Multi-Level Monte Carlo

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Outline

- Weak versus strong convergence
- Financial options
- Complexity of Monte Carlo

- Multi-level Monte Carlo (**Mike Giles**, Report NA-06/03, Oxford Un. Comp. Lab., 2006)

- New strong convergence results for path-dependent options—joint work with **Mike Giles** and **Xuerong Mao**

Weak versus Strong

SDE:

$$d\mathbf{S}(t) = a(\mathbf{S}(t))\,dt + b(\mathbf{S}(t))\,d\mathbf{W}(t)$$

$\mathbf{S}(0)$ given and $0 \leq t \leq T$

Euler–Maruyama

$$\mathbf{S}_{n+1} = \mathbf{S}_n + a(\mathbf{S}_n)h + b(\mathbf{S}_n)\Delta\mathbf{W}_n$$

$$\Delta\mathbf{W}_n := \mathbf{W}(t_{n+1}) - \mathbf{W}(t_n), \quad t_n = nh, \quad h = T/K$$

Weak versus Strong

Weak Convergence $\left|\mathbb{E}\left[\mathbf{S}(t_n)\right] - \mathbb{E}\left[\mathbf{S}_n\right]\right| \leq Ch$

Strong Convergence

$$\mathbb{E}\left[\sup_{0\leq n\leq K} \left|\mathbf{S}(t_n) - \mathbf{S}_n\right|\right] \leq Ch^{\frac{1}{2}}$$

Strong convergence $+$ Markov inequality $\Rightarrow$

$$\mathbb{P}\left(\left|\mathbf{S}(t_n) - \mathbf{S}_n\right| \geq h^{\alpha}\right) \leq Ch^{\frac{1}{2} - \alpha}$$

Continuous Time/Higher Moments

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left|\mathbf{S}(t) - \mathbf{S}(t)\right|^m\right] \leq C_{m,\delta} h^{\frac{m}{2} - \delta}$$

Weak versus Strong

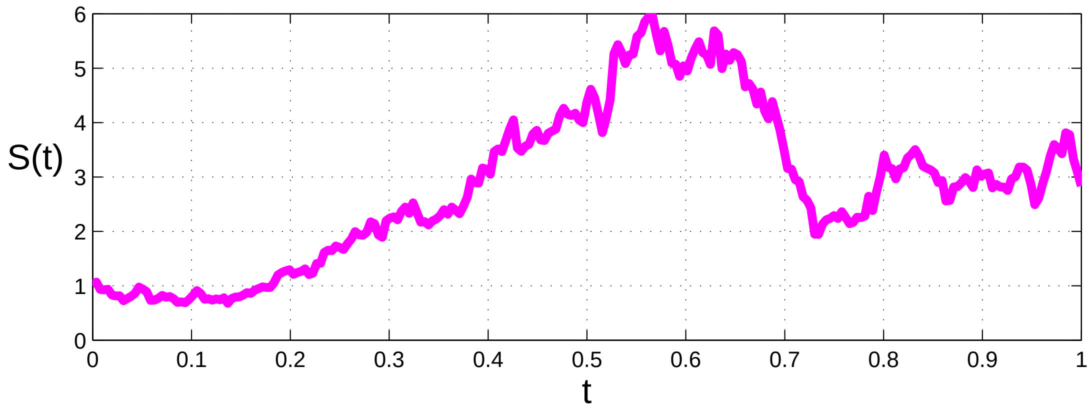
Which is more relevant, **weak** or **strong**?

Conventional wisdom:
Weak convergence is usually enough. Most problems require **expected value** type information.

Strong convergence covers cases where we want to **visualize paths** or generate **time series** (e.g. to test a filtering algorithm or a parameter fitting algorithm).

Financial Options

Now $\mathbf{S}(t)$ represents the **asset price**



Option Payoffs:

European call: $\max(\mathbf{S}(T) - E, 0)$

Digital: $\mathbf{1}_{\mathbf{S}(T)>E}$

Lookback: $\mathbf{S}(T) - \min_{0\le t\le T} \mathbf{S}(t)$

Up and out: $\max(\mathbf{S}(T) - E, 0) \times \mathbf{1}_{\left(\sup_{0\le t\le T} \mathbf{S}(t)\right)\le B}$

Task: compute $\mathbb{E}[\mathrm{Payoff}]$

Monte Carlo for SDEs

Approximate $\mathbb{E}[\mathbf{S}(T)]$ by applying E-M to get samples.

