Numerics for SDEs @ Changsha

0. Motivation / Introduction

See .pdf

1. Background material

<u>Goal</u>: Introduce or recall notions from probability theory.
This will be informal and not rigorous.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space

[$\Omega \to$ set of all possible outcomes of a random experiment
$\mathcal{A} \to$ set of all events
$\mathbb{P}: \mathcal{A} \to [0,1]$ probability measure]

ef. A random variable $X: \Omega \to \mathbb{R}$ is a variable whose possible values are the outcomes of a random phenomenon. Notation: RV

Ex Tossing a fair coin:
$\Omega = \{\text{Head}, \text{Tail}\} = \{H, T\}$.
Define RV: $X(H) = 1$, $X(T) = 0$ $\to$ assign values 1 for head, 0 for tail.
$\mathbb{P}(X=1) = \mathbb{P}(X=0) = \frac{1}{2} = \mathbb{P}(\{H\}) = \mathbb{P}(\{T\})$

ef. For a continuous RV X and $a<b$, $a,b \in \mathbb{R}$, we have
$$\mathbb{P}(a \leq X \leq b) = \int_a^b p_X(s)\,ds,$$
where p_X is called the probability density function of X. Notation: PDF

m: The above allows to compute the probability that a RV takes value in an interval $[a,b]$.

With the above, one can define 2 important quantities:

ef. • The expected value or mean of a RV X is given by
$$\mathbb{E}[X] = \int_{-\infty}^{\infty} s \cdot p_X(s)\,ds \qquad [\text{"long time behaviour of RV } X\text{"}]$$

• The variance of X reads
$$\mathrm{Var}[X] := \mathbb{E}[(X - \mathbb{E}(X))^2] \qquad [\text{"measure how } X \text{ spreads from mean"}]$$

Properties: $\forall$ RV X, Y and real α:

$\mathbb{E}[\alpha X + Y] = \alpha\, \mathbb{E}[X] + \mathbb{E}[Y]$

$\mathrm{Var}[\alpha X] = \alpha^2\, \mathrm{Var}[X]$

If X and Y are independent $\Rightarrow$ $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

Ex: Let $\mu, \sigma \in \mathbb{R}$ with $\sigma^2 > 0$. A RV X is normally distributed or is a normal RV or is a Gaussian RV if

$$p_X(s) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(s-\mu)^2}{2\sigma^2}}$$

Notation: $X \sim N(\mu, \sigma^2)$

Properties: $\mathbb{E}[X] = \mu$, $\mathrm{Var}[X] = \sigma^2$

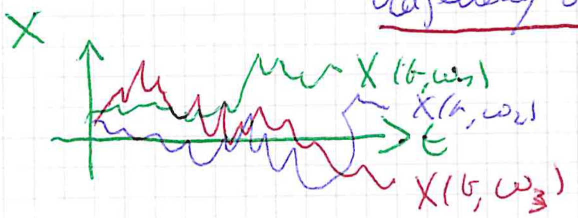
Def: A stochastic process is a family of RV $\{X(t), t \in \tau\} = \{X(t)\}_{t \in \tau}$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and indexed by a parameter t varying over a set τ. Notation: SP

Rem: • $X: \tau \times \Omega \to \mathbb{R}$

$(t, \omega) \longmapsto X(t, \omega)$ function of 2 variables

• $X(t, \cdot)$ is a RV for all fixed $t \in \tau$.

• $X(\cdot, \omega): \tau \to \mathbb{R}$ is called a sample path or realisation or trajectory of X for each fixed $\omega \in \Omega$.



Ex: A standard one-dimensional Brownian motion or Wiener process on $[0, T]$ is a real-valued stoch. process $\{W(t)\}_{t \in [0,T]}$ such that

(B0) Trajectories of $W(t)$ are a.s. continuous

(B1) $W(0) = 0$ a.s.

