MATHEMATICAL TRIPOS PART III (2016–17)

Local Fields - Example Sheet 2 of 4 **C. Johansson**

$|-|_p$ *denotes the* p*-adic absolute value on* $\mathbb{Q}_p$ *and* v_p *denotes the* p*-adic valuation.*

1. Let $K/\mathbb{Q}$ be a finite extension and let $\mathcal{O}_K \subseteq K$ be the subring of algebraic integers. It is a basic fact of algebraic number theory that every nonzero ideal $I \subseteq \mathcal{O}_K$ has a unique factorization
$$I = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}(I)}$$
where $\mathfrak{p}$ ranges through the maximal ideals of $\mathcal{O}_K$ and the $e_{\mathfrak{p}}(I) \in \mathbb{Z}_{\geq 0}$ are zero for all but finitely many $\mathfrak{p}$. Fix a maximal ideal $\mathfrak{p}$. Prove that the function $v_{\mathfrak{p}} : \mathcal{O}_K \to \mathbb{Z} \cup \{\infty\}$ defined by
$$v_{\mathfrak{p}}(x) = e_{\mathfrak{p}}(x\mathcal{O}_K)$$
if $x \neq 0$ and $v_{\mathfrak{p}}(0) = \infty$ defines a discrete valuation on $\mathcal{O}_K$. If $p \in \mathfrak{p}$, prove that $v_{\mathfrak{p}}|_{\mathbb{Z}}$ is equivalent to v_p. We denote the completion of K with respect to $v_{\mathfrak{p}}$ by $K_{\mathfrak{p}}$.

2. Using the previous exercise, or otherwise, find a valued field K and a finite extension L/K such that the absolute value on K has more than one extension to L.

3. Let K be a valued field with valuation ring $\mathcal{O}_K$ and let I be a finitely generated ideal. Show that I is principal. Deduce that $\mathcal{O}_K$ is a Noetherian ring if and only if K is a discretely valued field.

4. Let K be a complete valued field with valuation v and let $f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n \in K[x]$. Let $\alpha_1, \ldots, \alpha_n$ be the roots of f in a splitting field L of f over K, and let w be the valuation on L extending v. Prove that
$$f_r(x) = \prod_{i\,:\,w(\alpha_i)=r} (x - \alpha_i) \in K[x]$$
for any $r \in \mathbb{R}$ (if you find the general case tricky, try the case when L/K is separable first). Deduce that f has at least as many factors in $K[x]$ as there are line segments on its Newton polygon.

5. In this exercise you should use the conclusions of Exercise 4, even if you have not completed it. Work over $\mathbb{Q}_2$.
   (i) Consider $f(x) = 1 - x/2 - x^2/2 + x^3 \in \mathbb{Q}_2[x]$. Draw the Newton polygon of f and show that f has three roots in $\mathbb{Q}_2$. Can you prove this using Hensel's Lemma?
   (ii) Consider $g(x) = x^5 + 2x^2 + 4 \in \mathbb{Q}_2[x]$. Draw the Newton polygon of g and show that g has a factor of degree 2 and another factor of degree 3. Can you prove that both these factors are irreducible?

(iii) Give an example of a polynomial in $\mathbb{Q}_2[x]$ which is reducible and whose Newton polygon has a single line segment.

6. Let $p > 2$, $n \geq 1$ and let $\Phi_{p^n}(x) = x^{p^{n-1}(p-1)} + x^{p^{n-1}(p-2)} + \cdots + x^{p^{n-1}} + 1$ be the p^n-th cyclotomic polynomial. It is irreducible over $\mathbb{Q}_p$ (you may assume this). Let $K = \mathbb{Q}_p(\zeta_{p^n})$, where ζ_{p^n} is a primitive p^n-th root of unity, and let w denote the unique extension of v_p to K. Define $f_n(x) = \Phi_{p^n}(x+1)$. By computing the first and last coefficient of $f_n(x)$, prove that

$$w(\zeta_{p^n}^i - 1) = \frac{1}{p^{n-1}(p-1)}$$

for all n and all i coprime to p.

7. Compute the Mahler expansions of the polynomials x^3+4x+7 and $x^4+8x^3+6x^2+5$.

8. Let $K/\mathbb{Q}_p$ be a finite extension and let $a \in \mathfrak{m}_K$. We can define a function $\mathbb{Z}_p \to K$, denoted by $(1+a)^x$, by the Mahler expansion

$$(1+a)^x := \sum_{n=0}^{\infty} \binom{x}{n} a^n.$$

(i) When $x \in \mathbb{Z}_{\geq 0}$, prove that the right hand side above is equal to $(1+a)^n$ in the usual sense. Then show that $(1+a)^{x+y} = (1+a)^x(1+a)^y$ and $(1+a)^{xy} = ((1+a)^x)^y$ (make sense of the right hand side!) for all $x, y \in \mathbb{Z}_p$.

