MATHEMATICAL TRIPOS PART III (2016–17)

Local Fields - Example Sheet 1 of 4 **C. Johansson**

Note: Absolute values are non-trivial, but not necessarily non-archimedean. v_p *denotes the* p*-adic valuation on* $\mathbb{Q}_p$*, and* $|-|_p$ *denotes the* p*-adic absolute value on* $\mathbb{Q}_p$*.*

1. In lectures, we defined absolute values on fields. More generally, if R is a ring, an *absolute value* on R is function $|-| : R \to \mathbb{R}_{\geq 0}$ satisfying the axioms for an absolute value. Show that R is an integral domain, and that $|-|$ extends uniquely to an absolute value on the fraction field of R.

2. Let R be a ring with an absolute value $|-|$. Let C denote the set of all Cauchy sequences in R.

   (i) Show that the rules $(x_n)_n + (y_n)_n = (x_n + y_n)_n$ and $(x_n)_n \cdot (y_n)_n = (x_n y_n)_n$ define a ring structure on C.

   (ii) Show that set I of the of sequences tending to 0 form a prime ideal in C. Denote the quotient C/I by $\widehat{R}$, and show that the map $j : R \to \widehat{R}$ sending $x \in R$ to the equivalence class of the constant sequence $(x)_n$ is an injective ring homomorphism.

   (iii) If $(x_n)_n \in C$, show that $\lim_{n\to\infty} |x_n|$ exists and that the function $|(x_n)_n|' = \lim_{n\to\infty} |x_n|$ is constant on cosets of I, hence defines a function $|-|' : \widehat{R} \to \mathbb{R}_{\geq 0}$.

   (iv) Show that $|-|'$ is an absolute value on $\widehat{R}$. Show moreover that $|x| = |j(x)|'$ for all $x \in R$, that $j(R)$ is dense in $\widehat{R}$, and that $\widehat{R}$ is complete with respect to $|-|'$. We call $\widehat{R}$ the *completion* of R.

   (v) Show that if R is a field, then $\widehat{R}$ is a field. Find an example of an R which is not a field, but such that $\widehat{R}$ *is* a field.

3. If $c \in \mathbb{Z}_p$ satisfies $|c|_p < 1$ show that $(1+c)^{-1} = 1 - c + c^2 - c^3 + \ldots$ Hence or otherwise find $a \in \mathbb{Z}$ such that $|4a-1|_5 \leq 5^{-10}$.

4. Let $x = \sum_{n \geq N} a_n p^n \in \mathbb{Q}_p$, with $a_n \in \{0, 1, \ldots, p-1\}$, $N \in \mathbb{Z}$ and $a_N \neq 0$.

   (i) Show that $x \in \mathbb{Q}$ if and only if the sequence $(a_n)_n$ is eventually periodic.

   (ii) Assume that $x \in \mathbb{Z}$ and let $s_p(x) = \sum_n a_n$. Prove the formula $v_p(x!) = (x - s_p(x))/(p-1)$.

5. (Hensel's Lemma) Let K be a complete valued field with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. Let $f(x) \in \mathcal{O}[x]$ be a monic polynomial, and assume that $a_0 \in \mathcal{O}$ is such that $f(a_0) \in \mathfrak{m}$ but $f'(a_0) \notin \mathfrak{m}$. For each $n \geq 1$, define

$$a_n = a_{n-1} - \frac{f(a_{n-1})}{f'(a_{n-1})}.$$

   Prove that $a = \lim_{n\to\infty} a_n$ exists and is a simple root of f. Show also that there are no other roots of f which are congruent to a modulo $\mathfrak{m}$.

6. Show that the equation $x^3 - 3x + 4 = 0$ has a unique solution in $\mathbb{Z}_7$, but has no solutions in $\mathbb{Z}_5$ or in $\mathbb{Z}_3$. How many are there in $\mathbb{Z}_2$?

7. Consider the series
$$\text{“}\sqrt{1+15}\text{”} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} 15^n$$
where $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$. Show that the series converges to 4 with respect to the 3-adic absolute value, to -4 with respect to the 5-adic absolute value, and diverges with respect to all other absolute values on $\mathbb{Q}$.

8. Show that a subgroup of $\mathbb{Z}_p$ is open if and only if it has finite index.

9. Let $\widehat{R}$ be the completion of a ring R with respect to a non-archimedean absolute value $|-|$. Prove that $|\widehat{R}| = |R|$.

10. Let $|-|$ and $|-|'$ be two absolute values on a field K. Prove that the following are equivalent:

   (i) $|-|$ and $|-|'$ define the same topology on K.

   (ii) $|x| < 1 \implies |x|' < 1$ for all $x \in K$.

   (iii) There exists a real number $s > 0$ such that $|x|^s = |x|'$ for all $x \in K$.

   Deduce that the completion of K only depends to the equivalence class of the absolute value.

11. Let K be a field with an absolute value $|-|$. Prove that the following are equivalent:

   (i) $|-|$ is non-archimedean (i.e. satisfies the strong triangle inequality).

   (ii) The image of the natural map $\mathbb{Z} \to K$ is a bounded subset of K.

   (iii) $|x| \leq 1 \implies |x+1| \leq 1$ for all $x \in K$.

   Note that it is not necessary to use that $|-|$ satisfies the triangle inequality for the equivalence (i) $\iff$ (iii). Deduce that any absolute value on a field of positive characteristic is non-archimedean.

12. Let $X_1, X_2, ...$ be a sequence of Hausdorff topological rings with continuous homomorphisms $f_n : X_{n+1} \to X_n$ for $n \geq 1$. Let $X = \varprojlim_n X_n$. Show that X is closed inside the product topological ring $\prod_n X_n$.

13. Let R be a ring and let $x \in R$. Let S be the x-adic completion of R. Assume that R is x-torsionfree. Show that S is x-adically complete, and x-torsionfree.

14. Show that a non-archimedean absolute value on $\mathbb{Q}$ is equivalent to $|-|_p$ for a *unique* p. Show that any archimedean absolute value on $\mathbb{Q}$ is equivalent to the usual absolute value $|x| = \sqrt{x^2}$.

15. (Strassman's theorem) Let $f(T) = c_0 + c_1 T + c_2 T^2 + \ldots \in \mathbb{Z}_p[[T]]$ be a formal power series with $c_n \to 0$ as $n \to \infty$. Suppose that for some $N \geq 0$ we have $v_p(c_N) = 0$ and $v_p(c_n) > 0$ for all $n > N$. Show that $\#\{x \in \mathbb{Z}_p : f(x) = 0\} \leq N$.