(B2) $\forall\, s < t$ $(\in [0,T])$, the increments $W(t) - W(s)$ are $N(0, t-s)$

(B3) $W(t)$ has independent increments:

$\forall\, 0 \leq t_1 < t_2 \leq t_3 < t_4 \leq T$: $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are indep.

Properties: a) $\mathbb{E}[W(t)] = 0$ $\forall\, t \in [0, T]$

b) $\mathbb{E}[W(t) \cdot W(s)] = \min(t, s)$ $\forall\, t, s \in [0, T]$

c) $W(t)$ is a.s. nowhere differentiable!

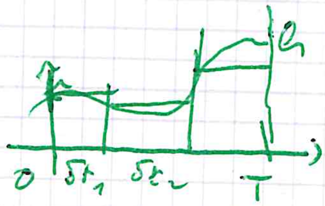
Notation: BM / WP $B(t)$ or $W(t)$.

We now want to integrate wrt BM $W(t)$!

Recall: Riemann integral for deterministic $h : \mathbb{R} \to \mathbb{R}$

For $L \gg 1$, let $\delta t = T/L$ and $t_i := i\delta t$, set

$$\int_0^T h(t)\,dt = \lim_{\delta t \to 0} \sum_{i=0}^{L-1} h(t_i)\,(t_{i+1} - t_i)$$



This motivates the following:

For h a nice SP, we define Ito stochastic integral by

$$\int_0^T h(t)\,dW(t) = \text{"}\lim_{\substack{\delta t \to 0 \\ \text{difficult!}}}\text{"} \sum_{i=0}^{L-1} h(t_i)\,\big(\underbrace{W(t_{i+1}) - W(t_i)}_{\sim N(0,\, t_{i+1} - t_i) \text{ by def of BM}}\big) \to \text{This is a RV!!}$$

Ex: Not the same as Riemann integrals!!

$$\int_0^T W(t)\,dW(t) = \frac{1}{2}W(t)^2 - \underbrace{\frac{1}{2}T}_{\text{extra term}}$$

Properties: $\forall$ stoch. process (nice) f, g $\forall$ real $a < b$, c:

(i) $\int_a^b (cf(t) + g(t))\,dW(t) = c\int_a^b f(t)\,dW(t) + \int_a^b g(t)\,dW(t)$

(ii) $\mathbb{E}\left[\int_a^b f(t)\,dW(t)\right] = 0$ $\forall$ nice SP f OBS: $\mathbb{E}[W(t)] = 0$

(iii) Ito's isometry $\mathbb{E}\left[\left(\int_a^b f(t)\,dW(t)\right)^2\right] = \int_a^b \mathbb{E}[f(t)^2]\,dt$ OBS: $\mathbb{E}[W(t)^2] = t$

We can now define SDE...

For a given BM $W(t)$ on some prob. space $(\Omega, \mathcal{A}, \mathbb{P})$. Let X_0 be a nice RV; $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be nice functions and $T > 0$.

Def: An Ito stochastic differential equations (SDE) on $[0,T]$ is defined as the integral equation:

$$X(t,\omega) = X_0(\omega) + \int_0^t f(s, X(s,\omega))\,ds + \int_0^t g(s, X(s,\omega))\,dW(s,\omega) \quad \forall\, 0 \le t \le T$$

Notation:

$$\boxed{\begin{cases} dX(t) = f(t, X(t))\,dt + g(t, X(t))\,dW(t) \\ X(0) = X_0 \end{cases}}$$

!This is just a notation!

Rem: • $X = X(t) = X(t,\omega)$ is a stoch. process.
• f is called the drift coefficient
g is called the diffusion coefficient

Ex: For $\mu, \sigma \in \mathbb{R}$, $(\sigma \neq 0)$. A SP X is called a geometric Brownian motion (gBM) if it solves the SDE:

$$\begin{cases} dX(t) = \underbrace{\mu X(t)}_{f} dt + \underbrace{\sigma X(t)}_{g} dW(t) \\ X(0) = X_0 \end{cases}$$

This SDE is used to model stock prices in the Black-Scholes model. The term $\sigma X dW$ describes the volatility of the price.