(ii) Consider $K = \mathbb{Q}_p(\zeta_{p^n})$ where ζ_{p^n} is a primitive p^n-th root of unity. Let $\lambda_i = \zeta_{p^n}^i - 1$ for any $i = 0, 1, ..., p^n - 1$; by Exercise 6 we may define $(1+\lambda_i)^x$ for any i. Fix $m \in \{0, 1, ..., p^n - 1\}$. Prove that

$$x \mapsto \frac{1}{p^n} \sum_{i=0}^{p^n-1} \zeta_{p^n}^{-im}(1+\lambda_i)^x$$

is equal to 1 if $x \in m + p^n\mathbb{Z}_p$, and 0 otherwise.

(iii) Use this to give a different proof that the Mahler expansion of any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ tends to zero.

9. (Continuity of roots) Let K be a complete valued field and let $\overline{K}$ be an algebraic closure of K with the extended absolute value. Let $f(x) = a_0 + a_1x + \cdots + x^n$ and $g(x) = b_0 + b_1x + \cdots + x^n$ be monic polynomials in $K[x]$, and let $\beta_1, ..., \beta_n \in \overline{K}$ be the roots of g. If $\alpha \in \overline{K}$ is a root of f, prove that there exists an i such that

$$|\alpha - \beta_i| \leq \max_{i=0,\ldots,n-1} \left(|a_i - b_i|^{1/n}|\alpha|^{i/n}\right).$$

(*Hint: Consider* $g(\alpha) - f(\alpha) = g(\alpha) = \prod_i(\alpha - \beta_i)$.) Reformulating it somewhat imprecisely, if the coefficients of g are close enough to those of f, then there is a root of g close to α.

10. (Krasner's Lemma) Let K be a complete valued field and let $\overline{K}$ be an algebraic closure of K with the extended absolute value. Let $\alpha \in \overline{K}$ be separable and let $\alpha_2, \ldots, \alpha_n \in \overline{K}$ be the K-conjugates of α. If $\beta \in \overline{K}$ is such that

$$|\alpha - \beta| < |\alpha - \alpha_i|$$

for $i = 2, ..., n$, show that $K(\alpha) \subseteq K(\beta)$. (*Hint: Let L be the Galois closure of $K(\alpha, \beta)$ over $K(\beta)$, and show that $|\alpha - \sigma(\alpha)| < |\alpha - \alpha_i|$ for all $i = 2, ..., n$ and $\sigma \in \mathrm{Gal}(L/K(\beta))$.*)

11. Let $L/\mathbb{Q}_p$ be a finite extension. Show, using the two previous exercises or otherwise, that we can find a finite extension $K/\mathbb{Q}$ and a maximal ideal $\mathfrak{p}$ containing p such that $L = K_{\mathfrak{p}}$ (in the notation of Exercise 1).

12. If you have never seen it before, prove (or look up) the Baire Category Theorem: If X is a complete metric space and U_i, $i = 1, 2, ...$ is a sequence of open dense subsets in X, then $\bigcap_{i=1}^{\infty} U_i \neq \emptyset$ (in fact it is dense in X). Use it to prove that an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete with respect to the extended absolute value.

13. Consider an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ with the extended absolute value. Let $\mathbb{C}_p$ denote its completion. Prove that $\mathbb{C}_p$ is algebraically closed.

14. This exercise is for enthusiasts. Let V be a Banach space over $\mathbb{Q}_p$, i.e. a complete normed vector space over $\mathbb{Q}_p$. Let $|| - ||$ be the norm on V. A collection e_i, $i \in I$, of elements in V is called an *orthonormal basis* for V if any $v \in V$ can be written uniquely as an expansion

$$v = \sum_{i \in I} a_i e_i$$

for $a_i \in \mathbb{Q}_p$ tending to zero, and moreover

$$||v|| = \max_{i \in I} |a_i|_p.$$

By "a_i tending to zero" we mean that for every $\epsilon > 0$, the set $\{i \in I \mid |a_i|_p > \epsilon\}$ is finite.

(i) Prove that if V has an orthonormal basis, then $||V|| = \{p^n \mid n \in \mathbb{Z}\} \cup \{0\}$, or in short $||V|| = |\mathbb{Q}_p|_p$.

(ii) Prove that, for any V, we can find a norm $|| - ||'$ equivalent to $|| - ||$ such that $||V||' = |\mathbb{Q}_p|_p$.

(iii) Prove the converse to the first part: If $||V|| = |\mathbb{Q}_p|_p$, then V has an orthonormal basis.

Thus, while there is no notion of a Hilbert space over $\mathbb{Q}_p$, Banach spaces over $\mathbb{Q}_p$ carry features analogous to those of Hilbert spaces over $\mathbb{R}$ or $\mathbb{C}$.