |
| 8 | / |

$\mathbb{Z}_5$: $x^3 - 3x + 4 \equiv x^3 + 2x - 1$

No roots.

| | |
|---|---|
| 0 | $-1$ |
| 1 | 2 |
| 2 | 1 |
| $-2$ | $-3$ |
| $-1$ | $-1$ |

$\mathbb{Z}_2$ : $x^3-3x+4 \equiv x^3-x \equiv x(x^2-1) \mod 2.$

Hensel
$\implies$ unique root $\equiv 0 \mod 2$.

$x^3-3x+4 \equiv x^3+x \equiv \cancel{x(x^2+1)} \mod 4$

$\cancel{0}\ \cancel{2}\ 1, -1$ : No roots $\equiv \pm 1$.

$\therefore$ One root.

7. $v_p\left(\binom{1/2}{n}\right)$.

$$2^n\binom{1/2}{n} = \frac{1(1-2)(1-4)\cdots(1-2n+2)}{n!} =$$

$$= (-1)^{n-1}\frac{1.3.5.7\cdots(2n-1)}{n!} =$$

$$= (-1)^{n-1}\frac{(2n-1)!}{n!\ 2.4.6.8\cdots 2n-2} =$$

$$= \left(-\frac{1}{2}\right)^{n-1}\frac{(2n-1)!}{n!\,(n-1)!}.$$

$$\therefore \quad 2^{2n-1}\binom{1/2}{n} = (-1)^{n-1}\,\frac{(2n-1)!}{n!\,(n-1)!}$$

$$v_p(\text{LHS}) = \begin{cases} 2n-1+v_p\left(\binom{1/2}{n}\right), & p=2 \\ v_p\left(\binom{1/2}{n}\right), & p>2. \end{cases}$$

$$v_p(\text{RHS}) = \frac{2n-1-s_p(2n-1)}{p-1} - \frac{n-s_p(n)}{p-1} - \frac{n-1-s_p(n-1)}{p-1} =$$

$$= \frac{s_p(n)+s_p(n-1)-s_p(2n-1)}{p-1}$$

<u>$p>2$</u>:

$n = p^k+1$, $n-1 = p^k$, $2n-1 = 2p^k+1$; then

get $v_p(\text{RHS}) = \dfrac{2+1-3}{p-1} = 0 \not\to \infty$.

$p=2$: $n = 2^k+1$, $n-1 = 2^k$, $2n-1 = 2.2^k+1 = 2^{k+1}+1$

$$v_p(\text{RHS}) = \frac{2+1-1}{2-1} = 1 \not\to 0$$

Have $$\Big(\underbrace{1 + \sum_{n=0}^{\infty} \binom{1/2}{n} 15^n}_{\alpha}\Big)^2 \overset{(*)}{=}$$

$$\overset{(*)}{=} 1 + \sum_{n=0}^{\infty} \binom{1}{n} 15^n = 16$$

$\Rightarrow \quad \alpha = \pm 4.$

~~$\alpha \equiv$~~ In $\mathbb{Z}_3$: $\quad \alpha \equiv 1 \mod 3 \Rightarrow$

$\Rightarrow \quad \alpha = 4.$

In $\mathbb{Z}_5$: $\quad \alpha \equiv 1 \mod 5 \Rightarrow$

$\Rightarrow \quad \alpha = -4.$

Proving $*$: Prove the relevant identity for binomial coefficients by proving it for $\mathbb{Z}_{\geq 1}$ & ~~d~~ extending to $\mathbb{Z}_p$ by density.

8. ~~$\mathbb{Z}_p$~~ Open $\Rightarrow$ finite by compactness.

$(\mathbb{Z}_p : G) = N \Rightarrow$

$\Rightarrow \quad G \supseteq N\mathbb{Z}_p = p^{v_p(N)}\mathbb{Z}_p$

$\Rightarrow \quad G$ open.

9. $\lim_{n\to\infty} |x_n|$ stabilizes if it doesn't tend to $0$.

10. We have (iii) $\Rightarrow$ (i) since open balls match up (only different radii)

& (i) $\Rightarrow$ (ii):

$|x| < 1 \iff x^n \xrightarrow[\text{in } |\cdot|]{} 0 \overset{(i)}{\iff} x^n \xrightarrow[\text{in } |\cdot|']{} 0 \iff$

$\iff |x|' < 1.$

(ii) $\Rightarrow$ (iii).