Exact sol.: $X(t) = X_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t) \right)$

Rem: $\mathbb{E}[X(t)] = \mathbb{E}[X_0] \exp(\mu t)$

⚠ Not always possible to find explicit sol. to SDE ⇒ need numerical methods to find good approximations of such solutions!!

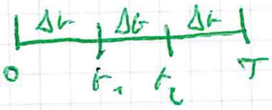
2) First numerical schemes and convergence types:

Goal: Define a couple of numerical schemes for SDE.
Discuss convergence.

Motivation: Consider ODE $\begin{cases} \frac{dX(t)}{dt} = f(X(t)) \\ X(0) = X_0 \end{cases}$ or $X(t) = X_0 + \int_0^t f(X(s))\,ds \quad \forall t \in [0,T]$.

Let $N > 0$ BIG integer, define step-size $\Delta t = \frac{T}{N}$ (small)

Consider discrete times $t_n = n \cdot \Delta t$



On a small interval $[t_n, t_{n+1}]$, we have

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(X(s))\,ds \underset{X(s) \approx X(t_n)}{\overset{\text{approx}}{\approx}} X(t_n) + f(X(t_n)) \cdot \int_{t_n}^{t_{n+1}} ds \approx$$

$$\approx X(t_n) + \Delta t\, f(X(t_n))$$

↳ Euler's scheme for ODE: $\begin{cases} X_{n+1} = X_n + \Delta t\, f(X_n) \\ X_0 = X(0) \end{cases}$

gives $\underbrace{X_n}_{\text{num. sol.}} \approx \underbrace{X(t_n)}_{\text{exact sol.}}$

In general, the order of convergence of Euler's scheme is $p=1$:

$$\underbrace{|X_N - X(T)|}_{\text{global error}} \leq C \cdot \Delta t^p \quad \text{with } p=1$$

We now proceed similarly for the SDE $\begin{cases} dX = f(X)dt + g(X)dW \\ X(0)=X_0 \end{cases}$

On the interval $[t_n, t_{n+1}]$, the exact sol. reads

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(X(s))ds + \int_{t_n}^{t_{n+1}} g(X(s))dW(s) \underset{X(s)\approx X(t_n)}{\overset{\text{approx}}{\approx}}$$

$$\approx X(t_n) + f(X(t_n))\int_{t_n}^{t_{n+1}} ds + g(X(t_n))\int_{t_n}^{t_{n+1}} dW(s)$$

$$\approx X(t_n) + f(X(t_n))\Delta t + g(X(t_n))(W(t_{n+1}) - W(t_n))$$

$$\approx X(t_n) + f(X(t_n))\Delta t + g(X(t_n))\Delta W_n \quad \text{, where } \Delta W_n = W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, \Delta t)$$

This motivates the Euler-Maruyama scheme (EM)

$$\boxed{\begin{aligned} X_{n+1} &= X_n + f(X_n)\Delta t + g(X_n)\Delta W_n \\ X_0 &= X(0) \end{aligned}}$$

gives numerical approximation $\underset{\text{num}}{X_n} \approx \underset{\text{exact}}{X(t_n)}$

A slight modification of EM, gives backward EM (BEM)

$$X_{n+1} = X_n + f(\underline{X_{n+1}})\Delta t + g(X_n)\Delta W_n$$

We will also consider Milstein's scheme. (Mil)

$$X_{n+1} = X_n + f(X_n)\Delta t + g(X_n)\Delta W_n + \frac{1}{2}g'(X_n)g(X_n)(\Delta W_n^2 - \Delta t)$$

? How do we measure the error $X_n - X(t_n)$ between 2 SP ??