Let $\mu = \frac{1}{N}\sum_{i=1}^{N} S_K^{[i]}$

Then

$$\begin{aligned}\mathbb{E}[\mathbf{S}(T)] - \mu &= \mathbb{E}[\mathbf{S}(T) - \mathbf{S}_K + \mathbf{S}_K] - \mu \\ &= \mathbb{E}[\mathbf{S}(T) - \mathbf{S}_K] + \mathbb{E}[\mathbf{S}_K] - \mu\end{aligned}$$

Confidence interval width is $O(h) + O(1/\sqrt{N})$

For confidence interval of $O(\epsilon)$, choose $h = 1/\sqrt{N} = \epsilon$

Computational cost is $N \times 1/h$

Hence, computational complexity is $O(\epsilon^{-3})$

Monte Carlo for SDEs

Extrapolated E-M (Talay and Tubaro): same cost of $N \times 1/h$ gives confidence interval width $O(h^2) + O(1/\sqrt{N})$

Hence, choose $h^2 = 1/\sqrt{N} = \epsilon$ to get computational complexity of $O(\epsilon^{-2.5})$

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The **Multi-level Monte Carlo** algorithm will achieve computational complexity of

$$O(\epsilon^{-2} \log(\epsilon)^2)$$

using E-M, and giving good results in practice

Key idea: Use a range of h values
many paths at large h, few paths at small h

Multi-level Monte Carlo

Consider payoff $f(\mathbf{S}(T))$, where f is globally Lipschitz.
ϵ is required accuracy (conf. int.)

Timesteps $h_l = M^{-l}T, \quad l = 0, 1, 2, \ldots, L$

M is fixed and $L = \frac{\log \epsilon^{-1}}{\log M}$, so that $h_L = O(\epsilon)$

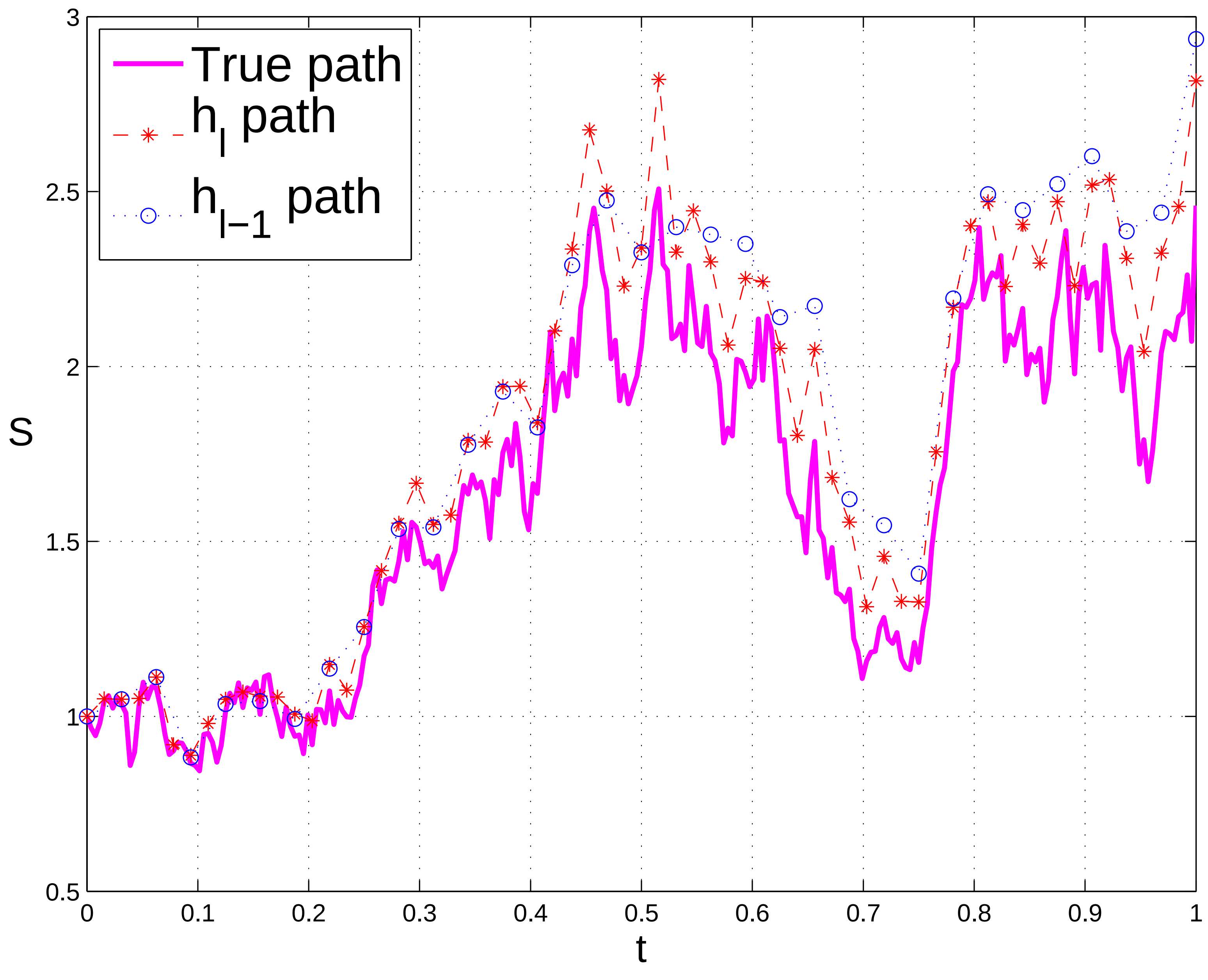
$\widehat{\mathbf{P}}_l$ denotes E-M approx. to $f(\mathbf{S}(T))$ using h_l. Clearly