Take $y \in K$, $|y| > 1$.

Let $x \in K$, $x \neq 0$. $\exists \alpha \in \mathbb{R}$ s.t

$|x| = |y|^{\alpha}$.

Take $\frac{m_i}{n_i} \downarrow \alpha$.

$\in \mathbb{Q}$, $n_i > 0$

$|x| = |y|^{\alpha} < |y|^{m_i/n_i} \Rightarrow$

$\Rightarrow \left| \frac{x^{n_i}}{y^{m_i}} \right| < 1 \overset{(ii)}{\Rightarrow} \left| \frac{x^{n_i}}{y^{m_i}} \right|' < 1$

$\therefore |x|' < |y|'^{m_i/n_i} \Rightarrow |x|' \leq |y|'^{\alpha}$.

Take $\frac{p_i}{q_i} \uparrow \alpha$, $|x| = |y|^{\alpha} > |y|^{p_i/q_i}$

$\in \mathbb{Q}$, $q_i > 0$

$\Rightarrow \left| \frac{y^{p_i}}{x^{q_i}} \right| < 1 \overset{(ii)}{\Rightarrow} \left| \frac{y^{p_i}}{x^{q_i}} \right|' < 1 \Rightarrow$

$\Rightarrow |y|'^{p_i/q_i} < |x|' \Rightarrow |y|'^{\alpha} \leq |x|'$.

$\therefore |x|' = |y|'^{\alpha}$.

~~$\therefore \log \alpha = \frac{\log |x|'}{\log |y|'} = \frac{\log |x|}{\log |y|}$~~

~~is constant as x varies~~

Thus we have $\dfrac{\log|x|'}{\log|y|'} = \dfrac{\log|x|}{\log|y|}$.

Put $s = \dfrac{\log|y|'}{\log|y|}$, then

$\log|x|' = s \cdot \log|x| = \log|x|^s$ so

$|x|' = |x|^s$.

For the final part, note that by (iii)

$$(K, |\cdot|) \underset{id}{\overset{id}{\rightleftarrows}} (K, |\cdot|')$$ are both

uniformly cts, so ~~comp~~ respects Cauchy sequences.

11. (i) $\Rightarrow$ (iii) is clear.

Assume (iii) & let $x, y \in K$. If $x = y = 0$,

then $0 = |x+y| \leq \max(|x|, |y|) = 0$.

If not, wlog $|x| \leq |y| \neq 0$. Then

$$|x+y| = \left|\frac{x}{y} + 1\right| |y| \leq 1 \cdot |y| = \max(|x|, |y|).$$

so (i) holds.

Clearly (i) $\Rightarrow$ (ii); indeed $|n| \leq |1|$ $\forall n \in \mathbb{Z}$.

<u>Converse:</u> Suppose $|n| \leq N$ $\forall n$.

Let $x, y \in K$ suppose that $|x| \geq |y|$. Then $|x|^k |y|^{n-k} \leq |x|^n$ $\forall k \in \{0, \dots, n\}$ so

$$|x+y|^n \leq \sum_{k=0}^{\infty} \left|\binom{n}{k}\right| |x|^k |y|^{n-k} \leq$$

$$\leq N \cdot (n+1) |x|^n$$

Hence $|x+y| \leq N^{1/n} (1+n)^{1/n} |x| \longrightarrow$

$\longrightarrow |x|$ as $n \to \infty$, so

$|x+y| \leq \max(|x|, |y|)$.

Last part is clear: char $K > 0$ $\Rightarrow$ $\mathrm{Im}(\mathbb{Z} \to K)$ is finite, ~~the~~ hence bounded, so by (ii) $\Rightarrow$ (i) $K$ is non-archimedean.

12. $f_m : X_{m+1} \longrightarrow X_m$.

$p_m : \prod X_m \longrightarrow X_m$.

Have $\prod X_n \xrightarrow{\alpha_m} X_m$

$\alpha_m = f_m p_{m+1} - p_m$

$X_m$ Hausdorff $\Rightarrow$ $\alpha_m^{-1}(\{0\})$ closed.

Now $\varprojlim_n X_n = \bigcap_{m=1}^{\infty} \alpha_m^{-1}(\{0\})$.

So closed.