Def: Given a function $\Phi: \mathbb{R}\to\mathbb{R}$, the weak error

$$err_{\Delta t}^{W} := \sup_{0\leq t_n\leq T} \left| \mathbb{E}[\Phi(X_n)] - \mathbb{E}[\Phi(X(t_n))] \right|$$

measures how well the numerical sol. $(X_n)_{n=1}^N$ approximates expectation of the exact sol. $\mathbb{E}[\Phi(X(t_n))]$.

Ex) $\Phi(X) = X \rightsquigarrow$ error in mean, $\Phi(X) = X^2 \rightsquigarrow$ error in second moment.

This type of error is often used in mathematical finance.

Def. • A numerical method $(X_n)_{n=1}^N$ converges weakly if for every fct Φ of a certain class, $err^W_{\Delta t} \to 0$ as $\Delta t \to 0$.

• A numerical method $(X_n)_{n=1}^N$ has weak order of convergence p if $\exists$ $k>0, \Delta t^* \geq 0$ s.t.

$$err^W_{\Delta t} \leq k \cdot \Delta t^p \qquad \forall\, 0 \leq \Delta t < \Delta t^*$$

Ex. In general $p_{EM} = 1$, $p_{Mil} = 1$, $p_{BEM} = 1$ (see below)

Def. The strong error of a numerical scheme reads

$$err^S_{\Delta t} := \sup_{0 \leq t_n \leq T} \mathbb{E}[\,|X_n - X(t_n)|\,]$$

Rem. • "strong error = mean of error, weak error = error in mean"

• Mean-square error $\sqrt{\mathbb{E}[\,|X_n - X(t_n)|^2]}$ is also used

Def. • A num. scheme converges strongly if $err^S_{\Delta t} \to 0$ for $\Delta t \to 0$.

• A " " has strong order of convergence q if

$$err^S_{\Delta t} \leq k \cdot \Delta t^q$$

$\forall$ small enough Δt.

Ex. In general $q_{EM} = \frac{1}{2}$ ⚠ $q_{EM} = \frac{1}{2} \neq 1 = p_{EM}$ ⚠

(see below)

$q_{BEM} = \frac{1}{2}$, $q_{Mil} = 1$

Rem. • Weak convergence is indeed weaker than strong convergence for Φ Lipschitz $(|\Phi(x) - \Phi(y)| \leq L|x-y|)$:

weak error $|\mathbb{E}[\Phi(X_n)] - \mathbb{E}[\Phi(X(t_n))]| \overset{\Delta}{\leq} \mathbb{E}[\,|\Phi(X_n) - \Phi(X(t_n))|\,]$

$$\overset{Lip}{\leq} L\, \underbrace{\mathbb{E}[\,|X_n - X(t_n)|\,]}_{\text{strong error!}}$$

• Not sharp since $q_{EM} = \frac{1}{2}$, $p_{EM} = 1$!

• Need to approximate $\mathbb{E}[RV]$ ⚠

⇝ may introduce statistical errors

⇝ Use Monte-Carlo algorithm: $\mathbb{E}[X] \approx \mathbb{E}^M[X] = \frac{1}{M} \sum_{k=1}^{M} X^k$

3 Strong convergence of Euler–Maruyama scheme:

Recall: Problem: SDE $\begin{cases} dX(t) = f(X(t))\,dt + g(X(t))\,dW(t) \\ X(0) = X_0 \end{cases}$ for $t \in [0,T]$

EM-scheme: $X_{n+1} = X_n + f(X_n)\Delta t + g(X_n)\Delta W_n$, where
$\Delta t = \frac{T}{N}$, $t_n = n\cdot\Delta t$ for $n = 0,1,2,\dots,N$ and $X_n \approx X(t_n)$.

Assumptions:

(A1) The coeff. $f,g:\mathbb{R}\to\mathbb{R}$ are globally Lipschitz:
$\exists L>0$ s.t. $|f(x)-f(y)| \leq L\cdot|x-y|$
$|g(x)-g(y)| \leq L\cdot|x-y|$ $\quad \forall x,y\in\mathbb{R}$

(A2) X_0 is non-random (simpler)

(A3) Exact and numerical sol. have bounded second moments:

$\mathbb{E}[X(t)^2]<\infty$ and $\mathbb{E}[X_n^2]<\infty$ $\quad \forall t\in[0,T]$ and $0\leq n\leq N$.