$$\mathbb{E}\left[\widehat{\mathbf{P}}_L\right] = \mathbb{E}\left[\widehat{\mathbf{P}}_0\right] + \sum_{l=1}^{L} \mathbb{E}\left[\widehat{\mathbf{P}}_l - \widehat{\mathbf{P}}_{l-1}\right]$$

$\widehat{Y}_0$ estimates $\mathbb{E}[\widehat{\mathbf{P}}_0]$ using N_0 paths, and
$\widehat{Y}_l$ estimates $\mathbb{E}[\widehat{\mathbf{P}}_l - \widehat{\mathbf{P}}_{l-1}]$ using N_l paths:

$$\widehat{Y}_l = \frac{1}{N_l} \sum_{i=1}^{N_l} \left(\widehat{P}_l^{[i]} - \widehat{P}_{l-1}^{[i]}\right)$$

Multi-level Monte Carlo ($M = 2$)



Multi-level Monte Carlo

Strong convergence of E-M $+$ glob. Lip. f give

$$\mathsf{var}\left[\widehat{\mathbf{P}}_l - f\left(\mathbf{S}(T)\right)\right] \leq \mathbb{E}\left[\left(\widehat{\mathbf{P}}_l - f\left(\mathbf{S}(T)\right)\right)^2\right] = O(h_l)$$

and

$$\mathsf{var}\left[\widehat{\mathbf{P}}_l - \widehat{\mathbf{P}}_{l-1}\right]$$

$$\leq \left(\sqrt{\mathsf{var}\left[\widehat{\mathbf{P}}_l - f\left(\mathbf{S}(T)\right)\right]} + \sqrt{\mathsf{var}\left[\widehat{\mathbf{P}}_{l-1} - f\left(\mathbf{S}(T)\right)\right]}\right)^2 = O(h_l)$$

So $\widehat{Y}_l = \frac{1}{N_l}\sum_{i=1}^{N_l}\left(\widehat{P}_l^{[i]} - \widehat{P}_{l-1}^{[i]}\right)$ has variance of $O(h_l/N_l)$

Recap: $\mathbb{E}\left[\widehat{\mathbf{P}}_L\right] = \mathbb{E}\left[\widehat{\mathbf{P}}_0\right] + \sum_{l=1}^{L} \mathbb{E}\left[\widehat{\mathbf{P}}_l - \widehat{\mathbf{P}}_{l-1}\right]$

Estimator for RHS is $\widehat{Y} := \widehat{Y}_0 + \sum_{l=1}^{L} \widehat{Y}_l$

For $l > 1$, $\widehat{Y}_l = \frac{1}{N_l} \sum_{i=1}^{N_l} \left(\widehat{P}_l^{[i]} - \widehat{P}_{l-1}^{[i]}\right)$ and var $\left[\widehat{Y}_l\right] = O(h_l/N_l)$

$\Rightarrow$ var $\left[\widehat{Y}\right] =$ var $\left[\widehat{Y}_0\right] + \sum_{l=1}^{L} O(h_l/N_l)$

Take $N_l = O(\epsilon^{-2} L h_l)$, to give var $\left[\widehat{Y}\right] = O(\epsilon^2)$

Because $h_L = O(\epsilon)$, the bias $\mathbb{E}\left[\widehat{\mathbf{P}}_L - f\left(\mathbf{S}(T)\right)\right] = O(\epsilon)$