13. Let $p_n : \varprojlim_m R/x^m R \longrightarrow R/x^n R$.

~~Have~~ Suffices to prove the case $n = 1$.

WTP: $p_n$ induces an isomorphism

$S/x^n S \xrightarrow{\sim} R/x^n R \quad \forall n$.

Put $p = p_1$. Have

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ S \arrow[r] & R/xR \end{tikzcd}$$

$\Rightarrow S \twoheadrightarrow R/xR$.

& clear $x \in \operatorname{Ker} p$, so $xS \subseteq \operatorname{Ker} p$.

WTP equality.

Let $s = (\overline{r_n}) \in \operatorname{Ker} p$, $\overline{r_n} \in R/x^nR$.

Let $r_n \in R$ s.t. $q_n(r_n) = \overline{r_n}$ $\quad q_n : R \to R/x^nR$ natural map,

$f_n : R/x^{n+1}R \to R/xR$.

By assumption $r_{n+1} - r_n \equiv 0 \mod x^n$

& $r_1 \equiv 0 \mod x$

so $r_n \equiv 0 \mod x \ \forall n$.

$\therefore$ $r_n = x t_n$, $r_{n+1} - r_n = x^n u_n$ for some $t_n, u_n \in R$.

Have $x(t_{n+1} - t_n) = x^n u_n \overset{x\text{-torsionfreeness}}{\Longrightarrow}$

$\Rightarrow t_{n+1} - t_n = x^{n-1} u_n$

Thus $\overline{t} = (q_1(t_2), q_2(t_3), \ldots) \in S$

and $x\overline{t} = (q_1(xt_2), q_2(xt_3), \ldots) =$

$= (q_1(r_2), q_2(r_3), \ldots) = (q_1(r_1), q_2(r_2), \ldots) = s$.

$\therefore \quad \operatorname{Ker} p \subseteq xS$.

~~We therefore~~

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}"] \arrow[d] & \varprojlim_n R/x^nR \\ \varprojlim_n S/x^nS \arrow[ur, "\sim"] & \end{tikzcd}$$

so $S \xrightarrow{\sim} \varprojlim_n S/x^nS$.

$S$ is $x$-torsionfree:

$x(\overline{r_n}) = 0$. Take lifts $r_n$.

Then $\quad r_{n+1} - r_n = x^n t_n$ and

$\qquad x r_n = x^n u_n$

WTP: $\quad r_n \equiv 0 \mod x^n \quad \forall n$.

Need to show $u_n \equiv 0 \mod x$.

Have $\quad x^{n+1} t_n = x r_{n+1} - x r_n = x^{n+1} u_{n+1} - x^n u_n$

$\implies \quad x t_n = x u_{n+1} - u_n \implies$

$\implies \quad u_n = x(u_{n+1} - t_n) \quad \checkmark$.

$\therefore \quad S \quad x$-torsionfree.

14. Inequivalence of the $|\cdot|_p$:

$p^n \to 0$ only in $|\cdot|_p$.

Let $|\cdot|$ be non-archimedean.

If $|p| = 1$ $\forall p$, we would have $|x| = 1$ $\forall x \in \mathbb{Q}^\times$ & $|\cdot|$ is trivial, so must have $|p| < 1$ for some $p$ ($|n| \leq 1$ $\forall n \in \mathbb{Z}$, so can't have $|p| > 1$).

Put $\mathfrak{a} = \{ n \in \mathbb{Z} \mid |n| < 1 \}$. This is a proper ideal & $p \in \mathfrak{a} \Rightarrow \mathfrak{a} = p\mathbb{Z}$.

Now if $a \in \mathbb{Z}$ & $a = bp^m$ with $(b, p) = 1$, then $|a| = \underbrace{|b|}_{=1} |p|^m = |a|_p^s$ where $s = -\log_p |p|$.

It follows (by multiplicativity) that this holds $\forall a \in \mathbb{Q}^\times$.

Now let $|\cdot|$ be archimedean.

Let $m, n > 1$ be integers.

Expand $m$ in base $n$:

$$m = a_0 + a_1 n + \dots + a_r n^r, \quad a_i \in \{0, 1, \dots, n-1\}$$

Note that $r \leqslant \frac{\log m}{\log n}$.