Rem. (A1) implies linear growth condition: $f(x)^2 \leq K\cdot(1+|x|^2)$ and for g.

$$\Big[\,|f(x)|^2 = |f(x)-f(0)+f(0)|^2 \overset{(a+b)^2\leq 2(a^2+b^2)}{\leq} 2\left(|f(x)-f(0)|^2 + |f(0)|^2\right) \overset{\text{Lip.}}{\leq}$$

$$\leq 2\left(L^2|x-0|^2 + |f(0)|^2\right) \leq K\cdot(1+|x|^2)\,\Big]$$

Th. Under (A1)–(A3), Euler–Maruyama's scheme has strong order of convergence $\frac{1}{2}$:
$$\sup_{0\leq n\leq N}\mathbb{E}[|X_n - X(t_n)|] \leq C\cdot\Delta t^{1/2}$$

Proof:

- Since EM only defined at grid point $t_n = n\Delta t$, define SP
$\bar{X}(t) := X_n$ for $t_n \leq t < t_{n+1}$ [piecewise constant process]

Define the quantity $Z(t) := \sup_{0\leq s\leq t}\mathbb{E}[|\bar{X}(s)-X(s)|^2]$ [error second moment]

Given any $s\in[0,T]$, we define n_s to be the integer such that
$s\in[t_{n_s}, t_{n_s+1}[$, i.e. $\bar{X}(s) = X_{n_s}$ by definition of $\bar{X}$
→ EM approx!!

For the error, we thus obtain

$$\bar{X}(s) - X(s) = X_{n_s} - X(s) \overset{\text{def } X(s)}{=} X_{n_s} - \left(X_0 + \int_0^s f(X(r))\,dr + \int_0^s g(X(r))\,dW(r) \right) =$$

$$\underset{\text{sum}}{\overset{\text{telescopic}}{=}} \sum_{i=0}^{n_s - 1} (X_{i+1} - X_i) - \int_0^s f(X(r))\,dr - \int_0^s g(X(r))\,dW(r) \overset{\text{Def EM}}{=}$$

$$= \sum_{i=0}^{n_s-1} f(X_i)\Delta t + \sum_{i=0}^{n_s-1} g(X_i)\Delta W_i - \int_0^s f(X(r))\,dr - \int_0^s g(X(r))\,dW(r) \quad (*)$$

By construction of the process $\bar{X}(t)$, we have:

$$\int_{t_i}^{t_{i+1}} f(\bar{X}(r))\,dr = \int_{t_i}^{t_{i+1}} f(X_i)\,dr = f(X_i)\Delta t \quad \text{and} \quad \int_{t_i}^{t_{i+1}} g(\bar{X}(r))\,dW(r) = g(X_i)\Delta W_i$$

We can thus rewrite (*) as follows:

$$\bar{X}(s) - X(s) = \sum_{i=0}^{n_s-1} \left(\int_{t_i}^{t_{i+1}} f(\bar{X}(r))\,dr + \int_{t_i}^{t_{i+1}} g(\bar{X}(r))\,dW(r) \right) - \int_0^s f(X(r))\,dr - \int_0^s g(X(r))\,dW(r) =$$

$$\overset{s \in [t_{n_s}, t_{n_s+1}[}{=} \int_0^{t_{n_s}} f(\bar{X}(r))\,dr - \int_0^{t_{n_s}} f(X(r))\,dr$$

$$+ \int_0^{t_{n_s}} g(\bar{X}(r))\,dW(r) - \int_0^{t_{n_s}} g(X(r))\,dW(r)$$

$$- \int_{t_{n_s}}^s f(X(r))\,dr - \int_{t_{n_s}}^s g(X(r))\,dW(r).$$

- We take the square of above and expectation.