Computational complexity is

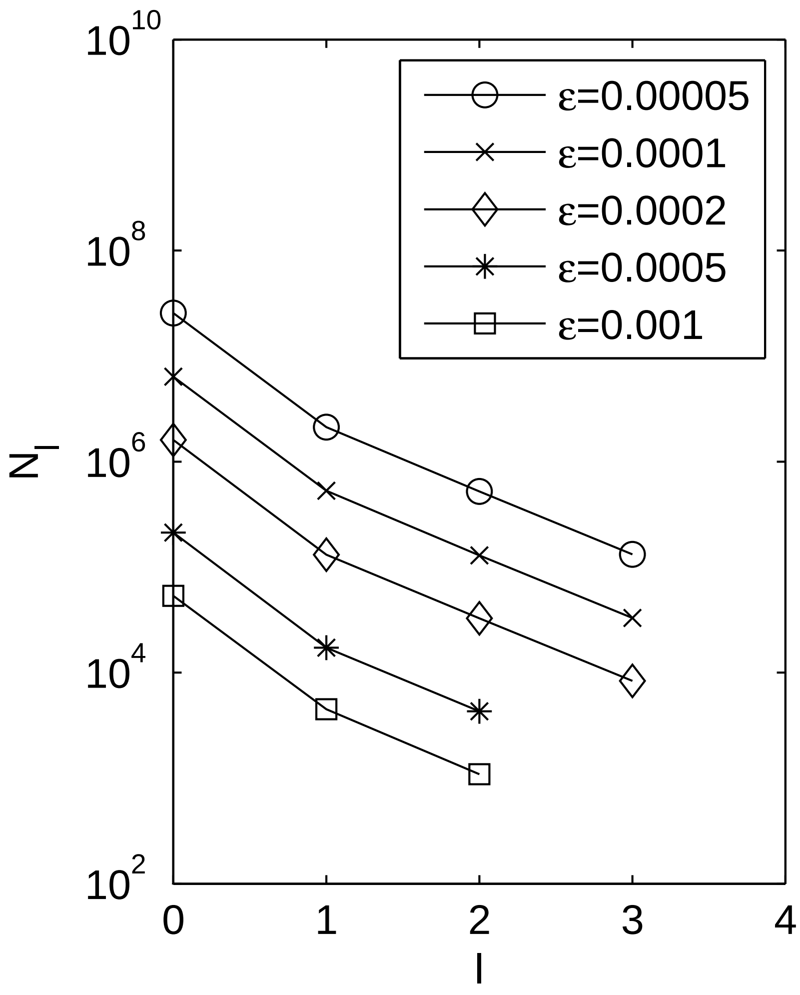
$$\sum_{l=0}^{L} N_l h_l^{-1} = \sum_{l=0}^{L} \epsilon^{-2} L h_l h_l^{-1} = L^2 \epsilon^{-2}$$

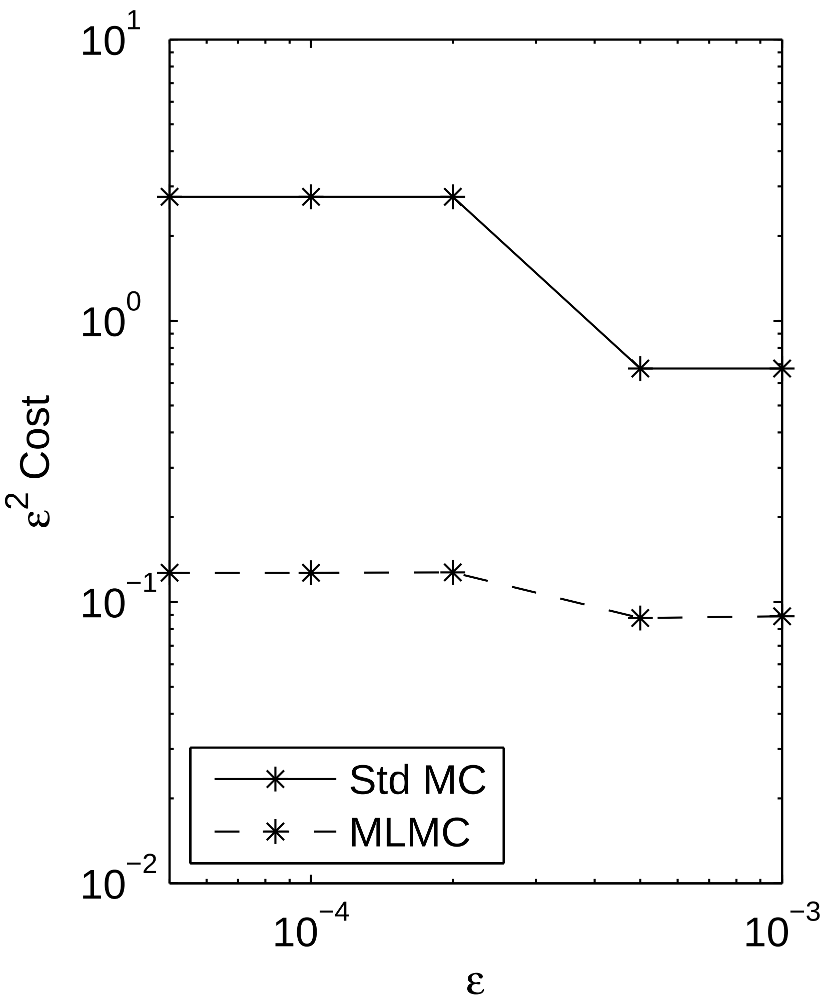
Since $L = \frac{\log \epsilon^{-1}}{\log M}$, this gives $O(\epsilon^{-2} (\log \epsilon)^2)$

