# Crash course on numerics for SDEs

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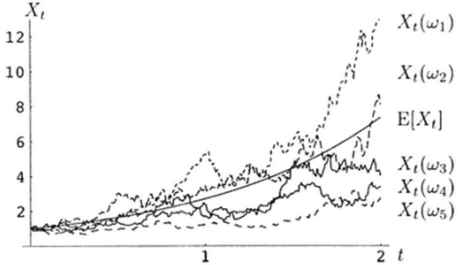
# Chapter 0: Introduction and Motivation




Thanks to Bernt Øksendal

# Ordinary differential equations

Ordinary Differential Equations (ODE) often appear in the dynamical description of deterministic systems in physics, chemistry, biology, etc.

Ordinary Differential Equation = An equation that contains some derivatives of an unknown function (here, $f$ and $y_0$ are given, $y$ is unknown):

$$\begin{cases} \dot{y} = \frac{\mathrm{d}}{\mathrm{d}t} y(t) = f(y(t)) \\ y(0) = y_0. \end{cases}$$

This can also be written in integral form (fundamental theorem of calculus "$\mathrm{d}y = f(y)\,\mathrm{d}t$"):

$$y(t) = y_0 + \int_0^t f(y(s))\,\mathrm{d}s.$$

# Stochastic differential equations

Recall: ODE $\frac{\mathrm{d}}{\mathrm{d}t}y(t) = f(y(t))$ or $\mathrm{d}y(t) = f(y(t))\mathrm{d}t$.
What happens if $f$ has some uncertainties or if the force acting on the model is random?
One must consider Stochastic Differential Equation (SDE):

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t$$

or in integral form

$$X_t = X_0 + \int_0^t f(X_s)\,\mathrm{d}s + \int_0^t g(X_s)\,\mathrm{d}W_s.$$

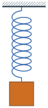
Here, the randomness is described by the term $g(X_t)\,\mathrm{d}W_t$. We thus first need to define what random means and what this $W_t$ is. Then, we should make sense of what it means that $X_t$ is a solution to the SDE. Finally, one needs to find good numerical methods to approximate such problems.

# Application in physics

Random harmonic oscillator: A model for a stochastic oscillator is the SDE ($\lambda, b, \sigma, x_0, x_1$ given)

$$\begin{cases} \ddot{X}_t = -\lambda^2 X_t - b\dot{X}_t + \sigma\zeta_t \\ X_0 = x_0 \\ \dot{X}_0 = x_1, \end{cases}$$

where $\zeta_t$ is a white noise (formally $\zeta_t = \dot{W}_t$).



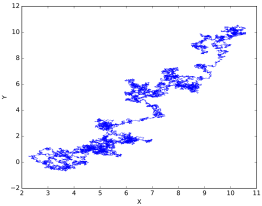
Thanks to wikipedia.org

# Application in physics

Particle in a gas: Consider a particle of unit mass moving with momentum $P_t$ at time $t$ in a gas.
It is subject to irregular bombardment by the ambient gas.
Newton's second law of motion gives us the SDE

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = -\lambda P_t + \sigma\zeta_t,$$

where $\lambda > 0$ is a dissipation constant and $\sigma\zeta_t$ a fluctuating force.




Thanks to wikipedia.org

# Application in finance

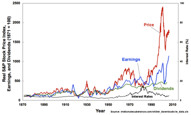
Financial modelling: Price $u(t)$ at time $t$ of a risk-free asset with interest rate $r$ obeys the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru.$$

On the stock market, stock prices fluctuate rapidly. We thus have to modify the constant $r$ by a stochastic process $r + \sigma\zeta_t$ with a given parameter $\sigma$ and a white noise $\zeta_t$. Here, $\sigma\zeta_t$ models the volatility of the market.

This gives us the SDE

$$\frac{\mathrm{d}u_t}{\mathrm{d}t} = (r + \sigma\zeta_t)u_t.$$



# Content of the crash course

SDE: $\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t$ or
$X_t = X_0 + \int_0^t f(X_s)\,\mathrm{d}s + \int_0^t g(X_s)\,\mathrm{d}W_s$.

Crash course on numerics for SDEs:

- Background material (stoch. process, stoch. integral, SDE)
- First numerical schemes and convergence types
- Strong convergence of Euler–Maruyama scheme
- Weak convergence of Euler–Maruyama scheme