Using $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, we then obtain:

$$\mathbb{E}\left[(\bar{X}(s) - X(s))^2 \right] \leq 4 \cdot \left\{ \mathbb{E}\left[\left| \int_0^{t_{n_s}} (f(\bar{X}(r)) - f(X(r)))\,dr \right|^2 \right] \right.$$

$$+ \mathbb{E}\left[\left| \int_0^{t_{n_s}} (g(\bar{X}(r)) - g(X(r)))\,dW(r) \right|^2 \right] + \mathbb{E}\left[\left| \int_{t_{n_s}}^s f(X(r))\,dr \right|^2 \right] + \mathbb{E}\left[\left| \int_{t_{n_s}}^s g(X(r))\,dW(r) \right.\right.$$

We next bound these 4 terms:

$$\mathbb{E}\left[\left| \int_0^{t_{n_s}} (f(\bar{X}(r)) - f(X(r)))\,dr \right|^2 \right] \overset{\text{Hölder ineq}}{\leq} t_{n_s} \cdot \int_0^{t_{n_s}} \mathbb{E}\left[|f(\bar{X}(r)) - f(X(r))|^2 \right] dr \leq$$

$$\underset{t_{n_s} \leq T}{\overset{\text{Lipschitz}}{\leq}} T \cdot L^2 \cdot \int_0^{t_{n_s}} \mathbb{E}\left[|\bar{X}(r) - X(r)|^2 \right] dr \overset{\text{def } z(r)}{\leq} T \cdot L^2 \cdot \int_0^s z(r)\,dr \quad \text{since } t_{n_s} \leq s.$$

Similarly, we obtain

$$\mathbb{E}\left[\left| \int_{t_{n_s}}^s f(X(r))\,dr \right|^2 \right] \overset{\text{Hölder}}{\leq} |s - t_{n_s}| \int_{t_{n_s}}^s \mathbb{E}\left[f(X(r))^2 \right] dr \overset{\text{linear growth condition}}{\leq}$$

$$\leq \underline{|s - t_{n_s}|} \cdot K \cdot \int_{t_{n_s}}^s \mathbb{E}\left[1 + X(r)^2 \right] dr \leq C \cdot \Delta t^2 \quad \text{since bounded second moment}$$

For the terms containing stoch. integrals, we use Itô isometry:

$$\mathbb{E}\Big[\Big|\int_0^{t_{n_s}}\big(g(\bar{X}(r)) - g(X(r))\big)dW(r)\Big|^2\Big] = \int_0^{t_{n_s}} \mathbb{E}\Big[\big|g(\bar{X}(r)) - g(X(r))\big|^2\Big]\,dr \overset{\text{Lipschitz}}{\leq}$$

$$\leq L^2 \int_0^{t_{n_s}} \mathbb{E}\big[|\bar{X}(r) - X(r)|^2\big]dr \leq L^2 \int_0^{s} Z(r)\,dr \quad \text{using } t_{n_s} \leq s \text{ and def. } Z(s).$$

Similarly, we have

$$\mathbb{E}\Big[\Big|\int_{t_{n_s}}^{s} g(X(r))\,dW(r)\Big|^2\Big] = \int_{t_{n_s}}^{s} \mathbb{E}\big[|g(X(r))|^2\big]\,dr \leq K^2 \int_{t_{n_s}}^{s} 1 + \underbrace{\mathbb{E}\big[|X(r)|^2\big]}_{\text{bounded by assumption}}\,dr \leq$$

$$\leq \text{Const.}\,\Delta t \quad \text{since } |s - t_{n_s}| \leq \Delta t$$

- Collecting the above estimates, we arrive at

$$Z(s) \leq 4 \cdot \Big\{ T L^2 \int_0^s Z(r)\,dr + C_1 \Delta t^2 + L^2 \int_0^s Z(r)\,dr + C_2 \cdot \Delta t \Big\} \leq$$

$$\leq D \cdot \Delta t + F \int_0^s Z(r)\,dr \quad \text{for } 0 \leq s \leq T.$$