Remarks

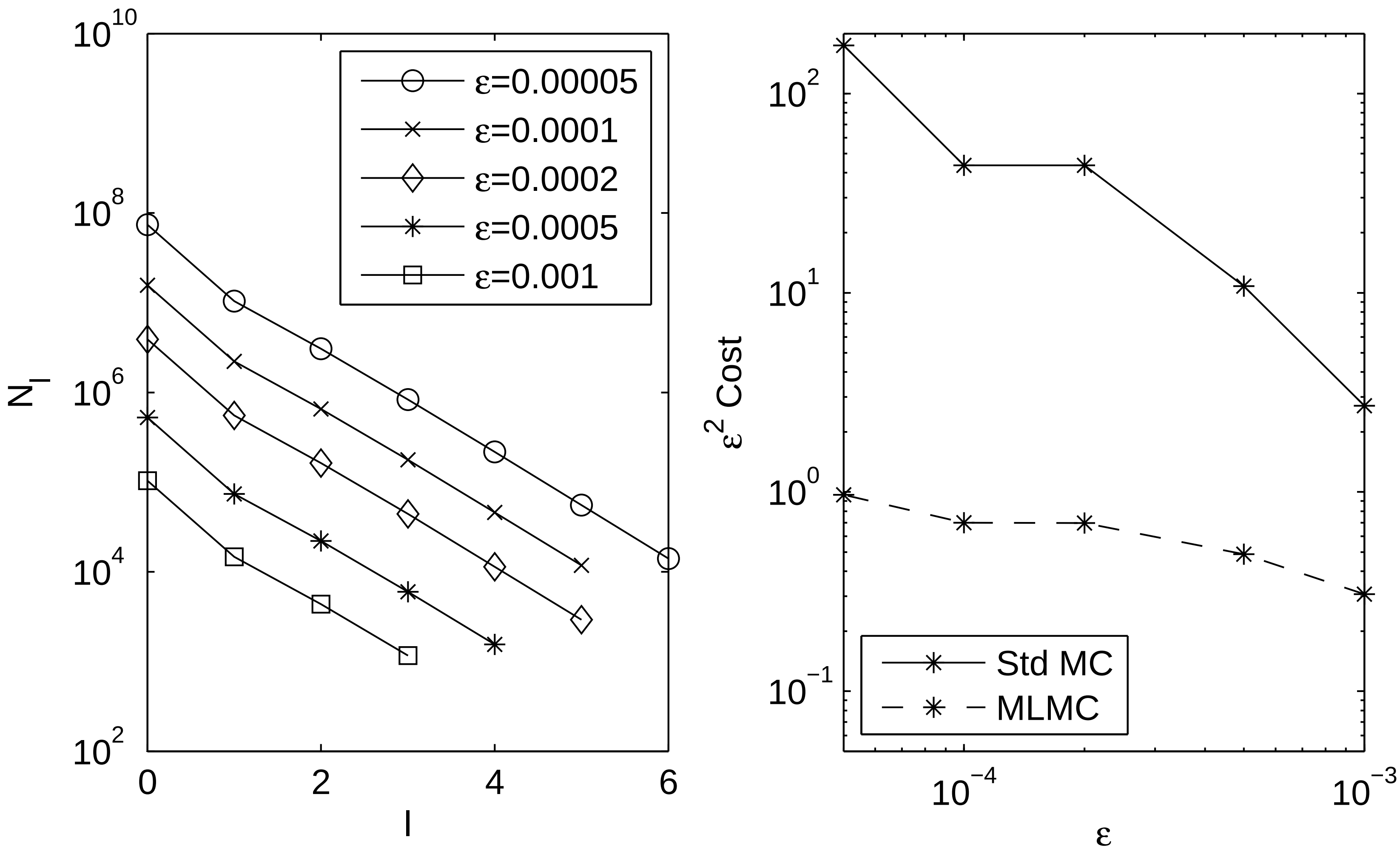
- Giles also gives an algorithm for adaptively choosing N_l and L
- Analysis uses **weak** and **strong** convergence of E-M
- Analysis was for payoff of the form $f\left(\mathbf{S}(T)\right)$, where f is globally Lipschitz