We have $|a_i| \leqslant a_i \leqslant n$, so

$$|m| \leqslant \sum |a_i| |n|^i \leqslant$$

<u>Claim</u> $|n| \geqslant 1 \ \ \forall n \in \mathbb{Z} \setminus 0$ &

Assuming this, we continue:

$$|m| \leqslant \sum |a_i| |n|^i \leqslant \sum |a_i| |n|^r \leqslant$$

$$\leqslant (1 + r) n \cdot |n|^r \leqslant$$

$$\leqslant \left(1 + \frac{\log m}{\log n}\right) n \cdot |n|^{\frac{\log m}{\log n}}$$

Replace $m$ by $m^k$ & take $\sqrt[k]{\phantom{x}}$:

$$|m| \leqslant \left(1 + k \frac{\log m}{\log n}\right)^{1/k} n^{1/k} |n|^{\frac{\log m}{\log n}}$$

Now take $k \to \infty$ : Get

$$|m| \leq |n|^{\frac{\log m}{\log n}}.$$

By symmetry $|m|^{1/\log m} = |n|^{1/\log n}$.

Put $c$ to be this constant ; then

$$|n| = c^{\log n} \quad \forall n \in \mathbb{Z}_{>0}$$

We have $c > 1$ (otherwise $|n|$ bounded).

Put $c = e^s$, $s > 0$, we have

$|n| = e^{s \cdot \log n} = n^s$ ; ~~this holds~~ valid $\forall n \in \mathbb{Z}_{>0}$.

Now for arbitrary $x = \pm \frac{m}{n} \in \mathbb{Q}$,

$$|x| = \left|\pm \frac{m}{n}\right| = \frac{m^s}{n^s} = |x|_\infty^s,$$

so $|x|$ equivalent to $|\,|_\infty$.

<u>Proof of Claim</u> :

If $|n| < 1$, then

$$|m| \leq \sum |a_i||n|^i \leq n \sum |n|^i \leq$$

$$\leq n \cdot \frac{1 - |n|^{r+1}}{1 - |n|} \leq \frac{n}{1 - |n|}$$

so it's bounded #.

15. Induction on $N$:

$N = 0$:

Have $|c_0| > |c_i| \quad \forall i \geq 1$.

then $\left|\sum_{i=1}^{\infty} c_i x^i\right| < |c_0| \quad \forall x \in \mathbb{Z}_p$, so

there can be ~~non~~ no zeroes.

$N \geq 1$:

Suppose that $\exists$ zero $\alpha$ (if not, done).

$$f(x) = f(x) - f(\alpha) = \sum_{n=1}^{\infty} c_n (x^n - \alpha^n) =$$

$$= (x-\alpha) \sum_{n \geq 1} \sum_{0 \leq j \leq n-1} c_n x^j \alpha^{n-1-j} =$$

$$= (x-\alpha) \sum_{k=0}^{\infty} d_k x^k$$

$$d_0 = \sum_{n\geq 1} c_n \alpha^{n-1}$$

$$d_1 = \sum_{n\geq 2} c_n \alpha^{n-2}$$

$$\vdots$$

$$d_k = \sum_{n\geq k+1} c_n \alpha^{n-1-k}$$

$$\vdots$$

We have $|d_k| \leq \max_n |c_n| = |c_N| \quad \forall k$,

and $|d_{N-1}| = \left|\sum_{n\geq N} c_n \alpha^{n-N}\right| = |c_N|$ and

$$|d_k| = \left|\sum_{n\geq k+1} c_n \alpha^{n-1-k}\right| < |c_N| \text{ for } k > N.$$

By our induction hypothesis, $g(T) = \sum_{k=0}^{\infty} d_k T^k$ has $\leq N-1$ zeroes. It follows that $f(T)$ has $\leq N$ zeroes.

## "High-brow" reason

## Weierstrass preparation theorem

$f(T) \in \mathbb{Q}_p\langle T\rangle$.

$f(T) = c_0 + c_1 T + \dots$

$N(f) =$ last $N$ s.t $|c_N| = \max_{n=0,1,\dots} |c_n|$.

Then $\exists!\, p$ monic polynomial of degree $N(f)$ in $\mathbb{Z}_p[T]$ & unique $u \in \mathbb{Q}_p\langle T\rangle$ with $N(u) = 0$

s.t $\quad f = p.u$

Moreover, $f \in \mathbb{Q}_p\langle T\rangle^\times \iff N(f) = 0$, in which case $f$ has no zeroes in $\mathbb{Z}_p$.