An application of Gronwall now gives

$$Z(s) \leq K \cdot \Delta t \quad \text{where} \quad K = D \cdot \exp(F \cdot T)$$

We have thus shown:

$$\mathbb{E}\big[|\bar{X}(s) - X(s)|^2\big] \leq K \cdot \Delta t \quad \forall\, 0 \leq s \leq T \text{ or in particular}$$

$$\mathbb{E}\big[|X_n - X(t_n)|^2\big] \leq K \cdot \Delta t \quad \forall\, 0 \leq t_n \leq T.$$

We conclude using Lyapunov's inequality $\Big[\ \mathbb{E}[X] \leq \sqrt{\mathbb{E}[X^2]}\ \Big]$

$$\sup_{0 \leq t_n \leq T} \mathbb{E}\big[|X_n - X(t_n)|\big] \leq K \cdot \Delta t^{1/2}$$

$\boxtimes$

Rem. One can relax the above assumptions on f and g → ask Xiaojie Wang.

) Weak convergence of Euler-Maruyama scheme:

Recall: (SDE) $\begin{cases} dX(t) = f(X(t))\,dt + g(X(t))\,dW(t) \\ X(0) = X_0 \end{cases}$ for $0 \leq t \leq T$.

Consider any numerical scheme $(X_n)_{n \geq 0}^{N}$, f. ex EM.

Weak error $\big|\mathbb{E}[\Phi(X_N)] - \mathbb{E}[\Phi(X(T_N))]\big| \leq ??$

Assumptions:

(A1) f, g globally Lipschitz

(A2) $f, g \in \mathcal{C}_p^{2p+2}(\mathbb{R}, \mathbb{R})$ for some p (integer or half integer)

Rem: $\mathcal{C}_p^{2p+2}(\mathbb{R},\mathbb{R}) = \{ h: \mathbb{R} \to \mathbb{R} : h \in \mathcal{C}^{2p+2}$ with polynomial growth, i.e.

$\exists\, C, k > 0$ s.t. $\forall m \le 2p+2, \forall x \in \mathbb{R}\ |h(x)| + |h^{(m)}(x)| \le C \cdot (1+|x|^k) \}$

Ex: $h(x) = x^5$ or $h(x) = x^8$ or $h(x) = \sin(x)$ are in $\mathcal{C}_p^{2p+2}$

Theorem [Taley, Tubaro, Platen, Tretyakov]

Under the above assumptions and if the numerical sol. $(X_n)_{n\ge 1}^{N}$ satisfies:

(i) Bounded moments: $\mathbb{E}[|X_n|^{2m}] \le C_m < \infty \quad \forall$ integers m, n

(ii) Local weak order p:

$$|\mathbb{E}[\Phi(X_1)] - \mathbb{E}[\Phi(X(\Delta t))]| \le C(1+|x|^k)\,\Delta t^{p+1} \quad \forall \Phi \in \mathcal{C}_p^{2p+2}$$

Then the global weak error satisfies

$$\sup_{0\le n\le N} |\mathbb{E}[\Phi(X(t_n))] - \mathbb{E}[\Phi(X_n)]| \le C \cdot \Delta t^p \qquad \forall \Phi \in \mathcal{C}_p^{2p+2},$$

where the constant C does not depend on n or Δt.

Application: We use the above to show that $p_{EM} = 1$, i.e. weak error for EM is 1.