European with geom. Brownian motion

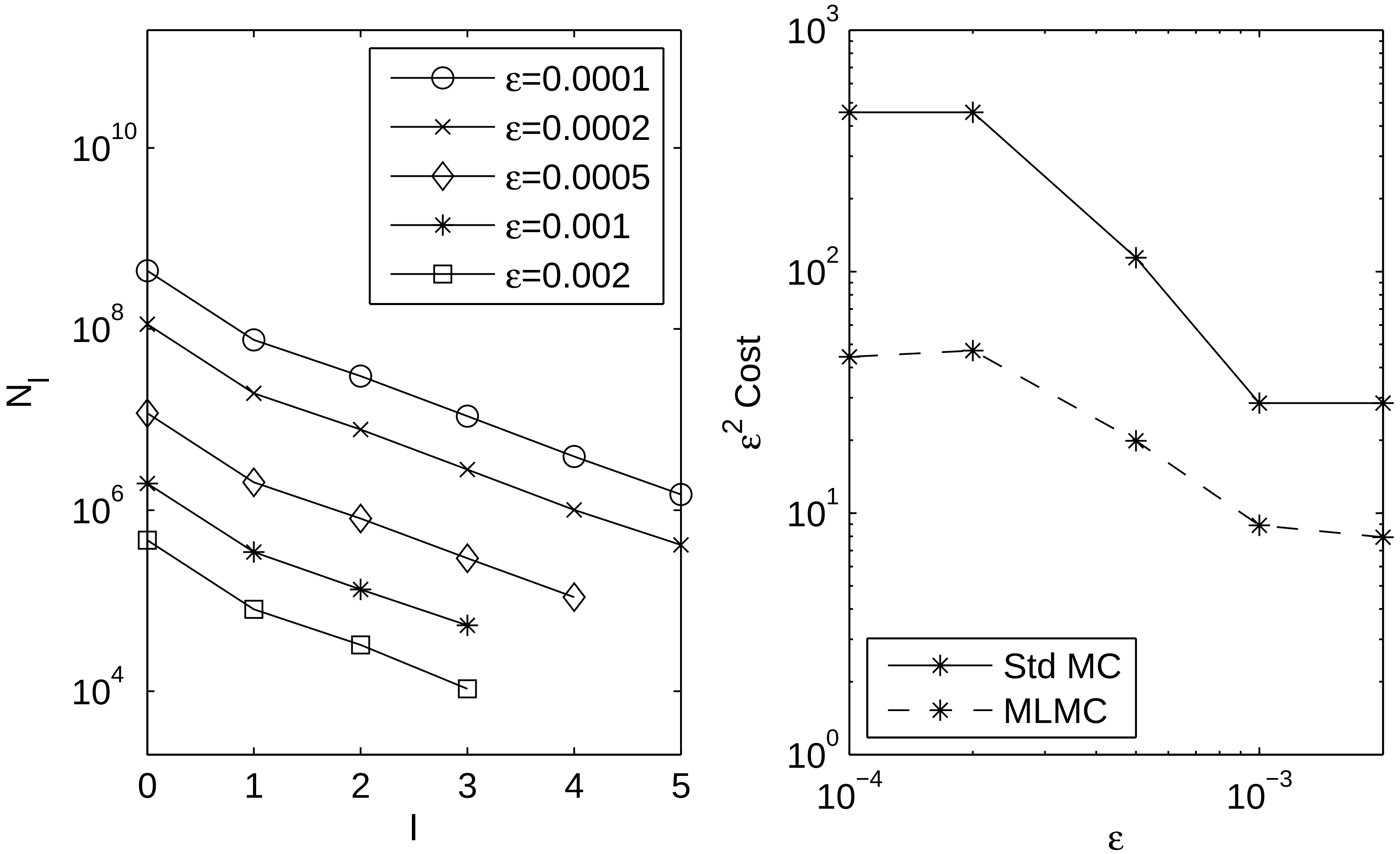




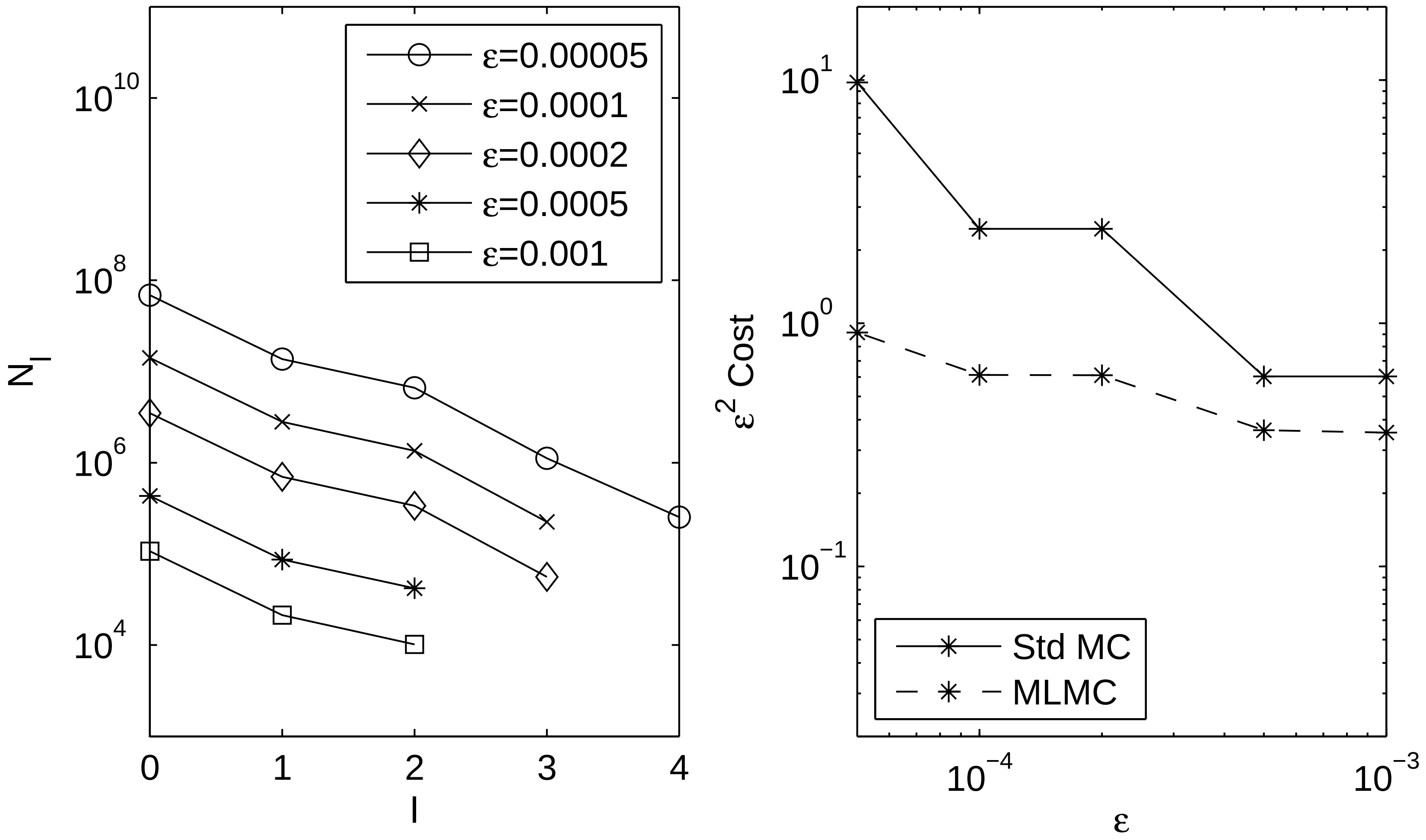
Lookback with geom. Brownian motion



Digital with geom. Brownian motion



European with Heston stoch. vol. model



Multi-level Monte Carlo

Giles (2006) analyses payoff $f\left(\mathbf{S}(T)\right)$, where f is glob. Lip.

Wish to analyse path-dependent options, e.g.

Digital: $\mathbf{1}_{\mathbf{S}(T)>E}$

Lookback: $\mathbf{S}(T) - \min_{0\leq t\leq T} \mathbf{S}(t)$

Up and out: $\max\left(\mathbf{S}(t) - E, 0\right) \times \mathbf{1}_{\left(\sup_{0\leq t\leq T} \mathbf{S}(t)\right)\leq B}$

Assume SDE coeffs are glob. Lip. (up and out fits well!)

New Analysis

Extending Giles (2006) reduces to getting

$$\mathbb{E}\left[\left(\mathbf{P} - \widehat{\mathbf{P}}\right)^2\right] \leq O\left(h^\beta\right)$$

where
$\mathbf{P}$ is true payoff,
$\widehat{\mathbf{P}}$ is Euler–Maruyama payoff

We know $\beta = 1$ for a **European call**

Numerical tests suggest $\beta = 1$ for **lookback**

but $\beta = \frac{1}{2}$ for **digital** & **up and out** ($\Rightarrow$ complexity $O(\epsilon^{2.5})$)

Lookback $\beta = 1$?

