MATHEMATICAL TRIPOS PART III (2016–17)

Local Fields - Example Sheet 3 of 4 **C. Johansson**

Except where stated otherwise: p is the residue characteristic of any local field considered. A local field K has valuation ring $\mathcal{O}_K$, normalised discrete valuation v_K, uniformiser π_K, and residue field k_K. We write ζ_n for a primitive nth root of unity.

1. Let L/K be a finite extension of local fields.

   (i) Let w be the extension of v_K to L. Let π_L be a uniformizer of L, and let $\mathfrak{m}_L = \pi_L \mathcal{O}_L$. Prove that
   $$e_{L/K}^{-1} = w(\pi_L) = \min_{x \in \mathfrak{m}_L} w(x).$$

   (ii) Let v' be any valuation on K (in the given equivalence class) and let w' be its extension to L. Show that $e_{L/K} = (w'(L^\times) : v'(K^\times))$. Use this to give a direct proof that if $M/L/K$ are finite extensions, then $e_{M/K} = e_{M/L} e_{L/K}$.

2. Let L/K be a finite extension and let $q = \# k_K$. If L/K is unramified of degree n, show that $L = K(\zeta_{q^n - 1})$. Conversely, if m is coprime to q and $L = K(\zeta_m)$, show that L/K is unramified and compute its degree.

3. Let L/K be a finite extension of local fields. We say that L/K is *tamely ramified* if $e_{L/K}$ is coprime to p. Let $a \in K$ and let $m \in \mathbb{Z}_{\geq 1}$ be coprime to p. Show that $K(\sqrt[m]{a})/K$ is tamely ramified.

4. Let L/K be a finite extension and let T/K be the maximal unramified subextension of L/K. Show that L/K is tamely ramified if and only if there are elements $a_1, \ldots, a_r \in T$ and positive integers $m_1, \ldots, m_r$ coprime to p such that $L = T(\sqrt[m_1]{a_1}, \ldots, \sqrt[m_r]{a_r})$.

5. If $K/\mathbb{Q}_p$ is a finite extension, show that $U_K^{(n)} \cong (\mathcal{O}_K, +)$ as (topological) groups for sufficiently large n, and find an explicit lower bound for n.

6. Let K be a local field and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathcal{O}_K[x]$ be a polynomial.

   (i) (Eisenstein's criterion) Assume that $\pi_K \mid a_i$, $i = 0, \ldots, n-1$ and $\pi_K^2 \nmid a_0$. Reformulate this condition in terms of the Newton polygon of f and show that f is irreducible.

   (ii) Let $\Phi_{p^n}(x) = x^{p^{n-1}(p-1)} + x^{p^{n-1}(p-2)} + \cdots + x^{p^{n-1}} + 1 \in \mathbb{Z}[x]$ be the p^n-th cyclotomic polynomial. Prove that $\Phi_{p^n}(x)$ is irreducible over $\mathbb{Q}_p$ for all $n \geq 1$.

   (iii) Find an optimal criterion for the shape of the Newton polygon of f alone to imply that f is irreducible. When this criterion is satisfied, what can you say about the extension of K given by adjoining a root of f?

7. Let L/K be a finite Galois extension of local fields, with Galois group $G = \mathrm{Gal}(L/K)$.

   (i) Show that the ramification groups $G_s := G_s(L/K)$ are normal subgroups of G for all s.

   (ii) Assume that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ and let $f(x) \in \mathcal{O}_K[x]$ be the minimal polynomial of α. Let v_L be the normalized valuation on L. Show that

   $$v_L(f'(\alpha)) = \sum_{1 \neq \sigma \in G} i_{L/K}(\sigma) = \sum_{s \in \mathbb{Z}_{\geq 0}} (\#G_s - 1).$$

   Deduce that the ideal $\mathfrak{D}_{L/K}$ of $\mathcal{O}_L$ generated by $f'(\alpha)$ is independent of the choice of α, and that it is equal to $\mathcal{O}_L$ if and only if L/K is unramified ($\mathfrak{D}_{L/K}$ is called the *different* of L/K).

8. Compute the ramification groups of $\mathbb{Q}_3(\zeta_3, \sqrt[3]{2})/\mathbb{Q}_3$.

9. Prove that $\mathbb{Q}_p$ has a unique Galois extension with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ if $p > 2$, and that $\mathbb{Q}_2$ has a unique Galois extension with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$. Compute the ramification groups in all cases (both with respect to the lower and upper numbering).

10. Prove that if L/K is a Galois extension of local fields with Galois group S_4, then the residue characteristic of K is 2. Construct a Galois extension $L/\mathbb{Q}_2$ with $\mathrm{Gal}(L/\mathbb{Q}_2) \cong S_4$.

11. Let $m \in \mathbb{Z}_{\geq 1}$. Compute the Galois group and all the ramification groups of $\mathbb{Q}_p(\zeta_m)$ in the lower and upper numbering (you may use the results in lectures in the case $m = p^n$, if you want to).

12. Let K be a local field. Prove that the abelian group structure on $U_K^{(1)} = 1 + \pi_K \mathcal{O}_K$ naturally extends to $\mathbb{Z}_p$-module structure. Let $q = \#k_K$. When K has characteristic 0, show that

    $$K^\times \cong \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$$

    as groups, for some $a \in \mathbb{Z}_{\geq 0}$. When K has characteristic p, show that

    $$K^\times \cong \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}_p^{\mathbb{Z}_{\geq 0}}.$$

13. Let K be a local field, let K^{sep} be a separable closure of K and let $n \in \mathbb{Z}_{\geq 1}$. If K has characteristic 0, show that there are only finitely many extensions $K \subseteq L \subseteq K^{sep}$ of degree n. What happens if K has characteristic p?