(i) We shall use the following result

Lemma (Milstein 1986)

Let $\{X_n\}_{n=0,1,2,}$ be a discrete stoch. process such that

(a) $|\mathbb{E}[X_0^{2m}]| < \infty \quad \forall m$

(b) $|\mathbb{E}[X_{n+1} - X_n \mid X_n = x]| \le k \cdot (1+|x|)\Delta t$ [conditional expectation]

(c) $|X_{n+1} - X_n| \le M_n (1+|X_n|)\sqrt{\Delta t}$ with $\mathbb{E}[M_n^{2m}] \le C_m \quad \forall m$

where C_m indep. of n and X_n,

Then, for all n, $\exists\, \hat{C}_m$ s.t. $\mathbb{E}[|X_n|^{2m}] \le \exp(\hat{C}_m n \Delta t)(\mathbb{E}[X_0^{2m}+1])$

In particular, $\forall n\Delta t \leq T$, one has $\mathbb{E}[|X_n|^{2m}] \leq \tilde{C}_m$ indep. of n, Δt.

The hypothesis of the Lemma are verified for EM:

(a) OK

(b) $\mathbb{E}[X_{n+1} - X_n \mid X_n = x] \overset{\text{Def EM}}{=} \mathbb{E}[f(X_n)\Delta t \mid X_n = x] + \underbrace{\mathbb{E}[g(X_n)\Delta W_n \mid X_n = x]}_{=0 \text{ since } \mathbb{E}[\Delta W_n]=0 \text{ and } X_n \text{ indep. of } \Delta W_n}$

$= f(x)\Delta t \underset{f \text{ linear growth}}{\leq} K\cdot(1+|x|)\Delta t$ ok

(c) $|X_{n+1} - X_n| \underset{\text{Def EM}}{\leq} \Delta t\,|f(X_n)| + |g(X_n)||\Delta W_n| \underset{f,g \text{ lin. growth}}{\leq} K\cdot\Delta t(1+|X_n|) + K\cdot(1+|X_n|)\cdot\left|\frac{\Delta W_n}{\sqrt{\Delta t}}\right|\sqrt{\Delta t}$

$\underset{\Delta t \leq \sqrt{\Delta t} \text{ for } \Delta t < 1}{\leq} M_n \cdot \sqrt{\Delta t}\cdot(1+|X_n|)$ ok

Using the fact that $\mathbb{E}[M_n^{2m}] := \mathbb{E}\left[\left|\frac{\Delta W_n}{\sqrt{\Delta t}}\right|^{2m}\right] \leq C_m$ since

$\frac{\Delta W_n}{\sqrt{\Delta t}} \sim N(0,1)$ and thus has bounded moments.

$\hookrightarrow$ EM has bounded moments

ii) local weak order $p=1$ of EM:

We compute

$\mathbb{E}[\Phi(X_1)] = \mathbb{E}[\Phi(X_0 + \Delta t f(X_0) + \Delta W_0 g(X_0))] \overset{\text{Taylor}}{=} \mathbb{E}[\Phi(X_0) + \Phi'(X_0)(\Delta t f(X_0) + \Delta W_0 g(X_0))$

$+ \frac{1}{2}\Phi''(X_0)(\Delta t f(X_0) + \Delta W_0 g(X_0))^2 + \text{Rest}] =$

$\overset{(*)}{=} \underbrace{\Phi(X_0) + \Delta t\,\Phi'(X_0) f(X_0) + \frac{\Delta t}{2}\Phi''(X_0) g^2(X_0)}_{\text{first term Taylor series exact solution! (no proof)}} + \mathcal{O}(\Delta t^2)$

(*) $\text{Rest} = \frac{1}{3!}\int_0^1 \Phi^{(4)}(X_0 + \theta\Delta t)\Delta x^4(1-\theta)^4\,d\theta$ write $X_1 := X_0 + \Delta x$

$\Rightarrow |\text{Rest}| \leq C\cdot(1 + (|X_0|^k + |\Delta x|^k)\Delta x^4)$ since $\Phi^{(4)}$ has polynomial growth

$\leq \tilde{C}\cdot(1 + \{|X_0|^{\tilde{k}}\})\Delta t^2$ since $|\Delta x| = |X_1 - X_0| \leq C\cdot(1+|X_0|)M_1\sqrt{\Delta t}$