$\mathbf{P} = \mathbf{S}(T) - \min_{0 \le t \le T} \mathbf{S}(t)$

$\widehat{\mathbf{P}} = \mathbf{S}(T) - \min_{0 \le t \le T} \mathbf{S}(t)$

$$\begin{aligned}
\mathbb{E}\left[\left(\mathbf{P} - \widehat{\mathbf{P}}\right)^2\right] &\le 2\mathbb{E}\left[(\mathbf{S}(T) - \mathbf{S}(T))^2\right] \\
&\quad + 2\mathbb{E}\left[\left(\min_{0 \le t \le T} \mathbf{S}(t) - \min_{0 \le t \le T} \mathbf{S}(t)\right)^2\right] \\
&\le O(h) + 2\mathbb{E}\left[\max_{0 \le t \le T} (\mathbf{S}(t) - \mathbf{S}(t))^2\right] \\
&= O(h^{1-\delta})
\end{aligned}$$

Confirms $\beta = 1 - \delta$

Digital Option $\beta = \frac{1}{2}$?

$\mathbf{P} = \mathbf{1}_{\mathbf{S}(T) > E}$

$\widehat{\mathbf{P}} = \mathbf{1}_{\mathbf{S}_K > E}$

Given any $0 < \epsilon < \frac{1}{2}$, choose m such that

$$\frac{1}{2m+1} < \epsilon$$

and let

$$\widehat{\beta} := \frac{1}{2} - \frac{1}{2m+1} > \frac{1}{2} - \epsilon$$

Now,

$$\mathbb{E}\left[\left(\mathbf{P} - \widehat{\mathbf{P}}\right)^2\right] = \mathbb{P}\left(\{\mathbf{S}(T) > E\} \cap \{\mathbf{S}_K \le E\}\right) + \mathbb{P}\left(\{\mathbf{S}(T) \le E\} \cap \{\mathbf{S}_K > E\}\right)$$

Digital Option $\beta = \frac{1}{2}$?

$$\mathbb{P}\left(\{\mathbf{S}(T) > E\} \cap \{\mathbf{S}_K \leq E\}\right) =$$

$$\mathbb{P}\left(\left\{E + h^{\widehat{\beta}} \geq \mathbf{S}(T) > E\right\} \cap \{\mathbf{S}_K \leq E\}\right)$$

$$+ \mathbb{P}\left(\left\{\mathbf{S}(T) > E + h^{\widehat{\beta}}\right\} \cap \{\mathbf{S}_K \leq E\}\right)$$

$$\leq \mathbb{P}\left(\left\{E + h^{\widehat{\beta}} \geq \mathbf{S}(T) > E\right\}\right)$$

$$+ \mathbb{P}\left(\left\{\mathbf{S}(T) - \mathbf{S}_K > h^{\widehat{\beta}}\right\}\right)$$

$$\leq O(h^{\widehat{\beta}}) + \mathbb{E}\left[\frac{|\mathbf{S}(T) - \mathbf{S}_K|^m}{h^{m\widehat{\beta}}}\right]$$

$$\leq O(h^{\widehat{\beta}}) + \frac{C_m h^{m/2}}{h^{m\widehat{\beta}}}$$

$$= O(h^{\widehat{\beta}}) + O(h^{m/(2m+1)})$$

$$= O(h^{\widehat{\beta}})$$

Digital Option $\beta = \frac{1}{2}$?

Similarly, ...

$$\mathbb{P}\left(\{\mathbf{S}(T) \leq E\} \cap \{\mathbf{S}_K > E\}\right) = O(h^{\widehat{\beta}})$$

giving

$$\mathbb{E}\left[\left(\mathbf{P} - \widehat{\mathbf{P}}\right)^2\right] = O(h^{\widehat{\beta}}) = O(h^{\frac{1}{2} - \epsilon})$$

Shows $\beta = \frac{1}{2} - \epsilon$

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More complicated arguments also give

$\beta = \frac{1}{2} - \epsilon$ for up/down-and-out/in calls/puts

Summary

- **Giles** (2006) Multi-level Monte Carlo method reduces Euler–Maruyama complexity from $O(\epsilon^{-3})$ to $O(\epsilon^{-2}\log(\epsilon)^2)$ for glob. Lip. Payoff $f\left(\mathbf{S}(T)\right)$
- **Practical algorithm** gives good results
- Analysis and algorithm exploit both **weak** and **strong** convergence properties

New analysis extends results to the cases of

- Lookback options
- Digital options
- Barrier options